Some remarks on a fixed point theorem of T. Kubiak

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Recently KUBIAK [2], extending fixed point theorems of KHAN and FISHER [1], RHOADES [3] and SINGH, TIWARI and GUPTA [7], gave a necessary and sufficient condition for the existence of a common fixed point of a pair of continuous mappings in 2-metric spaces. Our aim is to improve the result of KUBIAK [2] under more general conditions by using the concept of weak commutativity due to SESSA [5] in the context of metric spaces and a contractive condition due to SASTRY and NAIDU [4].

We first of all recall some well known definitions. Let X be a nonempty set and let $d: X \times X \times X \to [0, +\infty)$ ne a function such that

(i) $d(x, y, z) = 0$ if either $x = y$ or $x = z$ or $y = z$,

(ii) for each pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$,

(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$,

(iv) $d(x, y, z) \leqq d(x, y, w) + d(x, w, z) + d(w, y, z)$, for all $x, y, z, w \in X$.

The function d is called a 2-metric on X and (X, d) is called a 2-metric space. A sequence $\{x_n\}$ in X is said to be convergent to a point x in X if $\lim_{n \to \infty} d(x_n, x, a) = 0$ for all $a \in X$. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{m, n \to \infty} d(x_m x_n, a) = 0$ for all $a \in X$. A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

A 2-metric d on X is said to be continuous on X if it is sequentially continuous in two (and hence in each) of its three arguments. A mapping S of X into itself is said to be continuous at a point $x \in X$ if whenever a sequence $\{x_n\}$ converges to $x \in X$, then the sequence $\{Sx_n\}$ converges to Sx.

SESSA [5] introduced the concept of weak commutativity of a pair of mappings of an ordinary metric space into itself. We now extend this concept in a 2-metric space. Indeed, let S and A be two mappings of a 2-metric space (X, d) into itself. We say that S weakly commutes with A on X if

$$d(SAx, ASx, a) \leqq d(Ax, Sx. a)$$

for all $x, a \in X$. It is evident that two commuting mappings are also weakly commuting, but in general, two weakly commuting mappings do not commute as is shown in the following example.

Example 1. Let $\{(x_1, x_2): x_1, x_2 \geqq 0\}$ and let d be the 2-metric which expresses $d(x, y, a)$ as the area of the Euclidean triangle with vertices $x=(x_1, x_2)$, $y=(y_1, y_2)$ and $a=(a_1, a_2)$. Let A and S be two mappings of X into itself defined by

$$A(x_1, x_2) = (x_1/(x_1+4b), 0), \;\; S(x_1, x_2) = (x_1/b, 0)$$

for all $(x_1, x_2) \in X$, where $b \geqq 1$ is a constant. We have

$$d(SAx, ASx, a) = \frac{(b-1)x_1^2}{(x_1+4b^2)(bx_1+4b^2)} \cdot a_2 \leqq \frac{x_1^2+3bx_1}{bx_1+4b^2} \cdot a^2 = d(Ax, Sx, a)$$

for all $x, a \in X$. Thus S and A weakly commute on X but S and A do not commute if $b > 1$.

Drawing inspiration from a contractive condition of SASTRY and NAIDU [4], we now prove the following result.

Theorem 1. *Let (X, d) be a complete 2-metric space with d continuous on X and let S and T be two mappings of X into itself. If either S or T is continuous, then they have a common fixed point z in X if and only if there exists a mapping A of X into $T(X)$ and a mapping B of X into $S(X)$ such that A and B weakly commute with S and T respectively and satisfy the inequality*

$$(1) \qquad d(Ax, By, a) \leqq$$

$$\leqq \max\{cd(Sx, Ty, a), cd(Sx, Ax, a), cd(Ty, By, a), \; hd(Sx, Ty, a) + $$

$$+ kd(Ty, Ax, a)\}$$

for all $x, y, a \in X$, where $0 \leqq c < 1$, $h, k \geqq 0$,

$$(2) \qquad h+k < 1 \quad \text{and} \quad c \cdot \max\left\{\frac{h}{1-h}, \frac{k}{1-k}\right\} < 1.$$

Further, z is the unique common fixed point of A, B, S and T.

PROOF. This is similar to the first part of the proof of Theorem 3 of [6]. For the sake of completeness, we present the main steps of the proof in order to show how the weak commutativity and the continuity of S or T are essentially used. Of course the proof is modified in the details where the properties of the 2-metric d are taken into consideration. The necessity of the condition is proved as in [2].

We now show the converse implication. Let x_0 be an arbitrary point in X. Then since the ranges of S and T contain the ranges of B and A respectively, we define, as in [2], a sequence $\{x_n\}$ in X such that $Sx_{2n-1} = Bx_{2n-2}$ and $Tx_{2n} = Ax_{2n-1}$ for $n = 1, 2, \ldots$.

Adopting the same reasoning as in [2], it is not difficult to prove, using (1) and (2), that $\{Sx_{2n-1}\}$ is a Cauchy sequence and hence that it converges to a point $z \in X$, since X is complete. As in [2], it follows that

$$\lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Bx_{2n-2} = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Ax_{2n-1} = z.$$

Since

$$d(Ax_{2n-1}, Sx_{2n-1}, a) \leqq d(Ax_{2n-1}, z, Sx_{2n-1}) + d(Ax_{2n-1}, z, a) + \\ + d(Sx_{2n-1}, z, a)$$

for all $a \in X$, it follows that

$$\lim_{n\to\infty} d(Ax_{2n-1}, Sx_{2n-1}, a) = 0 \tag{3}$$

for all $a \in X$. Now assume that S is continuous. Then the sequences $\{S^2x_{2n-1}\}$ and $\{SAx_{2n-1}\}$ converge to Sz. Since S weakly commutes with A, we have

$$d(ASx_{2n-1}, Sz, a) \leqq$$
$$\leqq d(ASx_{2n-1}, Sz, SAx_{2n-1}) + d(ASx_{2n-1}, SAx_{2n-1}, a) + \\ + d(SAx_{2n-2}, Sz, a) \leqq$$
$$\leqq d(Ax_{2n-1}, Sx_{2n-1}, Sz) + d(Ax_{2n-1}, Sx_{2n-1}, a) + d(SAx_{2n-1}, Sz, a)$$

for all $a \in X$. By (3), this means that the sequence $\{ASx_{2n-1}\}$ also converges to Sz. Using (1) we now have

$$d(ASx_{2n-1}, Bx_{2n}, a) \leqq \max\{cd(S^2x_{2n-1}, Tx_{2n}, a), cd(S^2x_{2n-1}, ASx_{2n-1}, a), \\ cd(Tx_{2n}, Bx_{2n}, a), hd(S^2x_{2n-1}, Bx_{2n}, a) + kd(Tx_{2n}, ASx_{2n-1}, a)\}$$

for all $a \in X$. Since d is continuous, we have, on letting n tend to infinity in the foregoing inequality, that

$$d(Sz, z, a) \leqq \max\{c, h+k\} \cdot d(Sz, z, a)$$

for all $a \in X$ and by (2), we deduce that $Sz = z$.

From (1), we get

$$d(Az, Bx_{2n}, a) \leqq$$
$$\leqq \max\{cd(Sz, Tx_{2n}, a), cd(Az, Sz, a), cd(Tx_{2n}, Bx_{2n}, a), \\ hd(Sz, Bx_{2n}, a) + kd(Tx_{2n}, Az, a)\}$$

for all $a \in X$. Since d is continuous, this implies that, on letting n tend to infinity

$$d(Az, z, a) \leqq \max\{c, k\} \cdot d(z, Az, a)$$

for all $a \in X$. By (2), this means that $Az = z$.

Since $z = Az \in T(X)$, there exists a point $u \in X$ such that $Tu = z$. Using (1) again, we have

$$d(z, Bu, a) = d(Az, Bu, a) \leqq$$
$$\leqq \max\{cd(Sz, Tu, a), cd(Sz, Az, a), cd(Tu, Bu, a), hd(Sz, Bu, a) + kd(Tu, Az, a)\} =$$
$$= \max\{c, h\} \cdot d(z, Bu, a)$$

for all $a \in X$. By (2), we deduce that $Bu = z$.

Since T weakly commutes with B, we have

$$d(TBu, BTu, a) \leqq d(Bu, Tu, a) = d(z, z, a) = 0$$

for all $a \in X$. This implies that $Tz = TBu = BTu = Bz$ and from (1), it follows that

$$d(z, Bz, a) = d(Az, Bz, a) \leqq$$

$$\leqq \max\{cd(Sz, Tz, a), cd(Sz, Az, a), cd(Tz, Bz, a), hd(Sz, Bz, a) + kd(Tz, Az, a)\} =$$

$$= \max\{c, h+k\} \cdot d(z, Bz, a)$$

for all $a \in X$. By (2), we deduce that $z = Bz = Tz$. Thus z is a common fixed point of A, B, S and T. Of course, the proof is similar if we assume the continuity of T instead of S. The uniqueness of z is easily proved. This completes the proof of the theorem.

Remark 1. In Theorem 1 of [2] and Theorem 2 of [1], the authors assume the continuity of both S and T and also the commutativity of A and S and of B and T, but we assume only the continuity of either S or T and the weak commutativity of A and S and of B and T.

Remark 2. Kubiak [2] and Khan and Fisher [1] assume that A and B are mappings of X into $S(X) \cap T(X)$, but it suffices only to say that A maps X into $T(X)$ and B maps X into $S(X)$.

Remark 3. If we assume that $h = k = \frac{1}{2} c$, inequality (1) reduces to inequality (1) of [2].

The following example shows that the sufficiency condition of Theorem 1 of [2] is not applicable since S and T do not commute with A and B respectively, although both S and T are continuous.

Example 2. Let X be as in Example 1 and define A, B, S and T by

$$A(x_1, x_2) = (x_1/(x_1+16), 0), \quad B(x_1, x_2) = (x_1/(x_1+12), 0),$$

$$S(x_1, x_2) = (x_1/4, 0), \quad T(x_1, x_2) = (x_1/3, 0)$$

for all $(x_1, x_2) \in X$. By Example 1, we know that S and T weakly commute with A and B respectively. It is easily shown that A maps X into $T(X)$ and B maps X into $S(X)$ and that S and T are both continuous. Further

$$d(Ax, By, a) = \frac{4|3x_1 - 4y_1|}{(x_1+16)(y_1+12)} \cdot a_2 \leqq \frac{|3x_1 - 4y_1|}{48} = \frac{1}{4} \cdot d(Sx, Ty, a)$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2)$, $a = (a_1, a_2) \in X$. Thus inequality (1) holds with $c = 1/4$ and (0,0) is the unique common fixed point of A, B, S and T.

If neither of the mappings S and T are continuous then the following result holds.

Theorem 2. *Let (X, d) be a complete 2-metric space with d continuous on X and let S and T be two mappings of X into itself. Then S and T have a common fixed point z in X if and only if there exist mappings A of X into $T(X)$ and B of X into $S(X)$ such that either A or B is continuous, A and B weakly commute with S and T respectively, and* *satisfy the inequality* (1) *for all* $x, y, a \in X$, *where* $0 \leqq c < 1$, $h, k \geqq 0$ *and inequalities* (2) *hold. Then z is the unique common fixed point of A, B, S and T.*

PROOF. It is similar to the second part of the proof of Theorem 3 of [6] and we omit it for brevity.

Remark 4. Analogous results to Theorems 1 and 2 can be formulated in complete metric spaces.

Remark 5. Note that if we put $\alpha = \max\{c, h+k\}$, inequality (1) becomes

$$d(Ax, By, a) \leqq \tag{4}$$

$$\leqq \alpha \cdot \max\{d(Sx, Ty, a), d(Sx, Ax, a), d(Ty, By, a), d(Sx, By, a), d(Ax, Ty, a)\}$$

for all $x, y, a \in X$. Example 6 of [7], where $S = T$ is the identity mapping on X, proves that the analogous inequality in complete metric spaces does not in general guarantee the existence of a common fixed point of A, B, S and T.

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