On recurrent Finsler connections with deflection and torsion

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Summary: In 1934 E. CARTAN [1] published his monograph 'Les especes de Finsler' and fixed his method to define a notion of connection in the geometry of Finsler spaces. In 1966 his method was reconsidered by M. MATSUMOTO [4] and determined uniquely the Cartan's connection by assuming four elegent axioms,

(1) The connection is metrical,
(2) The deflection tensor field vanishes,
(3) The torsion tensor field T vanishes,
(4) The torsion tensor field S vanishes.

In 1969 M. HASHIGUCHI [2] replaced the condition (2) by some weaker condition and determined a Finsler connection with the given deflection tensor field. In 1975 [3] he also determined uniquely a Finsler connection by replacing the condition (3). In amost all these works it has been assumed that the connection is metrical so that covariant differentiation commutes with the raising and lowering of the indices.

The purpose of the present paper is to determine a Finsler connection which is not h-metrical. A Finsler connection will be called h-recurrent Finsler connection if the h-covariant derivative of the metric tensor is recurrent. In this paper we determine only those Finsler connections which are v-metrical and whose torsion tensor field S vanishes.

1. Introduction

A Finsler manifold (F^n, L) of dimension n is a manifold F^n associated with a fundamental function $L(x, y)$ where $x(=x^i)$ denote positional variable of F^n and $y(=y^i)$ denote the components of a tangent vector with respect to x^i. The metric tensor of (F^n, L) is given by $g_{ij}=\frac{1}{2}\,\dot\partial_i\dot\partial_j L^2$ where $\dot\partial_i=\dfrac{\partial}{\partial y^i}$.

A Finsler connection of (F^n, L) is a triad $(F^i_{jk}, N^i_k, C^i_{jk})$ of a v-connection F^i_{jk}, a non-linear connection N^i_k and a vertical connection C^i_{jk} [5]. If a Finsler connection is given, the h and v covariant derivatives of any tensor field V^i_j are defined as

$$V^i_{j|k} = d_k V^i_j + V^m_j F^i_{mk} - V^i_m F^m_{jk} \tag{1.1}$$

$$V^i_j|_k = \dot\partial_k V^i_j + V^m_j C^i_{mk} - V^i_m C^m_{jk} \tag{1.2}$$

where

$$d_k = \partial_k - N^m_k \dot\partial_m, \quad \partial_k = \frac{\partial}{\partial x^k}.$$

For any Finsler connection $(F^i_{jk}, N^i_k, C^i_{jk})$ we have five torsion tensors and three curvature tensors which are given by

(1.3) $$(h)\ h\text{-torsion}:\ T^i_{jk} = F^i_{jk} - F^i_{kj}.$$

(1.4) $$(v)\ v\text{-torsion}:\ S^i_{jk} = C^i_{jk} - C^i_{kj}.$$

(1.5) $$(h)\ hv\text{-torsion}:\ C^i_{jk} = \text{as the connection } C^i_{jk}.$$

(1.6) $$(v)\ h\text{-torsion}:\ R^i_{jk} = d_k N^i_j - d_j N^i_k.$$

(1.7) $$(v)\ hv\text{-torsion}:\ P^i_{jk} = \dot{\partial}_k N^i_j - F^i_{kj}.$$

(1.8) $$h\text{-curvature}:\ R^i_{hjk} = d_k F^i_{hj} - d_j F^i_{hk} + F^m_{hj} F^i_{mk} - F^m_{hk} F^i_{mj} + C^i_{hm} R^m_{jk}.$$

(1.9) $$hv\text{-curvature}:\ P^i_{hjk} = \dot{\partial}_k F^i_{hj} - C^i_{hk|j} + C^i_{hm} P^m_{jk}$$

(1.10) $$v\text{-curvature}:\ S^i_{hjk} = C^m_{hj} C^i_{mk} - C^m_{hk} C^i_{mj} + \dot{\partial}_k C^i_{hj} - \dot{\partial}_j C^i_{hk}.$$

The deflection tensor field D^i_k of a Finsler connection is given by

(1.11) $$D^i_k = y^j F^i_{jk} - N^i_k.$$

When a Finsler metric is given, various Finsler connections are determined from the metric. The well known examples are Cartan's connection, Rund's connection and Berwald's connection. We shall use Cartan's connection which will be denoted by $(\Gamma^{*i}_{jh}, G^i_k, C^i_{jk})$. This connection is uniquely determined from the metric function L by the following five axioms:

(C_1) The connection is h-metrical that is $g_{ij|k}=0$
(C_2) The connection is v-metrical, that is $g_{ij}\|_k=0$
(C_3) The deflection tensor field D^i_k vanishes
(C_4) The torsion tensor field T^i_{jk} vanishes
(C_5) The torsion tensor field S^i_{jk} vanishes

and are given by [6]

(1.12) $$\Gamma^{*i}_{jk} = \tfrac{1}{2} g^{ih}[d_k g_{jh} + d_j g_{kh} - d_h g_{jk}]$$

(1.13) $$\text{(a)}\quad G^i_k = \dot{\partial}_k G^i = Y^i_{0k} - 2C^i_{km} G^m$$

$$\text{(b)}\quad G^i = \tfrac{1}{2} Y^i_{00}$$

(1.14) $$C^i_{jk} = g^{ih} C_{jhk},\quad C_{jhk} = \tfrac{1}{2}\dot{\partial}_h g_{jk}$$

where

(1.15) $$Y^i_{jk} = \tfrac{1}{2} g^{ih}(\partial_k g_{jh} + \partial_j g_{kh} - \partial_h g_{jk})$$

is the Christoffel symbol of (F^n, L) and '0' denote contraction with y^j.

In this paper we replace the condition (C_1), (C_3) and (C_4) and investigate general Finsler connections with given deflection tensor field and torsion tensor field which are h-recurrent with respect to given vector field a_k that is $g_{ij|k} = a_k g_{ij}$. It should be noted from the axioms (C_2) and (C_5) that the vertical connection C^i_{jk} is uniquely determined from the metric function $L(x, y)$ which is given by (1.4) and it satisfies the so called first C-condition:

$$y^j C^i_{jk} = 0, \quad y^j C_{jhk} = 0 \tag{1.16}$$

and the v-curvature tensor (1.10) becomes

$$S^i_{hjk} = C^m_{hk} C^i_{mj} - C^m_{hj} C^i_{mk}.$$

We shall use the following lemma which has been proved in [3] by HASHIGUCHI for those Finsler connections which satisfy the axioms (C_2) and (C_5).

Lemma 1. *If a Finsler connection satisfies the first C-condition, it holds*

$$y^i_{|j} = D^i_j, \quad y^i|_j = \delta^i_j. \tag{1.17}$$

$$P^i_{0jk} = P^i_{jk} + D^i_m C^m_{jk} + D^i_j|_k. \tag{1.18}$$

2. h-recurrent connections with deflection and torsion

Since our Finsler connection is h-recurrent we have to notice that some formulae have the style different from the ones familiar to us. For example the h-recurrency, $g_{ij|k} = a_k g_{ij}$, of the metric tensor g_{ij} gives the formula

$$C_{ijk|l} = C^h_{ik|l} g_{hj} + a_l C_{ijk}. \tag{2.1}$$

Theorem 2.1. *If a Finsler connection is h-recurrent with respect to recurrence vector a_k and the connection coefficient C_{ijk} are symmetric, then it hold for the components $P_{ijkl} = g_{jh} P^h_{ikl}$ of the hv-curvature tensor field:*

$$P_{ijkl} + P_{jikl} + a_h g_{ij} C^h_{kl} + a_k|_l g_{ij} = 0 \tag{2.2}$$

$$P_{ijkl} = G_{(ij)}\{C_{jkl|i} - C_{jkm} P^m_{il} - a_i C_{jkl}\} + \lambda_{ijkl} \tag{2.3}$$

where

$$\begin{aligned} \lambda_{ijkl} = \tfrac{1}{2} \big[&(T_{ijk} + T_{kij} + T_{ikj} + a_j g_{ik} - a_k g_{ij} - a_i g_{kj})|l + \\ &+ (T_{ijm} - T_{jim} + T_{jmi} - a_m g_{ij}) C^m_{kl} + G_{(ij)}\{(T_{kim} + T_{ikm} + T_{kmi} + a_m g_{ki}) C^m_{jl}\} \big] \end{aligned} \tag{2.4}$$

where $T_{jhk} = g_{ih} T^i_{jk}$ and $G_{(ij)}$ { } denotes the interchange of the indices i and j and substraction.

PROOF. Applying the Ricci identity [6] for the metric tensor g_{ij} we get

$$g_{ij}|_{l|k} - g_{ij|k}|_l = g_{ij|h} C^h_{kl} + g_{ij}|_h P^h_{kl} + g_{hj} P^h_{ikl} + g_{ih} P^h_{jkl}$$

which in view of $g_{ij|k} = a_k g_{ij}$ and $g_{ij}|_k = 0$ gives (2.2).

Again contracting one of the Bianchi identity [6]

$$T^h_{ij}|_k - C^h_{mk} T^m_{ij} + G_{(ij)}\{T^h_{im} C^m_{jk} + C^h_{jk|i} + C^h_{im} P^m_{jk} - P^h_{ijk}\} = 0$$

with g_{hl} and applying Christoffel process with respect to i, l and j we get (2.3).

Theorem (2.2). *Given a non-linear connection N_k^i, a skew symmetric Finsler* (1, 2) *tensor field T_{jk}^i and a covariant vector field a_k in a Finsler space, there exists a unique Finsler connection $(F_{jk}^i, N_k^i, C_{jk}^i)$ satisfying the axioms* (C_2), (C_5) *and*

(C_1') *The connection is h-recurrent that is $g_{ij|k} = a_k g_{ij}$*
(C_3') *The non-linear connection is the given N_k^i*
(C_4') *The (h)-h torsion tensor field is the given T_{jk}^i.*

PROOF. From axioms (C_2) and (C_5), it follows that the vertical connection C_{jk}^i is the same as Cartan's vertical connection given by (1.14). From the axiom (C_1'), we have

$$\partial_k g_{ij} - N_k^m \dot{\partial}_m g_{ij} - g_{mj} F_{ik}^m - g_{im} F_{jk}^m = a_k g_{ij}.$$

Applying Christoffel process to the above equation and using axiom (C_4') and expression (1.3) for (h)-h torsion, we get

$$F_{jk}^i = Y_{jk}^i - (C_{km}^i N_j^m + C_{jm}^i N_k^m - g^{hi} C_{jkm} N_h^m) - \tag{2.5}$$
$$- \tfrac{1}{2}(a_j \delta_k^i + a_k \delta_j^i - a^i g_{jk}) + A_{jk}^i$$

where

$$A_{jk}^i = \tfrac{1}{2}(T_{kjh} g^{hi} + T_{jkh} g^{hi} + T_{jk}^i). \tag{2.6}$$

In view of (2.5) and axiom (C_3') it is clear that the Finsler connection $(F_{jk}^i, N_k^i, C_{jk}^i)$ is uniquely determined from the metric function L and from given vector fields a_k, T_{jk}^i.

For the above connection the deflection tensor field D_k^i defined in (1.11) is obtained by contraction of (2.5).

$$D_k^i = G_k^i + 2C_{km}^i G^m - C_{km}^i N_0^m - N_k^i - \tfrac{1}{2}(a_0 \delta_k^i + a_k y^i - a^i y_k) + A_{0k}^i. \tag{2.7}$$

Contracting (2.7) with y^k, we get

$$N_0^i = 2G^i - D_0^i - a_0 y^i + \tfrac{1}{2} a^i L^2 + A_{00}^i. \tag{2.8}$$

Substituting the value N_0^i in (2.7) and using (1.16) we get

$$N_k^i = G_k^i - C_{km}^i (A_{00}^m - D_0^m + \tfrac{1}{2} a^m L^2) + (A_{0k}^i - D_k^i) - \tfrac{1}{2}(a_0 \delta_k^i + a_k y^i - a^i y_k).$$

Hence we have the following:

Theorem (2.3). *Given a Finsler* (1,1) *tensor field D_k^i, a covariant vector field a_k and a skew-symmetric Finsler* (1, 2) *tensor field T_{jk}^i in a Finsler space there exists a unique Finsler connection $(F_{jk}^i, N_k^i, C_{jk}^i)$ satisfying the axioms* (C_1'), (C_2), (C_4'), (C_5) *and* (C_3'') *the deflection tensor field is the given D_k^i.*

The v-connection F_{jk}^i is given by (2.5), in which the non-linear connection is given by

$$N_k^i = G_k^i - C_{km}^i B_0^m + B_k^i \tag{2.9}$$

where

$$B^i_k = A^i_{0k} - D^i_k - \tfrac{1}{2}(a_0 \delta^i_k + a_k y^i - a^i y_k). \tag{2.10}$$

The vertical connection is given by (1.14).

As a special case of the above theorem if we impose the axiom (C_3) instead of (C_3''), the B^i_k in (2.10) become

$$B^i_k = A^i_{0k} - \tfrac{1}{2}(a_0 \delta^i_k + a_k y^i - a^i y_k) \tag{2.11}$$

and we have the following:

Theorem (2.4). *Given a skew-symmetric* (1, 2) *tensor field* T^i_{jk} *and a covariant vector field* a_k *in a Finsler space there exists a unique Finsler connection* $(F^i_{jk}, N^i_k, C^i_{jk})$ *satisfying the axioms* (C_1'), (C_2), (C_3), (C_4') *and* (C_5). *These coefficients are given by* (2.5), (1.14) *and*

$$N^i_k = G^i_k - C^i_{km}(A^m_{00} - \tfrac{1}{2} a^m L^2) + A^i_{0k} - \tfrac{1}{2}(a_0 \delta^i_k + a_k y^i - a^i y_k). \tag{2.12}$$

If we assume that $B^i_k = 0$, equation (2.9) reduces to $N^i_k = G^i_k$ and we have the following which gives the Finsler connection with deflection and torsion,

Theorem (2.5). *Given a skew symmetric Finsler* (1, 2) *tensor field* T^i_{jk} *and covariant vector field* a_k *in a Finsler space, there exists a unique Finsler connection* $(F^i_{jk}, N^i_k, C^i_{jk})$ *satisfying the axioms* (C_1'), (C_2), (C_4'), (C_5) *and*

(C_3''') *The non-linear connection* N^i_k *is the one given by* E. CARTAN.

The coefficient F^i_{jk} are given by

$$F^i_{jk} = Y^i_{jk} - (C^i_{km} G^m_j + C^i_{jm} G^m_k - g^{hi} C_{jkm} G^m_h) - \tfrac{1}{2}(a_j \delta^i_k + a_k \delta^i_j - a^i g_{jk}) + A^i_{jk}. \tag{2.13}$$

The deflection tensor field D^i_k is expressed as

$$D^i_k = A^i_{0k} - \tfrac{1}{2}(a_0 \delta^i_k + a_k y^i - a^i y_k). \tag{2.14}$$

As a special case of Theorem (2.4) if we impose the axiom (C_4) instead of (C_4') we have the following:

Theorem (2.6). *Given a covariant vector field* a_k *there exists a unique Finsler connection* $(F^i_{jk}, N^i_k, C^i_{jk})$ *satisfying the axioms* (C_1'), (C_2), (C_3), (C_4) *and* (C_5).

These coefficients are given by

$$F^i_{jk} = \Gamma^{*i}_{jk} + Q^i_{jk} \tag{2.15}$$

$$N^i_k = G^i_k + T^i_k \tag{2.16}$$

$$C^i_{jk} = \tfrac{1}{2} g^{ih} \dot\partial_h g_{jk} \tag{2.17}$$

where Γ^{*i}_{jl} *and* G^i_k *are given by* (1.12) *and* (1.13) *and*

(2.18) $$Q^i_{jk} = \tfrac{1}{2}\{a_0 C^i_{jk} + L^2(C^i_{jm}C^m_{kh} + C^i_{km}C^m_{jh} - C^m_{jk}C^i_{mh})a^h$$
where Γ^{*i}_{jl} *and* G^i_k *are given by* (1.12) *and* (1.13) *and*

(2.18) $$Q^i_{jk} = \tfrac{1}{2}\{a_0 C^i_{jk} + L^2(C^i_{jm}C^m_{kh} + C^i_{km}C^m_{jh} - C^m_{jk}C^i_{mh})a^h - -(C^i_{jh}y_k + C^i_{kh}y_j - C_{hjk}y^i)a^h - (a_k\delta^i_j + a_j\delta^i_k - a^i g_{jk})\},$$

(2.19) $$T^i_k = \tfrac{1}{2}(a^i y_k - a_k y^i - a_0\delta^i_k - L^2 C^i_{jk}a^j).$$

For simplicity we shall use the following terminology

1. A Finsler connection will be called an *h-recurrent Finsler connection* if it is h-recurrent and its (h)-h torsion tensor field vanishes.

2. A Finsler connection will be called an *h-recurrent Finsler connection with torsion* if it is h-recurrent and its (h)-h torsion tensor field does not vanish.

3. h-recurrent generalized Berwald spaces

A Berwald space is a Finsler space in which coefficients Γ^{*i}_{jk} of Cartan's connection depend on the position alone [6]. In (1975) M. Hashiguchi generalized the concept of Berwald space and defined a generalized Berwald space as a Finsler space in which the metrical Finsler connection F^i_{jk} with torsion depends on the position alone. We shall generalize these two concepts and give the following:

Definition 1. A Finsler space is called an *h-recurrent Berwald space* if there is possible to introduce an h-recurrent Finsler connection in such a way that the connection coefficients F^i_{jk} depends on the position alone.

Definition 2. A Finsler space is called an *h-recurrent generalized Berwald space* if there is possible to introduce an h-recurrent Finsler connection with torsion in such a way that the connection coefficient F^i_{jk} depends on the position alone.

Now we shall find the condition under which a Finsler space becomes an h-recurrent generalized Berwald space.

Theorem (3.1). *The h-recurrent Finsler connection* F^i_{jk} *with torsion satisfies the condition* $\dot\partial_l F^i_{jk} = 0$, *if and only if*

(3.1) $$C_{ijl}D^l_k = 0,$$

(3.2) $$C_{ijk|l} = C_{ijm}|_k D^m_l + a_l C_{ijk},$$

(3.3) $$\lambda_{ijkl} = 0,$$

where λ_{ijkl} *is given in* (2.4).

Proof. From (1.9) it follows that the condition

(3.4) $$\dot\partial_l F^i_{jk} = 0,$$

is equivalent to

(3.5) $$P^i_{jkl} = -C^i_{jl|k} + C^i_{jm}P^m_{kl}$$

which in view of (2.1) is equivalent to

$$P_{ijkl} = a_k C_{ijl} - C_{ijl|k} + C_{ijm} P^m_{kl}. \tag{3.6}$$

Substituting this value in (2.2) we get

$$(a_m C^m_{kl} + a_k|_l) g_{ij} + 2(a_k C_{ijl} - C_{ijl|k} - C_{ijm} P^m_{kl}) = 0. \tag{3.7}$$

Contracting this equation with y^i and using (1.16) and (1.17) we get

$$2C_{ijl} D^i_k + (a_m C^m_{kl} + a_k|_l) y_j = 0. \tag{3.8}$$

Again contracting with y^j and using (1.16) we get

$$a_m C^m_{kl} + a_k|_l = 0 \tag{3.9}$$

which, when substituted in (3.8), gives (3.1).

Substitution of (3.9) in (2.2) shows that P_{ijkl} is skew-symmetric in i and j, while (3.6) shows that P_{ijkl} is symmetric in i and j. Hence $P_{ijkl}=0$. Then equation (1.18) and (3.6) give

$$P^i_{kl} = -D^i_m C^m_{kl} - D^i_k|_l \tag{3.10}$$

$$C_{ijk|l} = a_k C_{ijl} + C_{ijm} P^m_{kl}. \tag{3.11}$$

$$P^i_{kl} = -D^i_m C^m_{kl} - D^i_k|_l$$

Hence substituting the va (3.11) and using (3.1), we get (3.2).

Again substituting $C_{ijk|l} = a_k C_{ijl} + C_{ijm} P^m_{kl}$ ıg (3.11), we get (3.3).

Conversely, let us assume that (3.1), (3.2) and (3.3) hold. Writing the formula P^j_{ikl} from (2.3), using (3.3) and contracting the resulting expression with y^i we get (after using (1.16), (1.17), (1.18) and (3.1))

$$P^j_{kl} + D^j_m C^m_{kl} + D^j_k|_l = C^j_{kl|0} - C^j_{km} P^m_{0l}. \tag{3.12}$$

Contraction of this equation with y^k and use of (3.1) and (1.16) yields

$$P^j_{0l} = -D^j_k|_l y^k. \tag{3.13}$$

Substituting (3.13) in (3.12) and using (3.1), and (3.2) we find that the right hand side (R. H. S.) of (3.12) vanishes. Thus we get

$$P^j_{kl} + D^j_m C^m_{kl} + D^j_k|_l = 0. \tag{3.14}$$

Thus from (3.1), (3.2) and (3.14) we get (3.11). Substituting (3.11) and (3.3) in (2.3) we get $P_{ijkl}=0$, which in view of (1.9), (3.11) and (2.1) give (3.4).

From the theorem (3.1) we have

Theorem (3.2). *A Finsler space is an h-recurrent generalized Berwald space if and only if there exists a Finsler* (1,1) *tensor field* D^i_k, *a covariant vector field* a_k *and a skew-symmetric Finsler* (1,2) *tensor field* $T^i_{jk} \neq 0$ *satisfying conditions* (3.1), (3.2) *and* (3.3), *where the connection is the one given from* D^i_k, a_k *and* T^i_{jk} *by Theorem* (2.3).

If we consider the Finsler connection without deflection we have

Theorem (3.3). *A Finsler space is an h-recurrent generalized Berwald space if there exists a covariant vector field a_k and a skew-symmetric Finsler* (1,2) *tensor field $T^i_{jk} \neq 0$ satisfying the condition* (3.3) *and*

$$C_{ijk|l} = a_l C_{ijk} \tag{3.15}$$

where the connection is one given from a_k and T^i_{jk} by Theorem (2.4).

Corresponding to the Theorem (2.5) we have

Theorem (3.4). *A Finsler space is an h-recurrent generalized Berwald space if there exists a covariant vector field a_k and a skew-symmetric* (1,2) *tensor field $T^i_{jk} \neq 0$ satisfying the condition* (3.3) *and the following two:*

$$C_{ijl}\{A^l_{0k} - \tfrac{1}{2}(a_0 \delta^l_k + a_k y^l - a^l y_k) = 0, \tag{3.16}$$

$$C_{ijk|l} = C_{ijm}|_k \{A^m_{0l} - \tfrac{1}{2}(a_0 \delta^m_l + a_m y^l - a^m y_l)\}, \tag{3.17}$$

where the connection is the one given from a_k and T^i_{jk} by Theorem (2.5).

Corresponding to Theorem (2.6) we have

Theorem (3.5). *A Finsler space is an h-recurrent generalized Berwald space if there exists a covariant vector field a_k satisfying the conditions*

$$C_{ijk|l} = a_l C_{ijk} \tag{3.18}$$

$$\dot{\partial}_j a_k = 0 \tag{3.19}$$

where the connection is the one given from a_k by Theorem (2.6).

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