On the n-dimensional SE-connection and its conformal change*

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Abstract, The purpose of the present paper is to investigate properties of the n-dimensional SE-connection and its change induced by the conformal change (5.2). In the present paper, we introduce the concept of the SE-connection, find a necessary and sufficient condition for the existence of a unique SE-connection, and derive its representations. We also derive a useful representation of its change induced by the conformal change.

I. Introduction

The conformal change in a generalized 4-dimenzional Riemannian space connected by an Einstein's connection was primarily studied by HLAVATÝ (1957). CHUNG (1968, 1974) also investigated the same topic in 4-dimensional $^*g$-unified field theory. In the present paper, we first introduce the concept of an n-dimensional SE-manifold, SEX_n, a generalized Riemannian space connected by the SE-connection $\Gamma^{\nu}_{\lambda\mu}$ which is both semi-symmetric and Einstein. In the sequel, we find a necessary and sufficient condition for the existence of a unique SE-connection and derive several representations of the unique SE-connection. In the last section we finally investigate the change $\Gamma^{\nu}_{\lambda\mu}\to\bar{\Gamma}^{\nu}_{\lambda\mu}$ of two n-dimensional SE-connections induced by the conformal change (5.2).

All considerations and results presented in the present paper hold for all classes, all possible indices of inertia, and an arbitrary $n>1$.

II. Preliminaries

This section is a brief collection of definitions, notations, and basic results which are needed in our subsequent considerations. The detailed proofs are given in CHUNG (1963, 1981, 1982), HLAVATÝ (1957), and MISHRA (1958).

Let X_n be a generalized n-dimensional Riemannian space referred to a real coordinate system x^ν, which obeys only coordinate transformations $x^\nu\to x^{\nu'}$ for which

$$\text{(2.1)}\qquad \mathrm{Det}\left(\left(\frac{\partial x'}{\partial x}\right)\right)\neq 0.$$

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The space X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$ (*):

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{2.2}$$

where

$$\mathfrak{g} \stackrel{\mathrm{df}}{=} \mathrm{Det}\left((g_{\lambda\mu})\right) \neq 0, \quad \mathfrak{h} \stackrel{\mathrm{df}}{=} \mathrm{Det}\left((h_{\lambda\mu})\right) \neq 0, \quad \mathfrak{k} \stackrel{\mathrm{df}}{=} \mathrm{Det}\left((k_{\lambda\mu})\right). \tag{2.3}$$

We may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu} h^{\lambda\nu} \stackrel{\mathrm{df}}{=} \delta^{\nu}_{\lambda}, \tag{2.4}$$

which together with $h_{\lambda\mu}$ will serve for raising and/or lowering indices of tensors in X_n in the usual manner.

The space X_n is connected by a general real connection $\Gamma^{\nu}_{\lambda\mu}$ with the following transformation rule:

$$\Gamma^{\nu'}_{\lambda'\mu'} = \frac{\partial x^{\nu'}}{\partial x^{\alpha}} \left(\frac{\partial x^{\beta}}{\partial x^{\lambda'}} \frac{\partial x^{\gamma}}{\partial x^{\mu'}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial^2 x^{\alpha}}{\partial x^{\lambda'} \partial x^{\mu'}} \right). \tag{2.5}$$

It may also be decomposed into its symmetric part $\Lambda^{\nu}_{\lambda\mu}$ and its skew-symmetric part $S^{\nu}_{\lambda\mu}$, called the torsion tensor of $\Gamma^{\nu}_{\lambda\mu}$:

$$\Gamma^{\nu}_{\lambda\mu} = \Lambda^{\nu}_{\lambda\mu} + S^{\nu}_{\lambda\mu}; \ \Lambda^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{(\lambda\mu)}, \ S^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{[\lambda\mu]}. \tag{2.6}$$

A connection $\Gamma^{\nu}_{\lambda\mu}$ is said to be *Einstein* if it satisfies the following Einstein equations:

$$\partial_{\omega} g_{\lambda\mu} - \Gamma^{\alpha}_{\lambda\omega} g_{\alpha\mu} - \Gamma^{\alpha}_{\omega\mu} g_{\lambda\alpha} = 0, \tag{2.7a}$$

or equivalently

$$D_{\omega} g_{\lambda\mu} = 2 S^{\alpha}_{\omega\mu} g_{\lambda\alpha} \tag{2.7b}$$

where D_{ω} is the symbolic vector of the covariant derivative with respect to $\Gamma^{\nu}_{\lambda\mu}$. The space X_n in this case is a generalization of the space-time X_4. A connection $\Gamma^{\nu}_{\lambda\mu}$ is said to be *semi-symmetric* if its torsion tensor $S^{\nu}_{\lambda\mu}$ is of the form

$$S^{\nu}_{\lambda\mu} = 2\delta^{\nu}_{[\lambda} X_{\mu]} \tag{2.8}$$

for an arbitrary vector X_{μ}. A connection $\Gamma^{\nu}_{\lambda\mu}$ which is both semi-symmetric and Einstein is called an *SE-connection*. An *n-dimensional SE-manifold, denoted by* SEX_n in our further considerations, is a space X_n on which the differential-geometric structure is imposed by $g_{\lambda\mu}$ through an *SE*-connection $\Gamma^{\nu}_{\lambda\mu}$.

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* Throughout the present paper, all Greek indices take the values $1, 2, \ldots, n$ and follow the summation convention.

The following quantities are frequently used in the present paper:

(2.9)a $$g \overset{\mathrm{df}}{=} \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k \overset{\mathrm{df}}{=} \frac{\mathfrak{k}}{\mathfrak{h}};$$

(2.9)b $$\sigma \overset{\mathrm{df}}{=} \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases};$$

(2.9)c $${}^{(0)}k_\lambda^\nu \overset{\mathrm{df}}{=} \delta_\lambda^\nu, \quad {}^{(p)}k_\lambda^\nu \overset{\mathrm{df}}{=} {}^{(p-1)}k_\lambda^\alpha k_\alpha^\nu, \quad (p = 1, 2, \ldots);$$

(2.9)d $$K_p \overset{\mathrm{df}}{=} k_{[\alpha_1}^{\alpha_1} k_{\alpha_2}^{\alpha_2} \ldots k_{\alpha_p]}^{\alpha_p}, \quad (p = 0, 1, 2, \ldots);$$

(2.9)e $$\overset{pqr}{T}_{\omega\mu\nu} \overset{\mathrm{df}}{=} {}^{(p)}k_\omega^\alpha \, {}^{(q)}k_\mu^\beta \, {}^{(r)}k_\nu^\gamma T_{\alpha\beta\gamma};$$

(2.9)f $$K_{\omega\mu\nu} \overset{\mathrm{df}}{=} \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu} + \nabla_\nu k_{\omega\mu}.$$

Here $T_{\omega\mu\nu}$ is an arbitrary tensor and ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbols $\left\{ {\nu \atop \lambda\mu} \right\}$ defined by $h_{\lambda\mu}$.

In a general X_n it may be easily shown that

(2.10)a $${}^{(p)}k_{\lambda\mu} \text{ and } {}^{(p)}k^{\lambda\nu} \text{ are } \begin{cases} \text{symmetric if } p \text{ is even} \\ \text{skew-symmetric if } p \text{ is odd}; \end{cases}$$

(2.10)b $$K = 1, \; K_n = k \quad \text{if } n \text{ is even}, \quad K_p = 0 \quad \text{if } p \text{ is odd};$$

(2.10)c $$\overset{000}{T}_{\omega\mu\nu} = T_{\omega\mu\nu}, \quad \overset{pqr}{T}_{\omega\mu\nu} = -\overset{qpr}{T}_{\mu\omega\nu} \quad \text{if} \quad T_{\omega\mu\nu} = -T_{\mu\omega\nu};$$

(2.10)d $$g = \sum_{s=0}^{n-\sigma} K_s;$$

(2.10)e $$\sum_{s=0}^{n-\sigma} K_s \, {}^{(n-s)}k_\lambda^\nu = 0 \quad \text{(Recurrence relation)}.$$

Here and in what follows we assume that *s takes only even integers* in the given range.

If the system (2.7) admits a solution $\Gamma_{\lambda\mu}^\nu$, it must be of the form

(2.11) $$\Gamma_{\lambda\mu}^\nu = \left\{ {\nu \atop \lambda\mu} \right\} + U_{\lambda\mu}^\nu + S_{\lambda\mu}^\nu,$$

where

(2.12) $$U_{\lambda\mu}^\nu = 2 \overset{001}{S}{}^\nu_{(\lambda\mu)}.$$

III. The *SE*-connection $\Gamma^{\nu}_{\lambda\mu}$

In this section we shall obtain a necessary and sufficient condition for a general connection $\Gamma^{\nu}_{\lambda\mu}$ of X_n to be a unique *SE*-connection and derive some useful representations of the *SE*-connection.

Theorem (3.1). *If there is an SE-connection $\Gamma^{\nu}_{\lambda\mu}$, it must be of the form*

$$\Gamma^{\nu}_{\lambda\mu} = \begin{Bmatrix} \nu \\ \lambda\mu \end{Bmatrix} + 2k^{\nu}_{(\lambda}X_{\mu)} + 2\delta^{\nu}_{[\lambda}X_{\nu]} \tag{3.1}$$

for a vector X_μ.

Proof. If $\Gamma^{\nu}_{\lambda\mu}$ is an *SE*-connection, it must be of the form (2.11). Substituting (2.8) and

$$U^{\nu}_{\lambda\mu} = 2h^{\nu\alpha}(\delta^{\beta}_{[\alpha}X_{\lambda]}k_{\mu\beta} - \delta^{\beta}_{[\alpha}X_{\mu]}k_{\lambda\beta}) = 2k^{\nu}_{(\lambda}X_{\mu)} \tag{3.2}$$

into (2.11), we have (3.1).

In the next Theorem we shall assume that the symmetric tensor

$$P_{\lambda\mu} \stackrel{\text{df}}{=} {}^{(2)}k_{\lambda\mu} - h_{\lambda\mu} \tag{3.3}$$

is of rank $n^{(*)}$, so that there exists a unique symmetric tensor $Q^{\lambda\nu} = Q^{\nu\lambda}$ satisfying

$$P_{\lambda\mu}Q^{\lambda\mu} \stackrel{\text{df}}{=} \delta^{\nu}_{\mu}. \tag{3.4}$$

Theorem (3.2). *There exists a unique SE-connection $\Gamma^{\nu}_{\lambda\mu}$ if, and only if there is a vector X_μ such that*

$$\nabla_{\omega}k_{\lambda\mu} + 2P_{\omega[\lambda}X_{\mu]} = 0. \tag{3.5}$$

The vector X_μ satisfying (3.5) is unique and may be given by

$$X_\mu = \frac{1}{1-n}Q^{\alpha\beta}\nabla_{\alpha}k_{\beta\mu}. \tag{3.6}$$

Proof. Multiply both sides of (3.5) by $Q^{\omega\nu}$ and contract for ν and λ in order to obtain (3.6). The uniqueness of the vector X_μ satisfying (3.5) is obvious.

Suppose that $\Gamma^{\nu}_{\lambda\mu}$ is an *SE*-connection. Then it must be of the form (3.1) and satisfy (2.7). Substituting (3.1) into (2.7)a, we have the condition (3.5). Conversely, assume now that there exists a vector X_μ satisfying (3.5). With this vector X_μ, define a semi-symmetric connection $\Gamma^{\nu}_{\lambda\mu}$ by (3.1). This connection $\Gamma^{\nu}_{\lambda\mu}$ is clearly Einstein since it satisfies (2.7)a in virtue of our assumption (3.5).

Besides the *SE*-connection $\Gamma^{\nu}_{\lambda\mu}$ defined by (3.1) and (3.5), assume that there exists another *SE*-connection

$${}^{*}\Gamma^{\nu}_{\lambda\mu} = \begin{Bmatrix} \nu \\ \lambda\mu \end{Bmatrix} + 2k^{\nu}_{(\lambda}{}^{*}X_{\mu)} + 2\delta^{\nu}_{[\lambda}{}^{*}X_{\mu]}, \quad {}^{*}X_\mu \neq X_\mu. \tag{3.7}$$

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* In Remark (4.3) we shall show that the rank of $P_{\lambda\mu}$ is n.

Then, in virtue of the above discussion, $^*X_\mu$ must satisfy

(3.8) $$\nabla_\omega k_{\lambda\mu} + 2P_{\omega[\lambda}{}^*X_{\mu]} = 0.$$

Applying the same method used to derive (3.6), we have from (3.8)

$$^*X_\mu = \frac{1}{1-n} Q^{\alpha\beta}\nabla_\alpha k_{\beta\mu} = X_\mu,$$

which contradicts the assumption (3.7). This proves the uniqueness of the SE-connection.

Remark (3.3). In virtue of the previous Theorems, we note that our n-dimensional SE-manifold SEX_n is connected by the *unique* SE-connection $\Gamma^\nu_{\lambda\mu}$ given by (3.1) and (3.6).

Theorem (3.4). *The following relations hold in* SEX_n:

(3.9) $$K_{\omega\mu\nu} = 2\nabla_\nu k_{\omega\mu}, \quad 2\nabla_{[\omega} k_{\mu]\nu} + \nabla_\nu k_{\omega\mu} = 0.$$

Proof. These assertions may be obtained from (2.9)f and (3.5).

In our subsequent considerations, we need the following tow vectors:

(3.10) $$S_\mu \stackrel{\text{df}}{=} S^\alpha_{\mu\alpha}, \quad U_\mu \stackrel{\text{df}}{=} U^\alpha_{\alpha\mu}.$$

Theorem (3.5) *In* SEX_n *the vectors* X_μ, S_μ, *and* U_μ *may be given by*

(3.11)a $$X_\mu = \frac{1}{2(n-1)} K_{\mu\alpha\beta} Q^{\alpha\beta},$$

(3.11)b $$S_\mu = (1-n)X_\mu = -{}^1\!/_2 K_{\mu\alpha\beta} Q^{\alpha\beta},$$

(3.11)c $$U_\mu = k^\alpha_\mu X_\alpha = \frac{1}{2(n-1)} \overset{100}{K}_{\mu\alpha\beta} Q^{\alpha\beta}.$$

Proof. Substituting for $\nabla_\alpha k_{\beta\mu}$ into (3.6) from (3.9), we have (3.11)a. The first representations of (3.11)b, c may be obtained by contracting for λ and μ in (2.8) and (2.12), respectively. The remaining representations are clear in virtue of (3.11)a and (2.9)e.

Remark (3.6). In a manifold connected with an Einstein connection, Hlavatý (1957) proved that the vector U_μ is a gradient of a scalar $\ln\sqrt{g}$. That is,

(3.12) $$U_\mu = {}^1\!/_2 \partial_\mu (\ln g).$$

Theorem (3.7). *The unique SE-connection* $\Gamma^\nu_{\lambda\mu}$ *of* SEX_n *may be given by*

(3.13)a $$\Gamma^\nu_{\lambda\mu} = \begin{Bmatrix} \nu \\ \lambda\mu \end{Bmatrix} + \frac{1}{1-n}(g_{\lambda\gamma}\nabla_\alpha k_{\beta\mu} - g_{\gamma\mu}\nabla_\alpha k_{\beta\lambda}) h^{\gamma\nu} Q^{\alpha\beta},$$

or equivalently

(3.13)b $$\Gamma^\nu_{\lambda\mu} = \begin{Bmatrix} \nu \\ \lambda\mu \end{Bmatrix} + \frac{1}{n-1}(\delta^\nu_{[\lambda} K_{\mu]\alpha\beta} + k^\nu_{(\lambda} K_{\mu)\alpha\beta}) Q^{\alpha\beta}.$$

PROOF. In virtue of (2.2), the representation (3.1) may be written as

$$\Gamma^{\nu}_{\lambda\mu} = \begin{Bmatrix} \nu \\ \lambda\mu \end{Bmatrix} + (g_{\lambda\alpha} X_{\mu} - g_{\alpha\mu} X_{\lambda}) h^{\alpha\nu}. \tag{3.14}$$

Substituting for X_{μ} into (3.14) from (3.6), we have (3.13)a. The representation (3.13)b is a consequence of (3.1) and (3.11)a.

Theorem (3.8). *The following five statements are equivalent in SEX_n:*

$$\text{(a)}\quad S_{\mu} = 0, \quad \text{(b)}\quad X_{\mu} = 0, \quad \text{(c)}\quad \nabla_{\omega} k_{\lambda\mu} = 0,$$

$$\text{(d)}\quad K_{\omega\lambda\mu} = 0, \quad \text{(e)}\quad \Gamma^{\nu}_{\lambda\mu} = \begin{Bmatrix} \nu \\ \lambda\mu \end{Bmatrix}.$$

PROOF. Since the tensor $Q^{\lambda\nu}$ is of rank n, the equivalence follows from the following diagram:

$$\text{(a)} \xleftrightarrow{(3.11)\text{b}} \text{(b)} \xleftrightarrow{(3.6)} \text{(c)} \xleftrightarrow{(3.9)} \text{(d)} \xleftrightarrow{(3.13)\text{b}} \text{(e)}$$

Remark (3.9). Note that Theorem (3.8) is very important for the study of field equations in SEX_n since the statement (a) is one of the field equations.

IV. The tensor $Q^{\lambda\nu}$

In this section, we first derive a representation of the tensor $Q^{\lambda\nu}$ in a general X_n *connected by Einstein's connection* and exhibit several useful representations of the unique n-dimensional SE-connection.

We need the following abbreviations for an arbitrary tensor $X^{\lambda\nu}$:

$$ {}^{(p)}X^{\lambda\nu} = {}^{(p)}k_{\alpha}^{\nu} X^{\lambda\alpha}, \quad (p = 0, 1, 2, \ldots) \tag{4.1)a}$$

$$\tilde{K}_s = 1 + K_2 + K_4 + \ldots + K_s. \tag{4.1)b}$$

Direct calculations show that

$$ {}^{(0)}X^{\lambda\nu} = X^{\lambda\nu}, \quad {}^{(q)}k_{\alpha}^{\nu}\, {}^{(p)}X^{\lambda\alpha} = {}^{(p+q)}X^{\lambda\nu}, \tag{4.2)a}$$

$$\tilde{K}_0 = 1, \quad \tilde{K}_s + K_{s+2} = \tilde{K}_{s+2}. \tag{4.2)b}$$

Theorem (4.1). *(Recurrence relation) An arbitrary tensor ${}^{(p)}X^{\lambda\nu}$ satisfies the following relation in SEX_n:*

$$\sum_{s=0}^{n-\sigma} K_s\, {}^{(n-s)}X^{\lambda\nu} = 0, \tag{4.3)a}$$

or

$$ {}^{(n)}X^{\lambda\nu} + K_2\, {}^{(n-2)}X^{\lambda\nu} + K_4\, {}^{(n-4)}X^{\lambda\nu} + \ldots \tag{4.3)b}$$

$$\ldots + K_{n-\sigma-2}\, {}^{(\sigma+2)}X^{\lambda\nu} + K_{n-\sigma}\, {}^{(\sigma)}X^{\lambda\nu} = 0.$$

PROOF. Multiplying $X^{\lambda\alpha}$ to both sides of (2.10)e, we get have (4.3).

Now, we are ready to derive a representation of $Q^{\lambda\nu}$.

Theorem (4.2). *In X_n the tensor $Q^{\lambda\nu}$ may be given by*

$$Q^{\lambda\nu}=-\frac{1}{g}M^{\lambda\nu}, \tag{4.4)a}$$

or

$$Q^{\lambda\nu}=\begin{cases}-\frac{1}{g}({}^{(n-2)}k^{\lambda\nu}+\tilde{K}_2{}^{(n-4)}k^{\lambda\nu}+\tilde{K}_4{}^{(n-6)}k^{\lambda\nu}+\dots\\ \quad\dots+\tilde{K}_{n-4}{}^{(2)}k^{\lambda\nu}+\tilde{K}_{n-2}h^{\lambda\nu}), \quad \textit{if } n \textit{ is even}\\ -\frac{1}{g}({}^{(n-1)}k^{\lambda\nu}+\tilde{K}_2{}^{(n-3)}k^{\lambda\nu}+\tilde{K}_4{}^{(n-5)}k^{\lambda\nu}+\dots\\ \quad\dots+\tilde{K}_{n-3}{}^{(2)}k^{\lambda\nu}+\tilde{K}_{n-1}h^{\lambda\nu}), \quad \textit{if } n \textit{ is odd,}\end{cases} \tag{4.4)b}$$

where $M^{\lambda\nu}$ is a symmetric tensor defined by

$$M^{\lambda\nu}\stackrel{\text{df}}{=}\sum_{s=0}^{n-1}\tilde{K}_s{}^{(n-2+\sigma-s)}k^{\lambda\nu}. \tag{4.5}$$

Proof. Consider a tensor $X^{\lambda\nu}$ defined by

$$P_{\alpha\mu}X^{\alpha\nu}\stackrel{\text{df}}{=}\delta_\mu^\nu. \tag{4.6}$$

Since the symmetric tensor $P_{\lambda\mu}$ is of rank n, the tensor $X^{\lambda\nu}=X^{\nu\lambda}$ exists uniquely and equals $Q^{\lambda\nu}$ in virtue of (3.4). Hence it suffices to find a representation of the tensor $X^{\lambda\nu}$. We first note that the tensor $X^{\lambda\nu}$ satisfies the following recurrence relations:

$$^{(p)}X^{\lambda\nu}={}^{(p-2)}X^{\lambda\nu}+{}^{(p-2)}k^{\lambda\nu}, \quad (p\geqq 2), \tag{4.7}$$

which may be obtained multiplying both sides of (4.6) by ${}^{(p-2)}k^{\mu\lambda}$ and making use of (3.3) and (4.1)a.

In order to derive (4.4), we now substitute for ${}^{(n)}X^{\lambda\nu}$ into (4.3)b from (4.7) and make use of (4.1)b to obtain

$$\tilde{K}_0{}^{(n-2)}k^{\lambda\nu}+\tilde{K}_2{}^{(n-2)}X^{\lambda\nu}+K_4{}^{(n-4)}X^{\lambda\nu}+\dots+K_{n-\sigma}{}^{(\sigma)}X^{\lambda\nu}=0. \tag{4.8)a}$$

Substituting again for ${}^{(n-2)}X^{\lambda\nu}$ into (4.8) a from (4.7), we have

$$\begin{aligned}&\tilde{K}_0{}^{(n-2)}k^{\lambda\nu}+\tilde{K}_2{}^{(n-4)}k^{\lambda\nu}+\tilde{K}_4{}^{(n-4)}X^{\lambda\nu}+\\ &+K_6{}^{(n-6)}X^{\lambda\nu}+\dots+K_{n-\sigma}{}^{(\sigma)}X^{\lambda\nu}=0.\end{aligned} \tag{4.8)b}$$

After $\frac{n-\sigma}{2}$ steps of repeated substitutions for ${}^{(p)}X^{\lambda\nu}$, we have

$$\begin{aligned}&\tilde{K}_0{}^{(n-2)}k^{\lambda\nu}+\tilde{K}_2{}^{(n-4)}k^{\lambda\nu}+\tilde{K}_4{}^{(n-6)}k^{\lambda\nu}+\dots\\ &\dots+\tilde{K}_{n-\sigma-4}{}^{(2+\sigma)}k^{\lambda\nu}+\tilde{K}_{n-\sigma-2}{}^{(\sigma)}k^{\lambda\nu}+K_{n-\sigma}{}^{(\sigma)}X^{\lambda\nu}=0.\end{aligned} \tag{4.8)c}$$

Now, multiply both sides of (4.8)c by ${}^{(\sigma)}k_\nu^\mu$ to obtain

$$\begin{aligned}&\tilde{K}_0{}^{(n-2+\sigma)}k^{\lambda\nu}+\tilde{K}_2{}^{(n-4+\sigma)}k^{\lambda\nu}+\tilde{K}_4{}^{(n-6+\sigma)}k^{\lambda\nu}+\dots\\ &\dots+\tilde{K}_{n-\sigma-4}{}^{(2+2\sigma)}k^{\lambda\nu}+\tilde{K}_{n-\sigma-2}{}^{(2\sigma)}k^{\lambda\nu}+K_{n-\sigma}{}^{(2\sigma)}X^{\lambda\nu}=0.\end{aligned} \tag{4.8)d}$$

Substituting

$$^{(2\sigma)}X^{\lambda\nu}={}^{(0)}X^{\lambda\nu}+\sigma{}^{(0)}k^{\lambda\nu}$$

into (4.8)d and making use of (2.9)c, (2.10)d, and (4.2), we finally get have a representation (4.4)b for the tensor $X^{\lambda\nu}=Q^{\lambda\nu}$. (4.4)b can be easily summarized as (4.4)a. Note, in particular, that the tensor ${}^{(n-2+\sigma-s)}k^{\lambda\nu}$ is symmetric according to (2.10)a since $n-2+\sigma-s$ is an even integer. Hence the tensor $M^{\lambda\nu}$ is also symmetric.

Remark (4.3). We note that according to Chung's previous results (Chung, 1981, 1985),

(4.9) $$P_{\lambda\mu}=-{}^*h_{\lambda\mu}=-{}^*g_{(\lambda\mu)},\quad Q^{\lambda\nu}=-{}^*h^{\lambda\nu}=-{}^*g^{(\lambda\nu)}.$$

Therefore, the tensors $P_{\lambda\mu}$ and $Q^{\lambda\nu}$ are both of rank n.

The next two Theorems are consequences of Theorem (4.2). They may be obtained from (3.11) and (3.13) making use of (3.9).

Theorem (4.4). *In SEX_n the vectors X_μ, S_μ, and U_μ may be given by*

(4.10)a $$X_\mu=\frac{1}{(n-1)g}M^{\alpha\beta}\nabla_\alpha k_{\beta\mu}=\frac{1}{2(1-n)g}K_{\mu\alpha\beta}M^{\alpha\beta},$$

(4.10)b $$S_\mu=-\frac{1}{g}M^{\alpha\beta}\nabla_\alpha k_{\beta\mu}=\frac{1}{2g}K_{\mu\alpha\beta}M^{\alpha\beta},$$

(4.10)c $$U_\mu=\frac{1}{(n-1)g}M^{\alpha\beta}k_\mu^\gamma\nabla_\alpha k_{\beta\gamma}=\frac{1}{2(1-n)g}\overset{100}{K}_{\mu\alpha\beta}M^{\alpha\beta}.$$

Theorem (4.5). *The unique SE -connection $\Gamma^\nu_{\lambda\mu}$ may be given by the following expressions:*

(4.11)a $$\Gamma^\nu_{\lambda\mu}=\begin{Bmatrix}\nu\\ \lambda\mu\end{Bmatrix}+\frac{1}{(n-1)g}(g_{\lambda\gamma}\nabla_\alpha k_{\beta\mu}-g_{\gamma\mu}\nabla_\alpha k_{\beta\lambda})h^{\nu\gamma}M^{\alpha\beta},$$

(4.11)*b* $$\Gamma^\nu_{\lambda\mu}=\begin{Bmatrix}\nu\\ \lambda\mu\end{Bmatrix}+\frac{1}{2(1-n)g}(g_{\lambda\gamma}K_{\mu\alpha\beta}-g_{\gamma\mu}K_{\lambda\alpha\beta})h^{\nu\gamma}M^{\alpha\beta},$$

(4.11)c $$\Gamma^\nu_{\lambda\mu}=\begin{Bmatrix}\nu\\ \lambda\mu\end{Bmatrix}+\frac{2}{(n-1)g}((\nabla_\alpha k_{\beta[\mu})\delta^\nu_{\lambda]}+(\nabla_\alpha k_{\beta(\lambda})k^\nu_{\mu)})M^{\alpha\beta},$$

(4.11)d $$\Gamma^\nu_{\lambda\mu}=\begin{Bmatrix}\nu\\ \lambda\mu\end{Bmatrix}+\frac{1}{(1-n)g}(\delta^\nu_{[\lambda}K_{\mu]\alpha\beta}+k^\nu_{(\lambda}K_{\mu)\alpha\beta})M^{\alpha\beta}.$$

Remark (4.6). In the following table we illustrate the representations of g and the tensor $M^{\lambda\nu}$ for the lower dimensional cases, which may be used to calculate the corresponding representations of the vectors X_μ, S_μ, and U_μ and the SE-connection $\Gamma^\nu_{\lambda\mu}$ for $n=2, 3, 4$.

(4.12)

n	g	$M^{\lambda\nu}$
2	$1+k$	$h^{\lambda\nu}$
3	$1+K_2$	$gh^{\lambda\nu}+{}^{(2)}k^{\lambda\nu}$
4	$1+K_2+k$	$K_2h^{\lambda\nu}+{}^{(2)}k^{\lambda\nu}$

V. Conformal change of SEX_n

In this final section we investigate the change $\Gamma^{\nu}_{\lambda\mu} \to \bar{\Gamma}^{\nu}_{\lambda\mu}$ of the SE-connection induced by a conformal change of the tensor $g_{\lambda\mu}$.

Consider two n-dimensional SE-manifolds SEX_n $(\overline{SEX}_n)$, on which the differential geometric structure is imposed by a general real tensor $g_{\lambda\mu}$ $(\bar{g}_{\lambda\mu})$ through the SE-connection $\Gamma^{\nu}_{\lambda\mu}$ $(\bar{\Gamma}^{\nu}_{\lambda\mu})$ given by (4.11) ((5.1)):

$$\bar{\Gamma}^{\nu}_{\lambda\mu} = \begin{Bmatrix} \bar{\nu} \\ \lambda\mu \end{Bmatrix} + \frac{1}{(1-n)\bar{g}} (\delta^{\nu}_{[\lambda} \bar{K}_{\mu]\alpha\beta} + \bar{k}^{\nu}_{(\lambda} \bar{K}_{\mu)\alpha\beta}) \bar{M}^{\alpha\beta}. \tag{5.1}$$

We say that SEX_n and $\overline{SEX}_n$ are *conformal* if, and only if

$$\bar{g}_{\lambda\mu}(x) = e^{\Omega} g_{\lambda\mu}(x), \tag{5.2}$$

where $\Omega = \Omega(x)$ is an at least twice differentiable function. This conformal change enforces a change of the connection. An explicit representation for $\Gamma^{\nu}_{\lambda\mu}$ will be exhibited in this section.

Agreement (5.1). Throughout this section, we agree that, if T is a function of $g_{\lambda\mu}$, then we denote by $\bar{T}$ the same function of $\bar{g}_{\lambda\mu}$. In particular, if T is a tensor, so is $\bar{T}$. Furthermore, the indices of T $(\bar{T})$ will be raised and/or lowered by means of $h^{\lambda\nu}(\bar{h}^{\lambda\nu})$ and/or $h_{\lambda\mu}(\bar{h}_{\lambda\mu})$.

The results listed in the following Theorem are immediate consequences of (5.2) and Agreement (5.1).

Theorem (5.2). *The conformal change* (5.2) *induces the following changes:*

$$ {}^{(p)}\bar{k}_{\lambda\mu} = e^{\Omega(p)} {}^{(p)}k_{\lambda\mu}, \quad {}^{(p)}\bar{k}^{\lambda\nu} = e^{-\Omega(p)} {}^{(p)}k^{\lambda\nu}, \quad {}^{(p)}\bar{k}^{\nu}_{\lambda} = {}^{(p)}k^{\nu}_{\lambda}; \tag{5.3a}$$

$$\bar{\mathfrak{h}} = e^{\Omega}\mathfrak{h}, \quad \bar{\mathfrak{k}} = e^{\Omega}\mathfrak{k}, \quad \bar{\mathfrak{g}} = e^{\Omega}\mathfrak{g}, \quad \bar{g} = g, \quad \bar{K}_p = K_p. \tag{5.3b}$$

$$(p = 0, 1, 2, \ldots)$$

Theorem (5.3). *The conformal change* (5.2) *induces the following changes:*

$$\begin{Bmatrix} \bar{\nu} \\ \lambda\mu \end{Bmatrix} = \begin{Bmatrix} \nu \\ \lambda\mu \end{Bmatrix} + \delta^{\nu}_{(\lambda}\Omega_{\mu)} - {}^{1}/_{2} h_{\lambda\mu} h^{\nu\alpha}\Omega_{\alpha}, \quad (\Omega_{\mu} \overset{\text{df}}{=} \partial_{\mu}\Omega) \tag{5.4a}$$

$$\bar{\nabla}_{\nu}\bar{k}_{\omega\mu} = e^{\Omega}(\nabla_{\nu}k_{\omega\mu} + k_{\nu[\omega}\Omega_{\mu]} - h_{\nu[\omega}k^{\alpha}_{\mu]}\Omega_{\alpha}), \tag{5.4b}$$

$$\bar{K}_{\omega\mu\nu} = e^{\Omega}(K_{\omega\mu\nu} - k_{\omega\mu}\Omega_{\nu} - 2h_{\nu[\omega}k^{\alpha}_{\mu]}\Omega_{\alpha}), \tag{5.4c}$$

$$\bar{M}^{\lambda\nu} = e^{-\Omega}M^{\lambda\nu}. \tag{5.4d}$$

Proof. The first relation follows by substituting (5.3)a into

$$\begin{Bmatrix} \bar{\nu} \\ \lambda\mu \end{Bmatrix} = \bar{h}^{\nu\alpha}(\partial_{(\mu}\bar{h}_{\lambda)\alpha} - {}^{1}/_{2}\partial_{\alpha}\bar{h}_{\lambda\mu}).$$

Substituting (5.3)a and (5.4)a into

$$\bar{\nabla}_{\nu}\bar{k}_{\omega\mu} = \partial_{\nu}\bar{k}_{\omega\mu} - \bar{k}_{\alpha\mu}\begin{Bmatrix} \bar{\alpha} \\ \omega\nu \end{Bmatrix} - \bar{k}_{\omega\alpha}\begin{Bmatrix} \bar{\alpha} \\ \mu\nu \end{Bmatrix},$$

we have (5.4)b. (5.4)c may be obtained by substituting (5.4)b into the corresponding representation for $\overline{K}_{\omega\mu\nu}$ in (2.9)f. The last relation is obvious in virtue of (4.5) and (5.3).

Theorem (5.4). *If SEX_n and $\overline{SEX}_n$ are conformal, the function Ω satisfies*

$$2k_{\nu[\omega}\Omega_{\mu]}+k_{\omega\mu}\Omega_\nu = 0. \tag{5.5}$$

PROOF. Since SEX and $\overline{SEX}_n$ are SE-manifolds, we have

$$\overline{K}_{\omega\mu\nu} = 2\overline{\nabla}_\nu \overline{k}_{\omega\mu}, \quad K_{\omega\mu\nu} = 2\nabla_\nu k_{\omega\mu}, \tag{5.6}$$

in virtue of Theorem (3.4). Our assertion follows from (5.6), (5.4)b, and (5.4)c.

Remark (5.5). In virtue of (5.5), we may simplify the relation (5.4)b as

$$\overline{\nabla}_\nu \overline{k}_{\omega\mu} = e^{\Omega}(\nabla_\nu k_{\omega\mu}-{}^1/_2\Omega_\nu k_{\omega\mu}-h_{\nu[\omega}k^\alpha_{\mu]}\Omega_\alpha). \tag{5.7}$$

Now, we are ready to derive representations of the changes $X_\mu \rightarrow \overline{X}_\mu$ and $\Gamma^\nu_{\lambda\mu} \rightarrow \overline{\Gamma}^\nu_{\lambda\mu}$ induced by the conformal change (5.2).

Theorem (5.6). *The vector X_μ is transformed by the conformal change (5.2) as follows:*

$$\overline{X}_\mu = X_\mu + \frac{1}{(n-1)g} M^{\alpha\beta}({}^1/_2 k_{\mu\beta}\Omega_\alpha - h_{\alpha[\beta}k^\gamma_{\mu]}\Omega_\gamma), \tag{5.8)a}$$

or equivalently

$$\overline{X}_\mu = X_\mu + \frac{1}{(n-1)g}\sum_{s=0}^{n-1}\tilde{K}_s \overset{(s)}{Z}_\mu, \tag{5.8)b}$$

where

$$\overset{(s)}{Z}_\mu \overset{\mathrm{df}}{=} ({}^{(n-1+\sigma-s)}k^\alpha_\mu - {}^1/_2\, {}^{(n-2+\sigma-s)}k^\beta_\beta k^\alpha_\mu)\Omega_\alpha . \tag{5.9}$$

PROOF. In virtue of (4.10)a and Agreement (5.1), we have

$$\overline{X}_\mu = \frac{1}{(n-1)\bar{g}}\overline{M}^{\alpha\beta}\overline{\nabla}_\alpha \overline{k}_{\beta\mu}. \tag{5.10}$$

The relation (5.8)a may be obtained by substituting (5.3)b, (5.4)d, and (5.7) into (5.10). The second relation follows from (5.8)a, using (2.9)c, (4.5), and (5.9). Note that the tensor ${}^{(n-2+\sigma-s)}k^{\alpha\beta}$ is symmetric.

Theorem (5.7). *The SE-connection $\Gamma^\nu_{\lambda\mu}$ is transformed by the conformal change (5.2) as follows:*

$$\overline{\Gamma}^\nu_{\lambda\mu} = \Gamma^\nu_{\lambda\mu}+\delta^\nu_{(\lambda}\Omega_{\mu)}-{}^1/_2 h_{\lambda\mu}h^{\nu\alpha}\Omega_\alpha + \frac{2}{(n-1)g}\sum_{s=0}^{n-1}\tilde{K}_s(\delta^\nu_{[\lambda}\overset{(s)}{Z}_{\mu]}+k^\nu_{(\lambda}\overset{(s)}{Z}_{\mu)}). \tag{5.11}$$

PROOF. In virtue of (3.1), (5.4)a, and (5.8)b, the relation (5.11) may be derived as follows:

$$\bar{\Gamma}^{\nu}_{\lambda\mu} = \left\{ \begin{matrix} \bar{\nu} \\ \lambda\mu \end{matrix} \right\} + 2\delta^{\nu}_{[\lambda}\bar{X}_{\mu]} + 2\bar{k}^{\nu}_{(\lambda}\bar{X}_{\mu)} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \delta^{\nu}_{(\lambda}\Omega_{\mu)} - {}^{1}/_{2} h_{\lambda\mu} h^{\nu\alpha}\Omega_{\alpha} +$$

$$+ 2\delta^{\nu}_{(\lambda}X_{\mu)} + \frac{2}{(n-1)g}\sum_{s=0}^{n-1}\tilde{K}_s \delta^{\nu}_{[\lambda}\overset{(s)}{Z}_{\mu]} + 2k^{\nu}_{(\lambda}X_{\mu)} + \frac{2}{(n-1)g}\sum_{s=0}^{n-1}\tilde{K}_s k^{\nu}_{(\lambda}\overset{(s)}{Z}_{\mu)} =$$

$$= (\text{The right-hand side of (5.11)}).$$

Remark (5.8). The relation (5.11) may be derived directly from (4.11), using (4,5) and (5.4).

Remark (5.9). As direct consequences of the above two theorems, we still state the representations of the changes of tensors $S^{\nu}_{\lambda\mu}$, $U^{\nu}_{\lambda\mu}$, and vectors S_{μ} and U_{μ} induced by the conformal change (5.2):

(5.12)a $$\bar{S}^{\nu}_{\lambda\mu} = S^{\nu}_{\lambda\mu} + \frac{2}{(n-1)g}\sum_{s=0}^{n-1}\tilde{K}_s \delta^{\nu}_{[\lambda}\overset{(s)}{Z}_{\mu]},$$

(5.12)b $$\bar{U}^{\nu}_{\lambda\mu} = U^{\nu}_{\lambda\mu} + \frac{2}{(n-1)g}\sum_{s=0}^{n-1}\tilde{K}_s k^{\nu}_{(\lambda}\overset{(s)}{Z}_{\mu)},$$

(5.12)c $$\bar{S}_{\mu} = S_{\mu} - \frac{1}{g}\sum_{s=0}^{n-1}\tilde{K}_s \overset{(s)}{Z}_{\mu},$$

(5.12)d $$\bar{U}_{\mu} = U_{\mu} + \frac{1}{(n-1)g}\sum_{s=0}^{n-1}\tilde{K}_s k^{\alpha}_{\mu}\overset{(s)}{Z}_{\alpha}.$$

Remark (5.10). The following table, which illustrates the tensor $\overset{(s)}{Z}_{\mu}$ for $n = 2, 3, 4$, may be used to calculate various representations of changes given in (5.8)b, (5.11), and (5.12) for the lower dimensional cases.

(5.13)

n \ $\overset{(s)}{Z}_{\mu}$	$\overset{(0)}{Z}_{\mu}$	$\overset{(2)}{Z}_{\mu}$
2	0	0
3	$({}^{(3)}k^{\alpha}_{\mu} + K_2 k^{\alpha}_{\mu})\Omega_{\alpha}$	$-{}^{1}/_{2} k^{\alpha}_{\mu}\Omega_{\alpha}$
4	$({}^{(3)}k^{\alpha}_{\mu} + K_2 k^{\alpha}_{\mu})\Omega_{\alpha}$	$-2k^{\alpha}_{\mu}\Omega_{\alpha}$

According to this table and (5.11), we have for $n=2$

(5.14) $$\bar{\Gamma}^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu} + \delta^{\nu}_{(\lambda}\Omega_{\mu)} - {}^{1}/_{2} h_{\lambda\mu} h^{\nu\alpha}\Omega_{\alpha}.$$

As an immediate consequence of (5.14), we have

Theorem (5.11). *In a 2-dimensional SE-manifold* SEX_2, *the tensors* $S_{\lambda\mu}^{\nu}$ *and* $U_{\lambda\mu}^{\nu}$ *are conformal invariant with respect to* (5.2). *That is,*

$$\bar{S}_{\lambda\mu}^{\nu} = S_{\lambda\mu}^{\nu}, \quad \bar{U}_{\lambda\mu}^{\nu} = U_{\lambda\mu}^{\nu}. \tag{5.15}$$

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