Weak convergence of sequences of random elements with multidimensional random indices

By Z. A. ŁAGODOWSKI and Z. RYCHLIK (Lublin)

Conditions are given for a randomly indexed sequence of random variables with multidimensional indices to converge weakly. We extend limit theorems with random indices to random fields. Functional central limit theorems with random multidimensional indices are also presented.

1. Introduction

Let Z^d, where $d \geqq 1$ is an integer, denote the positive integer d-dimensional lattice points. The points in Z^d will be denoted by m, n, etc., or sometimes, when necessary, more explicitly, by $(m_1, m_2, \ldots, m_d)$, $(n_1, \ldots, n_d)$, etc. The set Z^d is partially ordered by stipulating $m \leqq n$ iff $m_i \leqq n_i$ for each i, $1 \leqq i \leqq d$. We write 0 and 1 for the points $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ in Z^d, respectively, and $m < n$ iff $m_i < n_i$, $1 \leqq i \leqq d$.

Let (S, d) be a separable metric space equipped with its Borel σ-field $\mathscr{B}$. Let $\{Y_n, n \in Z^d\}$ be a random field, i.e., a collection of S-valued random elements defined on a probability space $(\Omega, \mathscr{A}, P)$.

Let $\{N_n, n \in Z^d\}$ be a set of Z^d -valued random variables defined on the same probability space $(\Omega, \mathscr{A}, P)$, i.e., for every $n \in Z^d$, the $N_n = (N_n^{(1)}, N_n^{(2)}, \ldots, N_n^{(d)})$, where $N_n^{(i)}$, $1 \leqq i \leqq d$, are positive integer — valued random variables.

If $n = (n_1, n_2, \ldots, n_d)$, let $|n|$ stand for the product $n_1 n_2 \ldots n_d$. In this paper the limit $n \to \infty$ will mean $\min\limits_{1 \leqq i \leqq d} n_i \to \infty$.

Assume that $Y_n \xrightarrow{\mathscr{D}} Y$, as $n \to \infty$, converges weakly to an S-valued random element Y with the distribution μ. This paper is aimed at presenting conditions on $\{Y_n, n \in Z^d\}$ and $\{N_n, n \in Z^d\}$ for $Y_{N_n} \xrightarrow{\mathscr{D}} \mu$ to hold, in the case where no assumption concerning the interdependence between $\{Y_n, n \in Z^d\}$ and $\{N_n, n \in Z^d\}$ is made.

We extend ALDOUS' results [1] to nonstationary random fields. The basic results are given in Theorems 1, 2 and 4. The presented results extend also the main theorems given in [3], [5], [6], [8] and [9].

Weak convergence of random fields has been studied by many authors. Situations in which such convergence arises can be found, for example, in [6]. Applications of random fields can also be found in biological investigations, in problems involving propagation of electromagnetic waves through random media and in the theory of turbulance. Random fields such as multiparameter stochastic processes, which we study in Section 3, play a prominent role in weak convergence of empirical processes. On the other hand, the theorems presented in Section 2 can be very useful in sequential analysis.

2. Extensions of Aldous' theorems to random fields

Let $\{k_n, n\in Z^d\}$ be a collection of positive numbers such that $k_n\to\infty$ as $n\to\infty$. We assume that $\{k_n, n\in Z^d\}$ is non-decreasing in the sense that for every $n, m\in Z^d$, $k_n\leqq k_m$ provided $n\leqq m$.

Definition 1. A random field $\{Y_n, n\in Z^d\}$ is said to satisfy the generalized Anscombe condition with norming family $\{k_n, n\in Z^d\}$ of positive numbers if for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\limsup_{n\to\infty} P\Big(\max_{i\in D_n(\delta)} d(Y_i, Y_n)\geqq\varepsilon\Big)\leqq\varepsilon, \tag{1}$$

where $D_n(\delta)=\{i\in Z^d\colon |k_i-k_n|\leqq\delta k_n\}$.

We note that in the case $d=1$ the concept of norming sequence and the generalized Anscombe condition have been introduced in [8].

Theorem 1. *Let $\{Y_n, n\in Z^d\}$ be a random field. The following conditions are equivalent:*

(i) *$\{Y_n, n\in Z^d\}$ satisfies the generalized Anscombe condition with $\{k_n, n\in Z^d\}$ and $Y_n\xrightarrow{\mathcal{D}}\mu$ as $n\to\infty$:*

(ii) *$Y_{N_n}\xrightarrow{\mathcal{D}}\mu$, as $n\to\infty$, for every random field $\{N_n, n\in Z^d\}$ such that*

$$k_{N_n}/k_{a_n}\xrightarrow{P}1 \quad \text{as} \quad n\to\infty \tag{2}$$

for some family $\{a_n: n\in Z^d\}$ such that $a_n\in Z^d$ and $a_n\to\infty$ as $n\to\infty$.

Proof. The implication (i)$\Rightarrow$(ii) can be proved similarly as that given in Proposition 1 [1]. On the other hand in the proof (ii)$\Rightarrow$(i) we can also follow the ideas presented in the proof of Proposition 1 [1], but in this case we need a total order relation in the set Z^d in order to introduce stopping random variables. Of course, the total order relation "$\prec$" can be defined as follows. Let $n=(n_1, \dots, n_d)\prec m=(m_1, \dots, m_d)$ mean that $\Sigma n=n_1+\dots+n_d<m_1+\dots+m_d=\Sigma m$ or $\Sigma n=\Sigma m$ and $n_1>m_1$ or $\Sigma n=\Sigma m$ and $n_1=m_1$ and $n_2>m_2$ or ... or $\Sigma n=\Sigma m$ and $n_1=m_1$ and $n_2=m_2$ and ... and $n_{d-2}=m_{d-2}$ and $n_{d-1}>m_{d-1}$ or $n=m$. Thus we omit further details.

Remark. Let us observe that in Theorem 1 it is not enough, in general, to assume $k_n\to\infty$ as $n\to\infty$ instead of $k_n\to\infty$ as $|n|\to\infty$, since under the assumption $k_n\to\infty$ as $n\to\infty$ the set $D_n(\delta)$ can be unbounded. To see this let us take $k_n=\min(n_1, \dots, n_d)$, where $n=(n_1, \dots, n_d)$. On the other hand if in (1), (i) and (2) the limit $n\to\infty$ means $|n|\to\infty$, then (ii) also holds in this sense.

As an example of an application of Theorem 1 we have the following.

Corollary. *Let* $\{X_n, n\in Z^d\}$ *be a collection of independent random variables such that* $EX_n=0$ *and* $0<ES_n^2=B_n^2<\infty$, $n\in Z^d$, *where* $S_n=\sum_{1\leqq k\leqq n} X_k$. *Assume* $B_n\to\infty$ *as* $n\to\infty$. *Then the following conditions are equivalent:*

$$S_n/B_n \xrightarrow{\mathcal{D}} \mu \quad as \quad n\to\infty, \tag{3}$$

$$S_{N_n}/B_{N_n} \xrightarrow{\mathcal{D}} \mu \quad as \quad n\to\infty \tag{4}$$

for every random field $\{N_n, n\in Z^d\}$ *such that* $B_{N_n}^2/B_{a_n}^2 \xrightarrow{P} 1$ *as* $n\to\infty$ *for some family* $\{a_n, n\in Z^d\}$ *such that* $a_n\in Z^d$ *and* $a_n\to\infty$ *as* $n\to\infty$.

Proof. Let $k_n=B_n^2$, $n\in Z^d$. Then, for every $\varepsilon>0$ and $\delta>0$, we have

$$P\big(\max_{i\in D_n(\delta)} |S_i/B_i-S_n/B_n|\geqq\varepsilon\big)\leqq$$

$$\leqq P\big(|S_n|\geqq\varepsilon B_n(1-\delta)^{1/2}/2\delta\big)+P\big(\max_{i\in D_n(\delta)}|S_i-S_n|\geqq\varepsilon(1-\delta)^{1/2}B_n/2\big)\leqq$$

$$\leqq 4\delta^2/\varepsilon^2(1-\delta)+P\big(\max_{i\in D_n(\delta)}|S_i-S_n|\geqq\varepsilon(1-\delta)^{1/2}B_n/2\big).$$

Now using Theorem 1 [12], and the above inequalities, we can easily prove that $\{S_n/B_n, n\in Z^d\}$ satisfies the generalized Anscombe condition with norming family $\{B_n^2, n\in Z^d\}$. Thus the Corollary follows from Theorem 1.

Let $\{Y_n, n\in Z^d\}$ be an S-valued random field defined on a probability space $(\Omega, \mathcal{A}, P)$. For $B\in\mathcal{A}$, let P_B be the restriction measure defined by the equality $P_B(A)=P(A\cap B)$, $A\in\mathcal{A}$, and let E_B denote expectation with respect to P_B. If $Y_n\xrightarrow{\mathcal{D}}\mu$ and for each $B\in\mathcal{A}$ there exists a measure μ_B such that for every continuous and bounded real function f defined on S

$$E_B f(Y_n)\to\int f\,d\mu_B \quad \text{as} \quad n\to\infty, \tag{5}$$

then we will write $Y_n\xrightarrow{\mathcal{D}}\mu$ (stably) (cf. [1]). In the special case when $\mu_B=\mu(\cdot)P(B)$ for all $B\in\mathcal{A}$ we write $Y_n\xrightarrow{\mathcal{D}}\mu$ (mixing) (cf. [2], [11]).

Let us observe that if in Theorem 1 $Y_n\xrightarrow{\mathcal{D}}\mu$ (stably), then $Y_{N_n}\xrightarrow{\mathcal{D}}\mu$ (stably), since if (1) holds then, for every $B\in\mathcal{A}$ with $P(B)>0$, (1) also holds with the measure $P_B(\,.\,)/P(B)$. The same is true in the case $Y_n\xrightarrow{\mathcal{D}}\mu$ (mixing).

Let $\{k_n, n\in Z^d\}$ be a collection of positive numbers such that $k_n\to\infty$ as $n\to\infty$. We assume that $n\leqq m$ implies $k_n\leqq k_m$, $n, m\in Z^d$.

Let $\{N_n, n\in Z^d\}$ be a set of Z^d-valued random variables.

Definition 2. A random field $\{N_n, n\in Z^d\}$ is said to satisfy the condition ($\triangle$) with norming family $\{k_n, n\in Z^d\}$ if for every $\varepsilon>0$ and $\delta>0$ there exists a finite and measurable partition $\{A_1, \dots, A_M\}$ of Ω and $a_{n_j}\in Z^d$, $1\leqq j\leqq M$, $n\in Z^d$, such that $a_{n_j}\to\infty$ as $n\to\infty$ and

$$\limsup_{n\to\infty}\sum_{j=1}^{M} P_{A_j}(|k_{N_n}-k_{a_{n_j}}|>\delta k_{a_{n_j}})\leqq\varepsilon. \tag{6}$$

We would like to note that the condition ($\triangle$) is a d-dimensional version of the condition (3.4) [1].

Theorem 2. *Assume* $Y_n \xrightarrow{\mathscr{D}} \mu$ *(stably). If for every* $\varepsilon>0$ *there exists* $\delta>0$ *such that for every* $A\in\mathscr{A}$

$$\limsup_{n\to\infty} P_A\big(\max_{i\in D_n(\delta)} d(Y_i, Y_n) \geqq \varepsilon\big) \leqq \varepsilon P(A), \tag{7}$$

then $Y_{N_n} \xrightarrow{\mathscr{D}} \mu$ *for every random field* $\{N_n, n\in Z^d\}$ *satisfying the condition* $(\triangle)$.

The proof of Theorem 2 is similar to that of Proposition 5 [1], so we omit the details.

Let us observe that in Theorem 2, in general, we may not assume $Y_n \xrightarrow{\mathscr{D}} \mu$ as $n\to\infty$ instead of $Y_n \xrightarrow{\mathscr{D}} \mu$ (stably) as $n\to\infty$, since we have the following.

Theorem 3. *Assume* $Y_{N_n} \xrightarrow{\mathscr{D}} \mu$ *as* $n\to\infty$ *for every random field* $\{N_n, n\in Z^d\}$ *satisfying the condition* $(\triangle)$. *Then* $Y_n \xrightarrow{\mathscr{D}} \mu$ *(stably) as* $n\to\infty$.

Theorem 3 can be proved analogously as Lemma 4 [1].

3. Random functional limit theorems

Let T denote the interval $[0, \infty)$ and T_d the d-fold Cartesian product of T. Let $D_d[0, \infty)$ denote the space of real functions defined on T_d, which are "continuous from above, with limits from below" (cf. [6], [7] and [10]). We consider the space $D_d[0, \infty)$ with the metric ϱ introduced in [10], therefore $(D_d[0, \infty), \varrho)$ is a complete separable metric space.

Let $q=(q_1, \dots, q_d)$, where $0<q_i<\infty$, $1\leqq i\leqq d$. For a given fixed $0<\alpha<\infty$ define $F_q\colon D_d[0, \infty)\to D_d[0, \infty)$ by $F_q x(t)=|q|^{-\alpha} x(q_1 t_1, \dots, q_d t_d)$. It is easily seen that the map $(q, x)\to F_q x$ is jointly continuous.

Let $Z=Z(t)$, $t\in T_d$, be a random element of $D_d[0, \infty)$ and let $k_n=(k_n^{(1)}, \dots, k_n^{(d)})$, $n\in Z^d$ be a collection of d-dimensional vectors such that $k_n^{(i)}\geqq 0$, $1\leqq i\leqq d$. We assume that $|k_n|=\prod_{i=1}^{d} k_n^{(i)}\to\infty$ as $n\to\infty$ and $n\leqq m$ implies $|k_n|\leqq|k_m|$. In what follows for $t=(t_1, \dots, t_d)$ and $k_n=(k_n^{(1)}, \dots, k_n^{(d)})$ we write tk_n for $(t_1 k_n^{(1)}, \dots, t_d k_n^{(d)})$ and k_n/k_m for $(k_n^{(1)}/k_m^{(1)}, \dots, k_n^{(d)}/k_m^{(d)})$.

For some fixed $0<\alpha<\infty$ and every $t=(t_1, \dots, t_d)\in T_d$ define

$$Y_n(t) = |k_n|^{-\alpha} Z(tk_n), \quad n\in Z^d. \tag{8}$$

Let us observe that the partial sum processes considered in [6], [9] and [5] are special cases of (8). Namely, it is enough to take $Z(t)=\sum_{k\leqq t} X_k$, where X_k, $k\in Z^d$ is some underlying family of random variables.

Theorem 4. *Let* $\{Y_n, n\in Z^d\}$ *be a family of* $D_d[0, \infty)$*-valued random elements defined by* (8). *Then the following conditions are equivalent*

(i) $$Y_n \xrightarrow{\mathscr{D}} \mu \text{ (stably)};$$

(ii) $Y_{N_n} \xrightarrow{\mathscr{D}} \mu$ *for every set* $\{N_n, n\in Z^d\}$ *of* Z^d*-valued random variables satisfying the condition* $(\triangle)$ *with the norming family* $\{|k_n|, n\in Z^d\}$.

PROOF. According to Theorem 3, (ii) implies (i). Thus suppose that (i) holds. Let for every $x\in D_d[0,\infty)$ and for every $q=(q_1,\dots,q_d)$, such that $0<|q|<\infty$, be $F_q x(t)=|q|^{-\alpha}x(qt)$, we recall that $qt=(q_1t_1,\dots,q_dt_d)$. It is easy to see that the random elements Y_n, defined by (8), satisfy the following equalities: $Y_n(t)=F_{k_n}Z(t)$ and $Y_{N_n}(t)=F_{k_{N_n}}Z(t)$. Moreover, the map $(q,x)\to F_q x$ is jointly continuous, so that (i) implies (ii) by the following Lemma, which is a d-dimensional version of Lemma 10 [1].

Lemma. *Let (S,d) be a complete separable metric space. For $q=(q_1,\dots,q_d)$, with $0<|q|<\infty$, let $F_q\colon S\to S$ be such that*

$F_q x$ is jointly continuous;

$$F_qF_\pi=F_{q\pi},$$

where, for $\pi=(\pi_1,\dots,\pi_d)$, $q\pi=(q_1\pi_1,\dots,q_d\pi_d)$. Let Z be a random element of S such that $F_{k_n}(Z)\xrightarrow{\mathcal{D}}\mu$ (stably). Then $F_{k_{N_n}}(Z)\xrightarrow{\mathcal{D}}\mu$ for any random field $\{N_n,n\in Z^d\}$ satisfying $(\triangle)$ with the norming sequence $\{|k_n|,n\in Z^d\}$.

Taking into account Theorem 1 and Theorems 2 and 4, we see that it is possible to weaken condition (2) on $\{N_n,n\in Z^d\}$ to condition $(\triangle)$, at the cost of strengthening the conditions on $\{Y_n,n\in Z^d\}$. On the other hand even in the case $d=1$ (cf. [1] or [8]) the condition $(\triangle)$ is the weakest possible condition on $\{N_n,n\in Z^d\}$ under which useful random indices limit theorems may be obtained, without imposing conditions on the interdependence between $\{Y_n,n\in Z^d\}$ and $\{N_n,n\in Z^d\}$. Thus Theorem 4, from this point of view, is the best result possible. Furthermore one can prove that Theorem 4 remains unchanged if the range is restricted from $D_d[0,\infty)$ to $D_d[0,1]$. Thus, for example, from Theorem 4 we get the following extensions of some results presented in [6], [9] and [5].

Let $\{X_n,n\in Z^d\}$ be a random field. For each $n\in Z^d$, let F_n be the σ-field generated by $\{X_k: n\not\leqq k\}$.

Theorem 5. *Let $\{X_n,n\in Z^d\}$ be a stationary, ergodic random field for which $E(X_n|F_m)=0$, whenever $m<n$, with probability 1 and for which $EX_n^2=1$. If, for $t\in T_d[0,1]$ (d-fold Cartesian product of [0, 1]),*

$$Y_n(t)=(|n|)^{-1/2}\sum_{k\leqq nt}X_k,$$

then $Y_{N_n}\xrightarrow{\mathcal{D}}W$, in $D_d[0,1]$, as $n\to\infty$ for every set $\{N_n,n\in Z^d\}$ of Z^d-valued random variables statisfying $(\triangle)$ with $k_n=|n|$, $n\in Z^d$, where W is the d-parameter Wiener process.

PROOF. By Theorem 1 [5] $Y_n\xrightarrow{\mathcal{D}}W$, in $D_d[0,1]$, as $n\to\infty$. Furthermore by adaptation, to multivariate time, of the proof of Theorem 1 [4] (cf. also Remark 4 [4]) one can prove that $Y_n\xrightarrow{\mathcal{D}}W$ (mixing) as $n\to\infty$. Thus Theorem 5 follows from Theorem 4 with $n=(n_1,\dots,n_d)$, $k_n=(n_1,\dots,n_d)$.

We remark that if $|N_n|/|n|\xrightarrow{P}\lambda$, as $n\to\infty$, where λ is a positive random variable, then $(\triangle)$ holds.

Theorem 6. *Let $\{X_n,n\in Z^d\}$ be a family of independent random variables with zero means and finite variances. Assume, for each $n=(n_1,\dots,n_d)$, $EX_n^2=b_{n_1}^{(1)}b_{n_2}^{(2)}\dots b_{n_d}^{(d)}$,*

$\max_{1 \leq r \leq d} (\max_{1 \leq i \leq n} b_i^{(r)}/B_n^{(r)}) \to 0$, *as* $n \to \infty$, *and for every* $\varepsilon > 0$

$$|B_n|^{-1} \sum_{k \leq n} EX_k^2 I(|X_k| \geq \varepsilon |B_n|^{1/2}) \to 0 \quad as \quad n \to \infty,$$

where $B_n^{(r)} = \sum_{i=1}^{n} b_i^{(r)}$ *and* $B_n = (B_{n_1}^{(1)}, \ldots, B_{n_d}^{(d)})$. *If, for every*

$$t = (t_1, \ldots, t_d) \in T_d[0, 1], \quad B_n(t) = (B_{n_1}^{(1)}(t_1), \ldots, B_{n_d}^{(d)}(t_d))$$

$$B_{n_i}^{(i)}(t_i) = \max \{1 \geq 0 : B_1^{(i)} \leq t_i B_{n_i}^{(i)}\}, \quad B_0^{(i)} = 0, \quad 1 \leq i \leq d,$$

$$Z_n(t) = |B_n|^{-1/2} (\sum_{k \leq B_n(t)} X_k),$$

then $Z_{N_n} \xrightarrow{\mathcal{D}} W$, *in* $D_d[0, 1]$, *as* $n \to \infty$ *for every set* $\{N_n, n \in Z^d\}$ *of* Z^d*-valued random variables satisfying* ($\triangle$) *with* $k_n = |B_n|$, $n \in Z^d$.

PROOF By a straightforward adaptation of Theorem 5 [6] we can prove that $Z_n \xrightarrow{\mathcal{D}} W$ (mixing). Hence Theorem 6 follows from Theorem 4 with $k_n = (B_{n_1}^{(1)}, \ldots, B_{n_d}^{(d)})$.

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INSTITUTE OF AGRICULTURAL MECHANIZATION
AGRICULTURAL UNIVERSITY
AL. PKWN 28
20—612 LUBLIN (POLAND)

INSTITUTE OF MATHEMATICS
M. CURIE-SKŁODOWSKA UNIVERSITY
PL. M. CURIE-SKŁODOWSKIEJ 1
20—031 LUBLIN (POLAND)

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