A theorem of Sjogren and Hartley on dimension subgroups

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In this note we give a direct proof of a generalized version of a result due to SJOGREN [5] and HARTLEY [4], and point out a far reaching application. Let $H=H_1 \geqq H_2 \geqq \dots$ and $K=K_1 \geqq K_2 \geqq \dots$ be series of normal subgroups of a group F and let $\{D_{k,l};\ 1 \leqq k < l\}$ be a family of normal subgroups of F such that (a) $D_{k,k+1}=H_k \cdot K_{k+1}$, (b) $H_k \cdot K_l \leqq D_{k,l}$ and (c) $D_{k,l+1} \leqq D_{k,l}$ for all $k<l$. We prove the following theorem whose foundation lies in [5] and [4].

Theorem A. *For each* $2 \leqq k+m \leqq n+1$, $k, m, n \geqq 1$, *let there exist a positive integer* $a(k)$, *depending only on* k *and* n, *such that* $(K_{k+m} \cap D_{k,k+m+1})^{a(k)} \leqq D_{k+1,k+m+1} \cdot H_k$. *Then* $(D_{1,n+2})^{a(1,n+1)} \leqq H_1 \cdot K_{n+2}$, *where* $a(1,n+1)=a(1)^{\binom{n}{1}} \dots a(n)^{\binom{n}{n}}$.

PROOF. We prove by induction on $1 \leqq m \leqq n$, that $(D_{k,k+m+1})^{a(k,m+1)} \leqq H_k \cdot K_{k+m+1}$, where $a(k,m+1)=a(k)^{\binom{m}{1}} \dots a(k+m-1)^{\binom{m}{m}}$; the proof of the theorem then follows with $k=1$, $m=n$. When $m=1$, we have $D_{k,k+2} \leqq D_{k,k+1}=H_k \cdot K_{k+1}$ and it follows that $D_{k,k+2} \leqq H_k \cdot K_{k+1} \cap D_{k,k+2} \leqq (K_{k+1} \cap D_{k,k+2}) \cdot H_k$, since $H_k \leqq D_{k,k+2}$. Thus by the given hypothesis $(D_{k,k+2})^{a(k)} \leqq D_{k+1,k+2} \cdot H_k = H_{k+1} \cdot K_{k+2} \cdot H_k = H_k \cdot K_{k+2}$ as required since $a(k)=a(k,2)$. For the inductive step, let $m \geqq 2$ and assume the result for $m-1$. Then we have $D_{k,k+m+1} \leqq D_{k,k+(m-1)+1}$ and the induction hypothesis yields $(D_{k,k+m+1})^{a(k,m)} \leqq H_k \cdot K_{k+m} \cap D_{k,k+m+1} \leqq (K_{k+m} \cap D_{k,k+m+1}) \cdot H_k$, since $H_k \leqq D_{k,k+m+1}$. By the given hypothesis it now follows that $(D_{k,k+m+1})^{a(k,m) \cdot a(k)} \leqq (K_{k+m} \cap D_{k,k+m+1})^{a(k)} \cdot H_k \leqq D_{k+1,k+m+1} \cdot H_k$. On the other hand, $D_{k+1,k+m+1}=D_{k+1,k+1+m}$ gives, by the induction hypothesis,

$$(D_{k+1,k+m+1})^{a(k+1,m)} \leqq H_{k+1} \cdot H_{k+m+1}.$$

This yields, in turn,

$$(D_{k,k+m+1})^{a(k,m) \cdot a(k) \cdot a(k+1,m)} \leqq H_k \cdot K_{k+m+1}.$$

It only remains to verify that $a(k,m+1)=a(k,m) \cdot a(k) \cdot a(k+1,m)$. Indeed, we have

$$a(k,m) = a(k)^{\binom{m-1}{1}} \dots a(k+m-2)^{\binom{m-1}{m-1}}$$

and

$$a(k+1,m) = a(k+1)^{\binom{m-1}{1}} \dots a(k+1+m-2)^{\binom{m-1}{m-1}}.$$

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Thus

$$a(k, m) \cdot a(k) \cdot a(k+1, m) =$$

$$= a(k)^{\binom{m-1}{0}+\binom{m-1}{1}} a(k+1)^{\binom{m-1}{1}+\binom{m-1}{2}} \dots a(k+m-2)^{\binom{m-1}{m-2}+\binom{m-1}{m-1}} a(k+m-1) =$$

$$= a(k)^{\binom{m}{1}} a(k+1)^{\binom{m}{2}} \dots a(k+m-2)^{\binom{m}{m-1}} a(k+m-1)^{\binom{m}{m}} =$$

$$= a(k, m+1).$$

An application. Let $G=F/RF''$ be a finitely generated metabelian group with F/RF' finite. Let $\mathbf{f}=\mathbf{Z}F(F-1)$ be the augmentation ideal of the free group ring $\mathbf{Z}F$. Define $\mathbf{r}(1)=\mathbf{r}=\mathbf{Z}F(R-1)$ and for $k\geqq 2$, $\mathbf{r}(k)=\sum_{i+j=k-1} \mathbf{f}^i\mathbf{r}\mathbf{f}^j$, $i,j\geqq 0$. Also, define $R(1)=R$, $R(k)=[R(k-1), F]$, $k\geqq 2$. Let $\mathbf{a}=\mathbf{Z}F(F'-1)$ and set $D(k, l)=$ $=F\cap(1+\mathbf{r}(k)+\mathbf{a}^2+\mathbf{f}^l)$ for all $1\leqq k<l$. It is easy to prove that $D(k, k+1)=$ $=R(k)\cdot F''\cdot\gamma_{k+1}(F)$. Thus, with $H_k=R(k)\cdot F''$, $K_l=\gamma_l(F)$ and $D_{k,l}=D(k, l)$, the family $\{D(k, l); 1\leqq k<l\}$, of normal subgroups of F, satisfies the conditions (a), (b) and (c) stated earlier. Let G be a finitely generated metabelian p-group and let $n\leqq 2p-3$. Then, for a suitable free presentation $1\to RF''\to F\to G\to 1$ of G, by a delicate blend of techniques from CLIFF—HARTLEY [1] and GUPTA [2] it is possible to choose $a(k)=k!$ for $k=1, \dots, p-1$ and $a(k)=(n-k+2)!$ for $k=p, \dots, n+1$ such that $(\gamma_{k+m}(F)\cap D(k, k+m+1))^{a(k)}\leqq D(k+1, k+m+1)\cdot R(K)\cdot F''$. It follows by Theorem A that $(D(1, n+2))^{a(1,n+1)}\leqq R\cdot F''\cdot\gamma_{n+2}(F)$, where $a(1, n+1)=$ $=a(1)^{\binom{n}{1}}\dots a(n)^{\binom{n}{n}}$ is coprime to p. Thus $F\cap(1+\mathbf{r}+\mathbf{a}^2+\mathbf{f}^{n+2})\leqq R\cdot F''\cdot\gamma_{n+2}(F)$ and consequently we have the following important result,

Theorem B. *Let G be a finite metabelian p-group. Then for $n\leqq 2p-1$ the n-th dimension subgroup $D_n(G)$ coincides with the n-th lower central subgroup $\gamma_n(G)$.*

[Details will be published elsewhere. The earlier best-known results in this direction are: $D_n(G)=\gamma_n(G)$ for $n\leqq p+2$, p odd (GUPTA—TAHARA [3]), and for arbitrary p-groups $D_n(G)=\gamma_n(G)$ for $n\leqq p+1$ (SJOGREN [5])].

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