A note on radicals and torsion theories

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For a subclass M of a universal class C of not necessarily associative rings we define as usual:

$$UM = \{A \in C \mid \text{ if } A \to B \neq 0 \text{ then } B \notin M\}, \text{ and}$$

$$SM = \{A \in C \mid \text{ if } 0 \neq B \lhd A \text{ then } BM\}, \text{ where}$$

$A \to B$ means B is a homomorphic image of A, and $B \lhd A$ means B is an ideal of A. In a recent paper [2] several possible properties of a class M were listed:

(a) M is homomorphically closed.

(a*) M is hereditary.

(A) M is "co-regular", that is $M \subseteq USM$.

(A*) M is "regular", that is $M \subseteq SUM$.

(A**) M is "hereditarily regular", that is $H(M) \subseteq SUM$, where $H(M)$ is the hereditary closure of M.

Note: In [3] a class M was called "co-regular" if $SUM \subseteq M$. However, it appears that the definition (A) given above is a preferable use of the term.

It is well-known that M regular is a sufficient condition that UM be a radical class. However, the following example shows that M can be co-regular with SM not semisimple and USM not radical:

Example 1. Let $M = \{Z_2^0$, the homomorphic closure of $Z_2[y]\}$, then $M = USM$ and note that except for $Z_2[y]$, all members of M are finite. Now let $R = Z_2[y, x_1, x_2, \ldots]$ where $yx_i = x_i y = x_{i+1}$ for all i, and $x_i^2 = x_i x_j = x_j x_i = 0$ for all i, j. It is easy to check that every non-zero ideal of R is infinite and contains nilpotents so is not in M. Thus $R \in SM$. Then if $I = \{\Sigma \alpha_i x_i\}$ then I is an ideal of R and any $0 \neq I/J$ contains an ideal $\cong Z_2^0 \in M$. Thus $I \in USM$ so SM is not semisimple. In fact, also we have $R/I \cong Z_2[y] \in USM$, so USM is not radical.

Also considered in [2] are pairs (E, F) of classes satisfying one or more conditions, including:

(1) $E \cap F = 0$.

(3) For any ring $A \in C$ there exists $B \lhd A$ with $B \in E$ and $A/B \in F$.

We will call (E, F) a "radical pair" if E is a radical class with F the corresponding semisimple class. It is well-known that (E, F) is a radical pair if and only if $E = UF$ and $F = SE$, and in [2] are given several equivalent conditions including:

II (E, F) satisfies (1), (3), and (a, A*) where (a, A*) means E satisfies (a) while F satisfies (A*).
In [2; Section 3.4] was asked:

Question 1. If the pair (E, F) satisfies

(α) Conditions (1), (3), and (A, A*), is (E, F) a radical pair? (The converse is, of course, clear.)

In [4] we defined a "torsion theory" as a pair (E, F) satisfying (1), (3), and (a, a*), and we showed (as is also clear from II) that a torsion theory is simply a radical pair whose semisimple class is hereditary. The two concepts thus coincide in any universal class in which all semisimple classes are hereditary (such as the class of all associative, or all alternative, rings). Another query in [2] was:

Question 2. If the pair of classes (E, F) satisfies:

(β) Conditions (1), (3), and (A, A**), is (E, F) a torsion theory? (Again the converse is clear.)

Since attempts to give affirmative answer to the above questions seem to fail, the next step is to try to construct a counter-example. In such a project it is useful to examine the properties required.

We will say that a pair of classes (E, F) is "entwined" in a ring R if there exists an infinite properly descending chain $R \rhd I_1 \rhd I_2 \rhd \dots$ and an infinite properly ascending chain (of accessible subrings) $0 = J_0 \lhd J_1 \lhd J_2 \lhd \dots$ such that for all $k \geqq 1$ each $J_k \lhd I_k$, $J_{k-1} \lhd I_k$ with $0 \neq I_k/J_{k-1} \in E$ and $0 \neq I_k/J_k \in F$. Then we have

Theorem 1. *If (E, F) satisfies (α) or (β) and $F \neq SUF$ then there exists a ring R in which (E, F) is entwined.*

Proof. By (A*) or (A**) we have $F \subseteqq SUF$ and if $F \neq SUF$ then there will exist a ring $R \in SUF$ with $R \notin F$. By (3) there is some $0 \neq I_1 \lhd R$ with $I_1 \in E$. Writing $J_0 = 0$ then $I_1/I_0 \in E$ and since $I_1/J_0 = I_1 \lhd R \in SUF$ it follows that I_1/J_0 has an image $I_1/J_1 \in F$. For induction assume we have a properly descending chain $R \rhd I_1 \rhd \rhd I_2 \rhd \dots \rhd I_k$ and a properly ascending chain $0 = J_0 \lhd J_1 \lhd J_2 \lhd \dots \lhd J_k$ such that for all $i = 1, 2, \dots, k$ we have $J_{i-1} \lhd I_i$ and $J_i \lhd I_i$ with $0 \neq I_i/J_{i-1} \in E$ and $0 \neq I_i/J_i \in F$. Since $E \subseteqq USE$ there is some $0 \neq I_{k+1}/J_k \lhd I_k/J_k$ such that $I_{k+1}/J_k \in E$ and since $E \cap F = 0$ we have $I_{k+1} \lhd I_k$ but not equal. But also $I_k/J_k \in F \subseteqq SUF$ so I_{k+1}/J_k has a proper image $I_{k+1}/J_{k+1} \in F$. Again from $E \cap F = 0$ we have $J_k \lhd J_{k+1}$ but not equal. The result then follows by induction.

Theorem 2. *If $(E. F)$ satisfies (α) or (β) and $E \neq USE$ then there exists a ring R in which (E, F) is entwined.*

Proof. Let $R \in USE$ with $R \notin E$ then by (3) there is some $J_0 \lhd R$ such that $J_0 \in E$ and $0 \neq R/J_0 \in F$. But $R \in USE$ so there is some $0 \neq I_1/J_0 \lhd R/J_0$ with $I_1/J_0 \in E$ and $I_1 \lhd R$ but not equal. Then since $R/J_0 \in F \subseteqq SUF$ it follows that I_1/J_0 has a proper image $I_1/J_1 \in F$. The induction is then the same as for Theorem 1 and (E, F) is entwined in R/J_0.

From the proof of the above theorems we see that

Corollary 1. *Let E be homomorphically closed. If (E, F) satisfies (α) then it is a radical pair and if (E, F) satisfies (β) then it is a torsion theory* (see [2; Theorem 3.3II and Theorem 2.3II]).

Corollary 2. *Let F be hereditary . If (E, F) satisfies (α) or (β) then it is a torsion theory* (see [2; Theorem 2.3III]).

We now proceed to the construction of a counter-example to Questions 1 and 2. The example will be constructed in the universal class C of all associative rings so from now on all rings will be assumed to be associative.

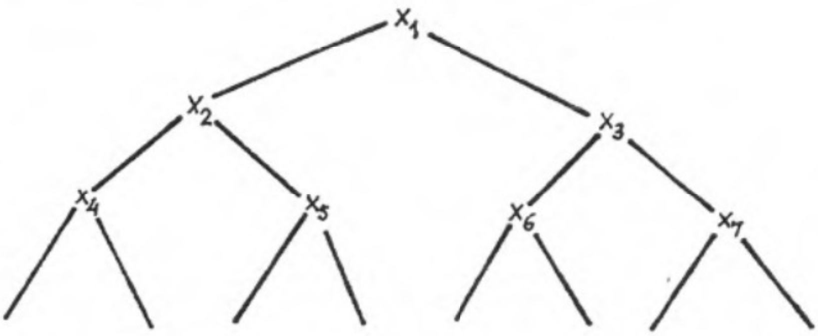
Write $A \lhd \lhd B$ when A is an accessible subring of B. Let R be a commutative ring with unit 1. For any integer $k \geqq 1$ we will designate as a "k-structure" any set $I_k \lhd \lhd I_{k-1} \lhd \lhd \ldots \lhd \lhd I_2 \lhd \lhd I_1 = R$ for which there is a set $\{f_1, f_2, \ldots, f_k\}$ where $f_1 = 1$, $f_i \in I_i$ and $f_i \notin J_i$ defined by

$$J_0 = 0,\ J_i = \{x \in I_i | x^n = u - f_i u + y \text{ for some } n \geqq 1, \text{ some } u \in I_i, \text{ and some } y \in J_{i-1}\}$$
$$\text{for all } i = 1, 2, \ldots, k.$$

Note that when R admits a k-structure then for $i = 1, 2, \ldots, k$ each I_i/J_i has a unit and zero nil ideal. We now define a class F as follows: F is the class of all commutative rings R with unit 1, zero nil ideal, and no minimal ideal such that for any k-structure contained in R if $I_{k+1}/J_k \lhd \lhd I_k/J_k$ then there exists $f_{k+1} \in I_{k+1}$ such that $f_{k+1} \notin J_{k+1}$ where $J_{k+1} = \{x \in I_{k+1} | x^n = u - f_{k+1}u + y$ for some $n \geqq 1$, some $u \in I_{k+1}$, and some $y \in J_k\}$, and such that I_{k+1}/J_{k+1} has no minimal ideal. Furthermore, for any $f \in I_{k+1}$ if $f \notin H$ where $H = \{x \in I_{k+1} | x^n = u - fu + y$ for some $n \geqq 1$, some $u \in I_{k+1}$, some $y \in J_k\}$ then I_{k+1}/H has no minimal ideal. As usual we assume 0 is a member of all classes, so also $0 \in F$.

To show that F is non-zero we note a ring given in [1] whose description we repeat here for completeness:

Example 2. Let R be the ring generated over Z_2 by commutative symbols $\{x_1, x_2, \ldots\}$ with relations described using the following diagram:



Each x_i is an identity for all x_j below it in the diagram, and (for $i < i$) $x_i x_j = 0$ if x_j is not below x_i. Also each x_i is the sum of the two symbols directly below. Thus $x_2 x_9 = x_9$ but $x_2 x_{12} = 0$, and $x_4 = x_8 + x_9$. The accessible subrings of R are the ideals of R and every ideal is a direct sum of copies of R. Also R is isomorphic with any finite direct sum of copies of R. Thus R is a commutative ring with unit, zero nil ideal, and no minimal ideal. Also if $I \lhd R$ and $J = \{u - fu$ for all $u \in I\}$ for any $f \in I$, then f is a finite sum of a set $\{x_i\}$ (all at the same level in the diagram) which then generate I modulo J, that is I/J is a finite direct sum of copies of R so $I/J \cong R$. Thus $R \in F$.

Theorem 3. $H(F) \subseteqq SUF$.

PROOF. Let $I \lhd A \lhd \lhd R \in F$. We have that R together with $f_1 = 1$ and $J_1 = 0$ is a k-structure in R with $k = 1$. Thus since $I \lhd \lhd R$, there is some $f \in I$ such that $f \notin J$ where $J = \{x \in I | x^n = u - fu$ for some $n \geqq 1$, and some $u \in I\}$, and such that I/J has no minimal ideal. Thus $\bar{I} = I/J$ is a ring with unit $\bar{f} = f + J$, zero nil ideal, and no minimal ideal. Suppose (for some $k \geqq 1$) $\bar{I}$ contains the k-structure $\bar{I}_k \lhd \lhd \dots \bar{I}_2 \lhd \lhd \bar{I}_1 = \bar{I}$ where we have $\{\bar{f}_1 = \bar{f}, \bar{f}_2, \dots, \bar{f}_k\}$ with $\bar{f}_i \in \bar{I}_i$ and $\bar{f}_i \notin \bar{J}_i$ where $\bar{J}_i = \{x \in \bar{I}_i | x^n = u - \bar{f}_i u + y$ for some $n \geqq 1$, some $u \in \bar{I}_i$, and some $y \in \bar{J}_{i-1}\}$ for all $i = 1, 2, \dots, k$. Then $I_k \lhd \lhd \dots \lhd \lhd I_1 = I \lhd \lhd I_0 = R$ with $\{f_0 = 1, f_1, \dots, f_k\}$ is a $(k+1)$-structure in R. If $\bar{I}_{k+1}/\bar{J}_k \lhd \lhd \bar{I}_k/\bar{J}_k$ then $I_{k+1}/J_k \lhd \lhd I_k/J_k$, so there exists $f_{k+1} \in I_{k+1}$, $f_{k+1} \notin J_{k+1}$ where $J_{k+1} = \{x \in I_{k+1} | x^n = u - f_{k+1}u + y$ for some $n \geqq 1$, some $u \in I_{k+1}$, and some $y \in J_k\}$. Then $\bar{f}_{k+1} \in \bar{I}_{k+1}$ and $\bar{f}_{k+1} \notin \bar{J}_{k+1}$ and moreover $\bar{I}_{k+1}/\bar{J}_{k+1} \cong I_{k+1}/J_{k+1}$ has no minimal ideal. Furthermore if $\bar{f} \in \bar{I}_{k+1}$ with $\bar{f} \notin \bar{H}$ where $\bar{H} = \{x \in \bar{I}_{k+1} | x^n = u - \bar{f}u + y$ for some $n \geqq 1$, some $u \in \bar{I}_{k+1}$, and some $y \in \bar{J}_k\}$, then $f \in I_{k+1}$ with $f \notin H$ so $\bar{I}_{k+1}/\bar{H} \cong I_{k+1}/H$ has no minimal ideal. Thus I has an image $\bar{I} = I/J \in F$ so $A \in SUF$. Then since A was an arbitrary accessible subring of $R \in F$ it follows that $H(F) \subseteqq SUF$.

Theorem 4. *If* $A, B \in F$ *then* $A \oplus B \in F$.

PROOF. Clearly $A \oplus B$ has a unit, zero nil ideal, and if it had a minimal ideal it would be of form $I \oplus J$ with either I or J non-zero and minimal in A or B. Now an accessible subring of $A \oplus B$ will be a subdirect sum $I \overset{s}{\oplus} J$ where I is an accessible subring of A and J accessible in B. Suppose $A \oplus B$ contains a k-structure $I_k \overset{s}{\oplus} H_k \lhd \lhd \dots \lhd \lhd I_2 \overset{s}{\oplus} H_2 \lhd \lhd I_1 \oplus H_1 = A \oplus B$, with $\{(f_1, g_1) = (1,1), (f_2, g_2), \dots, (f_k, g_k)\}$ where $(f_i, g_i) \in I_i \overset{s}{\oplus} H_i$, $(f_i, g_i) \notin J_i \overset{s}{\oplus} K_i = \{x \in I_i \overset{s}{\oplus} H_i | x^n = u - (f_i, g_i)u + y$ for some $n \geqq 1$, some $u \in I_i \overset{s}{\oplus} H_i$, and some $y \in J_{i-1} \overset{s}{\oplus} K_{i-1}\}$ for all $i = 1, 2, \dots, k$.

Now if for some r we have $f_r \in J_r$ so that $I_r = J_r$. Then for all $j \geqq r$ we would have $I_j \subseteqq I_r = J_r \subseteqq J_j$ so $I_j = J_j$. Thus in this case we would have $g_i \notin K_i$ for all i. We therefore have either $f_i \notin J_i$ or $g_i \notin K_i$ for all i, say $f_i \notin J_i$ for all i. Then $I_k \lhd \lhd \dots \lhd \lhd I_1 = A$ with $\{f_1 = 1, f_2, \dots, f_k\}$ is a k-structure in $A \in F$. Suppose $(I_{k+1} \overset{s}{\oplus} H_{k+1})/(J_k \overset{s}{\oplus} K_k) \lhd \lhd (I_k \overset{s}{\oplus} H_k)/(J_k \overset{s}{\oplus} K_k)$ then the projection $A \oplus B \to A$ induces $I_{k+1}/J_k \lhd \lhd I_k/J_k$. Therefore there exists some $f_{k+1} \in I_{k+1}$, $f_{k+1} \notin J_{k+1} = \{x \in I_{k+1} | x^n = u - f_{k+1}u + y$ for some $n \geqq 1$, some $u \in I_{k+1}$, and some $y \in J_k\}$. Furthermore I_{k+1}/J_{k+1} has no minimal ideal. If (f_{k+1}, g_{k+1}) is the inverse image of f_{k+1} under the projection we can define

$$J_{k+1} \overset{s}{\oplus} K_{k+1} = \{x \in I_{k+1} \overset{s}{\oplus} H_{k+1} | x^n = u - (f_{k+1}, g_{k+1})u + y \text{ for some } n \geqq 1,$$
$$\text{some } u \in I_{k+1} \overset{s}{\oplus} H_{k+1}, \text{ and some } y \in J_k \overset{s}{\oplus} K_k\}.$$

Now if $g_{k+1} \in K_{k+1}$ then $H_{k+1} = K_{k+1}$ and we would have

$$(I_{k+1} \overset{s}{\oplus} H_{k+1})/(J_{k+1} \overset{s}{\oplus} K_{k+1}) \cong I_{k+1}/J_{k+1}.$$

But if $g_{k+1} \notin K_{k+1}$ then since $B \in F$ we know that H_{k+1}/K_{k+1} has no minimal ideal.

Therefore $(I_{k+1} \overset{s}{\oplus} H_{k+1})/(J_{k+1} \overset{s}{\oplus} K_{k+1})$ also could not have a minimal ideal. Furthermore, for any $(f, g) \in I_{k+1} \overset{s}{\oplus} H_{k+1}$ such that $(f, g) \notin H$ where

$$H = \{x \in I_{k+1} \overset{s}{\oplus} H_{k+1} | x^n = u-(f, g)u+y \text{ for some } n \geqq 1, \text{ some } u \in I_{k+1} \overset{s}{\oplus} H_{k+1},$$
$$\text{and some } y \in J_k \overset{s}{\oplus} K_k\},$$

then by the same argument $I_{k+1} \overset{s}{\oplus} H_{k+1}/H$ does not have a minimal ideal. We have thus shown that $A \oplus B \in F$.

We now let E be the class of all (associative) rings not in F, so that $E \cap F = 0$. Then by definition

Lemma 1. $UF \subseteqq E$.

Also we have

Lemma 2. *Every non-zero ring contains a non-zero ideal from E.*

PROOF. Suppose there could exist $0 \neq R$ all ideals of which are in F. Then $R \in F$ so has a unit 1, and since R has no minimal ideal it has a proper ideal I. Thus $1 \notin I$ and let I be maximal relative to $1 \notin I$. Since $I \in F$ it has a unit so $R = I \oplus J$ for some $J \lhd R$. But then also $J \in F$ so has a proper ideal J_1. Then $J_1 \lhd R$ so by maximality we would have the contradiction $1 \in I \oplus J_1$.

Corollary 3. $E \subseteqq USE$

Corollary 4. *The pair (E, F) is entwined in every $0 \neq R \in F$.*

PROOF. We have $0 \neq R \in F$ and by Lemma 2 some $0 \neq I \lhd R$ with $I \in E$. Since $F \subseteqq SUF$ and $E \subseteqq USE$ the construction is the same as that of Theorem 1.

Theorem 5. *Every ring A has an ideal $I \in E$ such that $A/I \in F$.*

PROOF. We have shown that F is regular so UF is radical. We have $A/UF(A) \in SUF$ and $UF(A) \in E$, so if $A/UF(A) \in F$ we are done. Thus without loss of generality we may assume $A \in SUF$ but $A \notin F$. Then for some $I \lhd A$ we have $A/I \in F$. But if $I \in F$ we would have $A = I \oplus J$ for some $J \lhd A$. Then we would have $J = A/I \in F$ so by Theorem 4 the contradiction $A \in F$.

We now have a pair (E, F) of classes satisfying conditions (α) and (β). And to show it is not a radical pair we have

Theorem 6. *If a pair (E, F) of classes is a decomposition of a universal class C, admitting finite direct sums then (E, F) is a radical pair in C if and only if it is either $(0, C)$ or $(C, 0)$.*

PROOF. Suppose (E, F) is a radical pair and $0 \neq A \in E$ and $0 \neq B \in F$. But every ring is either radical or semisimple and if $A \oplus B$ is radical we would have the contradiction $A \oplus B \to B$, while if it is semisimple then $A \lhd A \oplus B$ would be a contradiction.

Corollary 4. *Property* (α) *does not define a radical pair and* (β) *does not define a torsion theory.*

We now give another example of an F-ring.

Example 3. Let $R=Z_2[x_1, x_2, ...]$, so that R is a commutative ring with unit, zero nil ideal, and no minimal ideal. Now suppose R has the k-structure $I_k \lhd \lhd ... ... \lhd \lhd I_2 \lhd \lhd I_1 = R$ and $(f_1=1, f_2, ..., f_k\}$ with $f_i \in I_i$ and $f_i \notin J_i$ where $J_0=0$, $J_i=\{x\in I_i | x^n=u-f_iu+y$ for some $n\geqq 1$, some $u\in I_i$, and some $y\in J_{i-1}\}$.

Lemma 3. *We may assume that* $f_iz\in I_i$ *for all* $z\in R$ *and for any* $g\in I_i$ *there is some h congruent to g modulo* J_i *such that* $hz\in I_i$ *for all* $z\in R$.

PROOF. Since f_i is a unit of I_i modulo J_i the same is true of any power of f_i, and J_i can be generated by any such power. Since $f_i^n z\in I_i$ for sufficiently large n, we can assume that $f_iz\in I_i$ for all $z\in R$. Also any $g\in I_i$ is congruent modulo J_i to $h=gf_i$ and $gf_iz\in I_i$ for all $z\in R$.

Note that if $x\in I_i$ is such that $x^n=u-f_iu+y$ for some $n\geqq 1$, some $u\in I_i$, and some $y\in J_{i-1}$, we may assume n is even (multiply by x if n is odd).

Lemma 4. *Let* $t>$ *all subscripts of all symbols appearing in any of the terms of any of* $\{f_2, f_3, ..., f_k\}$. *For all* $k\geqq i$ *if* $w\in J_i$ *and* $w=\sum_0^r w_jx_t^j$ *with each* w_j *free of* x_t *then* $f_kw_j\in J_i$ *for all j.*

PROOF. This is certainly true for $J_1=0$ and assume for induction that the result is true for $i-1$. Let $w\in J_i$ so $w^n=u-f_iu+y$ for some (even) n, some $u\in I_i$, and some $y\in J_{i-1}$. Writing $u=\Sigma u_jx_t^j$ and $y=\Sigma y_jx_t^j$, then since n is even and R has characteristic 2, we have

$$\sum_{j=0}^{r} w_j^n x_t^{nj} = \Sigma(u_j-f_iu_j)x_t^j+\Sigma y_jx_t^j.$$

Equating coefficients of x_t yields

$$w_j^n = u_{nj}-f_iu_{nj}+y_{nj}.$$

Now $f_k^n u_{nj}\in I_k\subseteqq I_i$ and by the induction hypothesis $f_ky_{nj}\in J_{i-1}$ so since $f_k\in I_k\subseteqq I_{i-1}$ we have $f_k^n y_{nj}\in J_{i-1}$. Thus $f_kw_j\in J_i$ so the result follows by induction.

Corollary 5. *Let* $f\in I_k, f\notin J_k$ *be such that* $fz\in I_k$ *for all* $z\in R$. *Let* $t>$ *all subscript of all symbols appearing in any of the terms of any of* $\{f, f_2, f_3, ..., f_k\}$ *then* $fx_t\in I_k$, $fx_t\notin J_k$.

PROOF. Since f if free of x_t then if $fx_t\in J_k$ it would follow from Lemma 4 that $f_kf\in J_k$. But f_kf is congruent to f modulo J_k so the contradiction $f\in J_k$.

Proposition 1. *If* $0\neq I_{k+1}/J_k\lhd\lhd I_k/J_k$ *then there exists some* $f_{k+1}\in I_{k+1}$, $f_{k+1}\notin J_{k+1}=\{x\in I_{k+1}|x^n=u-f_{k+1}u+y$ *for some* $n\geqq 1$, *some* $u\in I_{k+1}$, *and some* $y\in J_k\}$.

PROOF. Let $f\in I_{k+1}$, $f\notin J_k$ and such that $fz\in I_{k+1}$ for all $z\in R$, and let $t>$ all subscripts of all symbols appearing in any term of any of $\{f, f_2, ..., f_k\}$. By Corollary 5 we have $fx_t\in I_k$, $fx_t\notin J_k$. Let $f_{k+1}=fx_t$ so if $I_{k+1}=J_{k+1}$ then $f\in J_{k+1}$ so for some

$v \in I_{k+1}$ we have $f^n-(v-fx_t v) \in J_k$. Writing $v=v_0+v_1x_t+\ldots+v_r x_t^r$, where since $f_k v$ is congruent to v modulo J_k, we can assume each $v_i \in I_k$. Then by Lemma 4 (again using congruence modulo J_k) we have

$$f^n-v_0 \in J_k$$

$$v_1-v_0 f \in J_k$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$v_r-v_{r-1} f \in J_k$$

$$v_r f \in J_k.$$

But then we obtain $f^{r+n+1} \in J_k$ so the contradiction $f \in J_k$. Thus $f_{k+1} \notin J_{k+1}$.

Proposition 2. *I_{k+1}/J_{k+1} has no minimal ideal.*

Proof. Let A/J_{k+1} be a minimal ideal of I_{k+1}/J_{k+1}. Let $g \in A$, $g \notin J_{k+1}$. Since f_{k+1} is a unit of I_{k+1} modulo J_{k+1} where $f_{k+1}z \in I_{k+1}$ for all $z \in R$ then $gf_{k+1}z \in A$ for all $z \in R$. Thus we may assume $gz \in A$ for all $z \in R$. By Corollary 5 for sufficiently large t we have $gx_t \in A$ with $gx_t \notin J_{k+1}$. But then gx_t would generate A modulo J_{k+1} so that for some $w \in A$ we would have $g-wgx_t \in J_{k+1}$. By Lemma 4 this would give the contradiction $g \in J_{k+1}$.

Corollary 6. *For any $f \in I_{k+1}$ if $f \notin H=\{x \in I_{k+1} | x^n=u-fu+y$ for some $n \geqq 1$, some $u \in I_{k+1}$, and some $y \in J_k\}$ then I_{k+1}/H has no minimal ideal.*

Proof. Again since f is a unit of I_{k+1} modulo H we may assume $fz \in I_{k+1}$ for all $z \in R$. The construction is then the same as for Proposition 1 and the proof is the same as that of Proposition 2.

References

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