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Sasakian anti-holomorphic submanifolds of locally conformal Kaehler manifolds

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Abstract. Some necessary and sufficient conditions for an anti-holomorphic submanifold in a locally conformal Kaehler manifold to be a Sasakian submanifold are obtained.

1. Introduction

The differential geometry of the CR submanifolds of a locally conformal Kaehler (l.c.K.) manifold have been studied in the last ten years (cf. [2]–[5]). A. Bejancu introduced the concept of Sasakian anti-holomorphic submanifolds in a Kaehler manifold [1]. The Sasakian anti-holomorphic submanifolds in an l.c.K. manifold are studied by F. Verroca [7], she gives some characterizations for them under the condition that the normal connection is flat. In the present paper we make a further study of the Sasakian anti-holomorphic submanifolds of an l.c.K. manifold, we give some characterizations for them so that the results in §4 of [7] are still true without supposing that the normal connection is flat.

2. Preliminaries

Let $(\bar{M}, g, J)$ be a Hermitian manifold of complex dimension n with Kaehler 2-form Ω_0, i.e. $\Omega_0(X,Y) = g(X, JY)$, $X, Y \in T\bar{M}$. Then $\bar{M}$ is a

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locally conformal Kaehler (l.c.K.) manifold if there exists a closed 1-form ω_0 on $\bar{M}$ such that [6]

$$d\Omega_0 = \omega_0 \wedge \Omega_0. \tag{2.1}$$

The 1-form ω_0 is called the Lee form, then the Lee vector field is the vector field B_0 such that $g(B_0, X) = \omega_0(X)$. If $\bar{\nabla}$ denotes the Riemannian connection of $\bar{M}$, then one has:

$$(\bar{\nabla}_X J)Y = \frac{1}{2}(\theta_0(Y)X - \omega_0(Y)JX - \Omega_0(X,Y)B_0 - g(X,Y)A_0) \tag{2.2}$$

for any $X, Y \in T\bar{M}$, where $\theta_0 = \omega_0 \cdot J$ is the anti-Lee 1-form and $A_0 = -JB_0$ is the anti-Lee vector field.

An m-dimensional submanifold M of $\bar{M}$ is called a CR submanifold if the tangent bundle TM is expressed as a direct sum of two distributions D and $D^\perp$, such that D is holomorphic (i.e. $J_x D_x = D_x$, $x \in M$) and $D^\perp$ is totally real (i.e. $J_x D_x^\perp \subset T_x^\perp M$), in particular, if $J_x D_x^\perp = T_x^\perp M$, then M is called anti-holomorphic submanifold. Denote by P and Q the projection morphism of TM to D and $D^\perp$, respectively, then, restricted to M, $P + Q = I$.

For $X \in T_x M$, denote

$$JX = \phi X + FX \tag{2.3}$$

where ϕX and FX are, respectively the tangent part and the normal part of JX, then we have

$$\phi = J \cdot P, \quad F = J \cdot Q, \quad F \cdot \phi = 0, \quad \phi \cdot Q = 0 \tag{2.4}$$

$$\phi^2 = -P = -I + Q. \tag{2.5}$$

The Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \text{and} \quad \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N \tag{2.6}$$

respectively, where $X, Y \in TM$, $N \in T^\perp M$. Now ∇, h, A and $\nabla^\perp$ are the induced connection, the second fundamental form, the Weingarten operator and the normal connection, respectively. Denote by θ, ω and Ω the forms induced on M by θ_0, ω_0 and Ω_0, respectively. Then one has

$$\theta = \omega \cdot \phi + \omega_0 \cdot F, \qquad \Omega(X,Y) = g(X, \phi Y), \quad X, Y \in TM. \tag{2.7}$$

For $X, Y \in TM$, define

$$dF(X,Y) = \frac{1}{2}(\nabla^{\perp}_X FY - \nabla^{\perp}_Y FX - F[X,Y]). \tag{2.8}$$

From (2.2), we have

$$\begin{aligned}\nabla^{\perp}_X FY &= \nabla^{\perp}_X JQY = (\bar{\nabla}_X JQY)^{\perp} = ((\bar{\nabla}_X J)QY + J\bar{\nabla}_X QY) \\ &= F\nabla_X QY - \frac{1}{2}(\omega(QY)FX + g(X,QY)A_0).\end{aligned} \tag{2.9}$$

Let $E_1, \dots, E_p, JE_1, \dots, JE_p$ be an orthonormal basis for D, then the normal vector field

$$H_D = \frac{1}{2p}\sum_{i=1}^{p}(h(E_i,E_i) + h(JE_i,JE_i))$$

is well defined and is called the D-mean curvature vector of M.

Definition 2.1. Let M be an anti-holomorphic submanifold of an l.c.K. manifold $\bar{M}$; M is called normal if for any $X, Y \in TM$

$$N^{(1)}(X,Y) \equiv [\phi,\phi](X,Y) - 2J(dF)(X,Y) = 0. \tag{2.10}$$

Here $[\phi,\phi]$ is the Nijenhuis torsion of ϕ. M is called contact if $H_D \neq 0$ and

$$(dF)(X,Y) = -\Omega(X,Y)H_D, \qquad X, Y \in TM. \tag{2.11}$$

A normal contact anti-holomorphic submanifold of an l.c.K. manifold is called a Sasakian anti-holomorphic submanifold.

Proposition 2.1. *If M is a normal (or contact) anti-holomorphic submanifold of an l.c.K. manifold $\bar{M}$, then*

$$N^{(2)}(X,Y) = dF(\phi X,Y) - dF(\phi Y,X) = 0. \tag{2.12}$$

Proof. If M is a contact anti-holomorphic submanifold of $\bar{M}$, then from (2.11) we obtain (2.12) immediately. Now suppose that M is a normal anti-holomorphic submanifold for $\xi \in D^{\perp}$; from (2.10) we obtain

$$(\phi^2[X,\xi] - \phi[\phi X,\xi]) - J(\nabla^{\perp}_X F\xi - \nabla^{\perp}_{\xi} FX - F[X,\xi]) = 0, \tag{2.13}$$

so we have

$$\nabla^{\perp}_{X} F\xi - \nabla^{\perp}_{\xi} FX - F[X, \xi] = 0. \tag{2.14}$$

Replacing X by ϕX in (2.14) we have

$$\nabla^{\perp}_{\phi X} F\xi = F[\phi X, \xi]. \tag{2.15}$$

Substituting (2.9) into (2.15), we get

$$0 = \nabla^{\perp}_{\phi X} F\xi - F\nabla_{\phi X}\xi + F\nabla_{\xi}\phi X = F\nabla_{\xi}\phi X. \tag{2.16}$$

On the other hand, from (2.4) and (2.5) we can derive

$$\begin{aligned} 0 &= N^{(1)}(X, \phi Y) = [\phi, \phi](X, \phi Y) - 2JdF(X, \phi Y) \\ &= [\phi X, QY] - [\phi X, Y] - \phi[\phi X, \phi Y] + \phi[X, PY] \\ &\quad - [X, \phi Y] + J\nabla^{\perp}_{\phi Y} FX. \end{aligned} \tag{2.17}$$

Projecting to $D^{\perp}$ we obtain

$$J\nabla^{\perp}_{\phi Y} FX - Q[\phi X, Y] + Q[\phi X, QY] - Q[X, \phi Y] = 0. \tag{2.18}$$

Operating (2.18) by J we have

$$-\nabla^{\perp}_{\phi Y} FX + F[\phi Y, X] + F[\phi X, QY] - F[\phi X, Y] = 0. \tag{2.19}$$

From (2.15) we get

$$F[\phi X, QY] = \nabla^{\perp}_{\phi X} FY. \tag{2.20}$$

From (2.19) and (2.20) we have

$$-\nabla^{\perp}_{\phi Y} FX + F[\phi Y, X] + \nabla^{\perp}_{\phi X} FY - F[\phi X, Y] = 0,$$

i.e. $N^{(2)}(X, Y) = dF(\phi X, Y) - dF(\phi Y, X) = 0$.

Lemma 2.1. [7]. *Let M be a CR submanifold of the l.c.K. manifold $\bar{M}$. Then we have*

$$\begin{aligned} 2g((\nabla_X \phi)Y, Z) = {} & 3(d\Omega)(X, \phi Y, \phi Z) - 3(d\Omega)(X, Y, Z) + g([\phi, \phi](Y, Z), \phi X) \\ & + 2g((dF)(\phi Y, Z), FX) + 2g((dF)(\phi Y, X), FZ) \\ & - 2g((dF)(\phi Z, X), FY) - 2g((dF)(\phi Z, Y), FX). \end{aligned}$$

Proposition 2.2. *If M is a contact anti-holomorphic submanifold of an l.c.K. manifold $\bar{M}$, and M is orthogonal to the Lee vector field B_0, then*

$$2g((\nabla_X\phi)Y,Z) = g(N^{(1)}(Y,Z),\phi X) + 2g(N^{(2)}(Y,Z),FX) + 2g(dF(\phi Y,X),FZ) - 2g(dF(\phi Z,X),FY). \tag{2.21}$$

Proof. Since M is orthogonal to the Lee vector field B_0 for $X, Y, Z \in TM$, from Lemma 2.1, (2.10) and (2.12) we can obtain (2.21).

3. Sasakian anti-holomorphic submanifolds

Theorem 3.1. *Let M be an anti-holomorphic submanifold of an l.c.K. manifold, and M orthonormal to the Lee vector field B_0. If M is a Sasakian anti-holomorphic submanifold, then*

$$(\nabla_X\phi)Y = g(\phi X,\phi Y)JH_D + g(FY,H_D)PX. \tag{3.1}$$

Conversely, if there exists $\xi \in T^\perp M$ such that

$$(\nabla_X\phi)Y = g(\phi X,\phi Y)J\xi + g(FY,\xi)PX \tag{3.1$'$}$$

then M is a Sasakian anti-holomorphic submanifold and $\xi = H_D$.

Proof. Suppose M is a Sasakian anti-holomorphic submanifold. By Proposition 2.1 and Proposition 2.2 we have

$$\begin{aligned} g((\nabla_X\phi)Y,Z) &= g(dF(\phi Y,X),FZ) - g(dF(\phi Z,X),FY) \\ &= g(\phi Y,\phi X)g(-H_D,FZ) + g(\phi Z,\phi X)g(H_D,FY) \\ &= g(\phi Y,\phi X)g(JH_D,Z) + g(H_D,FY)g(PX,Z) \end{aligned} \tag{3.2}$$

for any Z, so we get (3.1).

Conversely, suppose that $(3.1)'$ holds, then from (2.9) and (2.5) we get

$$\begin{aligned} 2dF(X,Y) &= \nabla^\perp_X FY - \nabla^\perp_Y FX - F[X,Y] \\ &= F(\nabla_X QY - \nabla_Y QX - \nabla_X Y - \nabla_Y X) \\ &= F(\nabla_X \phi^2 Y - \nabla_Y \phi^2 X) = F((\nabla_X\phi)\phi Y - (\nabla_Y\phi)\phi X) \\ &= (-g(\phi X,\phi^2 Y) + g(\phi Y,\phi^2 X))\xi = -2\Omega(X,Y)\xi. \end{aligned} \tag{3.3}$$

In the sequel we prove that $\xi = H_D$. Let $E_1, \dots, E_p, JE_1, \dots, JE_p$ be an orthonormal basis for D, then

$$\begin{aligned} -h(E_i, E_i) &= F(Jh(E_i, E_i)) = F(J(\bar{\nabla}_{E_i} E_i)) = F(Q\bar{\nabla}_{E_i} JE_i) \\ &= F(Q\bar{\nabla}_{E_i}\phi E_i) = F(Q(\nabla_{E_i}\phi)E_i) = F(Q(\nabla_{E_i}\phi)E_i) = -\xi. \end{aligned}$$

Similarly we have $h(JE_i, JE_i) = \xi$, so $\xi = H_D$ is the D-mean curvature vector of M, and M is a contact anti-holomorphic submanifold.

On the other hand

$$\begin{aligned} [\phi,\phi](X,Y) &= [\phi X, \phi Y] + \phi^2[X,Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \\ &= (\nabla_{\phi X}\phi)Y - (\nabla_{\phi Y}\phi)X + \phi(\nabla_Y\phi)X - \phi(\nabla_X\phi)Y \\ &= g(\phi^2 X, \phi Y)J\xi - g(\phi^2 Y, \phi X)J\xi = -2\Omega(X,Y)J\xi \\ &= -2\Omega(X,Y)JH_D. \end{aligned} \tag{3.4}$$

Combining with (3.3), (3.4) and $\xi = H_D$ we have $N^{(1)} = 0$, M is normal, and this completes the Proof of Theorem 3.1.

By direct computation we have

$$(\nabla_X \Omega)(Y,Z) = g(Y, (\nabla_X\phi)Z). \tag{3.5}$$

From (3.1) and (3.5) we obtain the

Theorem 3.2. *Let M be an anti-holomorphic submanifold of an l.c.K. manifold $\bar{M}$. If M is orthonormal to the Lee vector field B_0, then M is a Sasakian anti-holomorphic submanifold if and only if for any $X, Y, Z \in TM$*

$$(\nabla_X\Omega)(Y,Z) = g(\phi X, \phi Y)g(H_D, FZ) - g(\phi X, \phi Z)g(H_D, FY). \tag{3.6}$$

Remark. From Theorem 3.1 and Theorem 3.2 we know that the results in §4 of [7] still hold without the assumption that the normal connection is flat.

Theorem 3.3. *Let M be a Sasakian anti-holomorphic submanifold of an l.c.K. manifold, and $\xi \in D^\perp$. If M is orthonormal to the Lee vector field B_0, then*

$$P\nabla_X\xi = g(F\xi, H_D)\phi X. \tag{3.7}$$

In particular, if $F\xi$ is parallel with respect to the normal connection, then

$$\nabla_X \xi = g(F\xi, H_D)\phi X. \tag{3.8}$$

Proof. From (3.1) we have

$$0 = \phi\nabla_X\xi + (\nabla_X\phi)\xi = \phi\nabla_X\xi + g(F\xi, H_D)PX. \tag{3.9}$$

Operating (3.9) by ϕ and combining with (2.5), we get (3.7).

Using (2.9) again, we have

$$\nabla_X\xi = P\nabla_X\xi + Q\nabla_X\xi = g(F\xi, H_D)\phi X - J\nabla^{\perp}_X F\xi,$$

and this implies (3.8).

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