On a structure of preorder relations on a topological space

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Abstract: The following main result is proved in this paper. If X is a topological space and P is a preorder on X then P is the union of two preorders P_1 and P_2 on X such that (i) the smallest closed preorder on X containing P_1 is an equivalence relation (ii) P_2 is "equivalence-free" in P (i.e. if L is an equivalence relation on X and the smallest closed preorder on X containing the intersection of P and L is an equivalence relation then $P_2 \cap L$ is trivial). The decomposition is unique. A class $\mathcal{R}$ of transitive relations is defined for each relation of which maximal elements exist in every compact set. It contains the largest possible class of preorders on topological vector spaces, determined by convex cones, with regard to existence of maximal elements. The class $\mathcal{R}$ includes also the class of closed preorders, the class of preorders admitting continuous utility functions and the class of "equivalence-free" preorders. Moreover, it is closed under inverses and arbitrary intersections.

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1. Introduction and preliminaries

The paper is motivated by results obtained in [6, 8] and it has its roots un multicriteria optimization. In [6] the author defined the class $\mathcal{C}$ of convex cones in real topological vector spaces (a nonempty convex subset C is a convex cone if $tC \subseteqq C$ for $t \geqq 0$), which is the largest possible with regard to existence of maximal elements (Definition 4.1 below) in vector optimization problems. In [8] Theorem 2.1 we described some geometrical and topological structure of convex cones in an arbitrary topological vector space. We proved that every convex cone can be decomposed to two convex cones C_1 and C_2 such that $C_1 \in \mathcal{C}$, and the closure of C_2 is a vector subspace which meets C_1 at the origin. The aim of the present paper is to show that a similar decomposition can be given for an arbitrary preorder relation P on a topological space (Theorem 2.6 below). In the case when P is a preorder on a real topological vector space determined by a convex cone C, the decomposition is induced by the appropriate decomposition of C (Remark 3.2 below).

We define the class $\mathcal{R}$ (Section 3) of preorders on topological spaces, which contains the class of preorders determined by convex cones from the class $\mathcal{C}$. The class carries over the property of the existence of maximal elements with respect to every member of $\mathcal{R}$. Moreover, $\mathcal{R}$ includes the class of closed preorders and the class of preorders admitting continuous utility functions [2]. In mathematical economics preorders are interpreted as preference relations whereas utility functions are numerical utility indicators. The class $\mathcal{R}$ is closed under inverses and arbitrary intersections (Proposition 3.5 below).

In what follows X is a topological space, $\bar{A}$ denotes the closure of a subset A in X, $\Delta=\{(x,x): x\in X\}$ and $P\subseteqq X\times X$ is a preorder relation on X, i.e. P is reflexive and transitive. We write frequently xPy instead of $(x,y)\in P$ for $x,y\in X$. For a given P, the inverse P^{-1} is defined by $xP^{-1}y$ if and only if yPx for $x,y\in X$. If $x\in X$, the set $P(x)=\{y\in X: xPy\}$ is called an upper section of P. The union and intersection of preorders is defined in the usual set-theoretic way; however, the union usually is not a transitive relation.

For every preorder P there exists the largest equivalence relation $\ell(P)$ contained in P, namely $\ell(P)=P\cap P^{-1}$.

Let $\mathscr{K}$ denote the class of all preorders on topological vector spaces determined by convex cones. That is, a preorder P on a topological vector space X belongs to $\mathscr{K}$ if there is a convex cone $C\subseteqq X$ such that $P=P_C$, where xP_Cy means $y-x\in C$ for $x,y\in X$. Observe that $\ell(P_C)=P_{C\cap(-C)}$.

Since the topological closure of a preorder P on X may fail to be a transitive relation we define the closure of P in the following way.

1.1 Definition. If P is a preorder on X, the closure $\tilde{P}$ of P is the intersection of all closed preorders on X containing P. For large formulae we shall use notation $(\cdot)^{\sim}$.

The definition is meaningful because $X\times X$ is a closed preorder containing every preorder P on X.

The properties of the closure of a preorder summarized in the following lemma are easy to verify.

1.2 Lemma *Let P be a preorder on X. Then*

(i) *$\tilde{P}$ is a preorder on X.*

(ii) *$\tilde{\tilde{P}}=\tilde{P}$ and $(P^{-1})^{\sim}=(\tilde{P})^{-1}$.*

(iii) *If P_1 and P_2 are preorders on X then $P_1\subseteqq P_2$ implies $\tilde{P}_1\subseteqq\tilde{P}_2$ and $(P_1\cap P_2)^{\sim}\subseteqq\tilde{P}_1\cap\tilde{P}_2$.*

(iv) *P is closed if and only if $P=\tilde{P}$.*

(v) *$\ell(\tilde{P})$ is closed.*

(vi) *If P is an equivalence relation so is $\tilde{P}$.*

(vii) *$\tilde{P}$ is the smallest closed preorder containing P.*

(viii) *If X is a topological vector space and C is a convex cone in X then $(P_C)^{\sim}=P_{\bar{C}}$.*

2. On the structure of preorders

Let $\mathscr{P}$ be the set of all preorders on X. Define the operation $T:\mathscr{P}\to\mathscr{P}$ by $T(P)=P\cap\ell(\tilde{P})$. For every ordinal α put $T^\alpha(P)\equiv T(T^{\alpha-1}(P))$ if $\alpha-1$ exists and $T^\alpha(P)\equiv\bigcap_{\beta<\alpha}T^\beta(P)$ otherwise. The reader may verify that $T^\alpha(P)=P\cap\ell((T^{\alpha-1}(P))^{\sim})$ whenever $\alpha-1$ exists and $T^\alpha(P)=P\cap\bigcap_{\beta<\alpha}\ell((T^\beta(P))^{\sim})$ when α is a limit ordinal. Observe that $\{T^\alpha(P)\}$ is a chain with respect to inclusion of subsets of $X\times X$. Since by the Kuratowski—Zorn Lemma there exists a maximal chain in every ordered set we must have an ordinal $\alpha(P)$ such that $T^{\alpha(P)}(P)=T^\alpha(P)$ for every $\alpha\geqq\alpha(P)$. We shall denote $T^{\alpha(P)}(P)$ by $\mathscr{k}(P)$.

The preorder $\mathscr{k}(P)$ has the following properties.

2.1 Lemma. (i) $\ell(P) \subseteqq k(P) \subseteqq P$.

(ii) $(k(P))^{\sim}$ *is an equivalence relation.*

(iii) $k(P) = P \cap (k(P))^{\sim}$.

(iv) *If* L *is an equivalence relation on* X *such that* $(P \cap L)^{\sim}$ *is an equivalence relation then* $P \cap L = T^{\alpha}(P) \cap L$ *for every ordinal* α; *in particular* $P \cap L = k(P) \cap L$.

(v) *If* $k(P)$ *is an equivalence relation then* $k(P) = \ell(P)$.

Proof. Using Lemma 1.2 the idea of the proof is similar to that of Lemma 2.1 in [8]. We shall show for example (ii) and (iii). First observe that $k(P) = T^{\alpha(P)+1}(P) = P \cap \ell((k(P))^{\sim})$. Thus $k(P) \subseteqq \ell((k(P))^{\sim}) \subseteqq (k(P))^{\sim}$, which completes the proof of (ii) and (iii). □

Note that a preorder given by a ubiquitous cone in an infinite dimensional topological vector space X [4 p. 9] is an example of a preorder P for which $\ell(P) = \Delta$, $k(P) = T(P) = P$ and $(k(P))^{\sim} = P^{\sim} = X \times X$.

2.2 Lemma. *Let* (x, y) *and* $(y, z) \in P$. *If* $(x, z) \in k(P)$ *then* (x, y) *and* (y, z) *belong to* $k(P)$.

Proof. Since $(k(P))^{\sim}$ is an equivalence relation (Lemma 2.1 (ii)) and $(x, z) \in k(P)$ we must have that $(z, x) \in (k(P))^{\sim}$. Hence (z, x), (x, y) and $(y, z) \in \tilde{P}$. Using transitivity of $\tilde{P}$ we obtain that (z, y), $(y, x) \in \tilde{P}$. Consequently (x, y) and (y, z) belong to $\ell(\tilde{P})$. Using transfinite induction one can show that (x, y), $(y, z) \in \ell((T^{\alpha}(P))^{\sim})$ for every ordinal α. Hence (x, y), $(y, z) \in P \cap \ell((k(P))^{\sim}) = k(P)$ by Lemmy 2.1 (ii) and (iii). □

The following notion is auxiliary.

2.3. Definition. (i) We say that a preorder $P_\emptyset \subseteqq P$ on X is equivalence-free in P if for every equivalence relation L on X, such that $(P \cap L)^{\sim}$ is an equivalence relation, $P_\emptyset \cap L = \Delta$. (ii) The preorder $P_\emptyset$ is equivalence-free if it is equivalence-free in itself.

A preorder $P_C \in \mathscr{K}$ is equivalence-free if $\bar{C}$ is pointed, i.e. $\bar{C} \cap (-\bar{C}) = \{\emptyset\}$ (the converse is not true). However, it follows from Proposition 2.1 in [8] and Proposition 2.5 below that in a finite dimensional Hausdorff topological vector space X, $P_C \in \mathscr{K}$ is equivalence-free if and only if $\bar{C} \cap (-\bar{C}) = \{\emptyset\}$ (see also Proposition 3.4 (iv) (c) below).

2.4 Proposition. *If* $P_\emptyset \subseteqq P \subseteqq Q$ *are preorders on* X *and* $P_\emptyset$ *is equivalence-free in* Q *then it is equivalence-free in* P; *in particular* $P_\emptyset$ *is equivalence-free.*

Proof. Let $(P \cap L)^{\sim}$ be an equivalence relation for some equivalence relation L on X. Then since

$$(P \cap L)^{\sim} \subseteqq (Q \cap (P \cap L)^{\sim})^{\sim} \subseteqq ((P \cap L)^{\sim})^{\sim} = (P \cap L)^{\sim},$$

$(Q \cap (P \cap L)^{\sim})^{\sim}$ is an equivalence relation and since $P_\emptyset$ is equivalence-free in Q we must have that $P_\emptyset \cap (P \cap L)^{\sim} = \Delta$. Since $P_\emptyset \cap L \subseteqq P_\emptyset \cap (P_\emptyset \cap L)^{\sim} \subseteqq P_\emptyset \cap (P \cap L)^{\sim}$ we obtain that $P_\emptyset \cap L = \Delta$, which proves that $P_\emptyset$ is equivalence.free in P. □

2.5 Proposition. *A preorder* P *is equivalence-free if and only if* $k(P) = \Delta$. *In particular, if* P *is equivalence-free then* $\ell(P) = k(P) = \Delta$.

Proof. Assume that P is equivalence-free. Using Lemma 2.1 (ii) and (iii) we obtain that $(P\cap(\mathscr{k}(P))^{\sim})^{\sim}=(\mathscr{k}(P))^{\sim}$ is an equivalence relation. Hence again by Lemma 2.1 (iii) we have that

$$\Delta = P\cap(\mathscr{k}(P))^{\sim} = \mathscr{k}(P).$$

Conversely, assume that $\mathscr{k}(P)=\Delta$ and $(P\cap L)^{\sim}$ is an equivalence relation for some equivalence relation L on X. Applying Lemma 2.1 (iv) we have that $P\cap L= =\mathscr{k}(P)\cap L=\Delta\cap L=\Delta$.

The second part follows from Lemma 2.1 (i). □

The next theorem is the main result concerning the decomposition of an arbitrary preorder.

2.6 Theorem. (i) *Let P be a preorder on X. Then the relation $\varrho(P)$ on X defined by*

$$\varrho(P) = \big(P\setminus(\mathscr{k}(P))^{\sim}\big)\cup\Delta$$

is a preorder on X which is equivalence-free in P and

$$P = \mathscr{k}(P)\cup\varrho(P)$$

In particular $\mathscr{k}(\varrho(P))=\Delta$ and $(\mathscr{k}(P))^{\sim}\cap\varrho(P)=\Delta$.

(ii) *If $P=P_1\cup P_2$ where P_1 and P_2 are preorders on X such that $\tilde{P}_1$ is an equivalence relation and P_2 is equivalence-free in P then $P_1=\mathscr{k}(P)$ and $P_2=\varrho(P)$.*

Proof. Since $P\cap(\mathscr{k}(P))^{\sim}=\mathscr{k}(P)$ (Lemma 2.1 (iii)) we must have that $P=\mathscr{k}(P)\cup \cup\varrho(P)$.

Let $(x, y), (y, z)\in\varrho(P)$. If $(x, z)\notin\varrho(P)$ then $(x, z)\in\mathscr{k}(P)$ and by Lemma 2.2 $(x, y), (y, z)\in\mathscr{k}(P)$, which is a contradiction. Thus $(x, z)\in\varrho(P)$, which shows that $\varrho(P)$ is transitive.

Let L be an equivalence relation on X such that $(P\cap L)^{\sim}$ is also an equivalence. Using Lemma 2.1 (iv) gives $P\cap L=\mathscr{k}(P)\cap L$, thus $\varrho(P)\cap L\subseteq\varrho(P)\cap\mathscr{k}(P)=\Delta$, which proves that $\varrho(P)$ is equivalence-free in P.

The second part of the theorem is straightforward. □

Let $\mathscr{R}$ denote the class of preorders P with $\mathscr{k}(P)=\ell(P)$. The class $\mathscr{R}$ will be subject of investigation in Sections 3 and 4.

3. Properties of the class $\mathscr{R}$

The following proposition gives equivalent characterizations of the class $\mathscr{R}$.

3.1 Proposition. *Let P be a preorder on X. The following conditions are equivalent.*

(i) *$P\in\mathscr{R}$.*

(ii) *For every equivalence relation L on X, $P\cap L$ is equivalence relation whenever $(P\cap L)^{\sim}$ is an equivalence relation.*

(iii) *For every closed equivalence relation L on X, $P\cap L$ is an equivalence relation whenever $(P\cap L)^{\sim}$ is an equivalence relation.*

(iv) *For every closed equivalence relation L on X such that $L\subseteq\ell(\tilde{P})$, $P\cap L$ is an equivalence relation whenever $(P\cap L)^{\sim}$ is an equivalence relation.*

PROOF. The implications (ii)⇒(iii)⇒(iv) are obvious. The implication (i)⇒(ii) follows from Lemma 2.1 (iv). In order to show that (iv) implies (i) observe that $(k(P))^{\sim}$ is a closed equivalence relation in $\ell(\tilde{P})$ and apply Lemma 2.1 (iii) twice and then Lemma 2.1 (i). □

3.2 Remark. If X is a topological vector space and we restrict our consideration only to the class $\mathscr{K}$, i.e. we use equivalence relations $L\in\mathscr{K}$ in Proposition 3.1, then we obtain exactly the class of preorders $P_C\in\mathscr{K}$ with C satisfying the following condition $(*)$ introduced in [6]:

$(*)$ For every closed vector subspace L of X $C\cap L$ is a vector subspace whenever $\overline{C\cap L}$ is a vector subspace.

The class of convex cones satisfying $(*)$ was called $\mathscr{C}$ in [8]. Thus $\mathscr{K}\cap\mathscr{R}=\{P_C\}_{C\in\mathscr{C}}$. Moreover $C\in\mathscr{C}$ if and only if $k(P_C)=C\cap(-C)$ (see Proposition 2.1 in [8]). □

Hence, by Proposition 3.1 we obtain the following corollary.

3.3 Corollary. *Let X be a topological vector space and C be a convex cone in X. Then the following conditions are equivalent.*

(i) *For every closed vector subspace L contained in $\bar{C}\cap(-\bar{C})$, $C\cap L$ is a vector subspace whenever $\overline{C\cap L}$ is a vector subspace.*

(ii) *For every equivalence relation L on X, $P_C\cap L$ is an equivalence relation whenever $(P_C\cap L)^{\sim}$ is an equivalence relation.*

We say that a real-valued function f on X is a utility function for a preorder P on X if $f(x)\leqq f(y)$ whenever xPy and $f(x)\neq f(y)$ if xPy and not yPx (see for instance [2,5]).

3.4 Proposition. *The following classes of preorders are contained in $\mathscr{R}$.*

(i) *The class of closed preorders.*

(ii) *The class of preorders admitting continuous utility functions.*

(iii) *The class of equivalence-free preorders; in particular $\varrho(P)\in\mathscr{R}$ for every P.*

(iv) *If X is a topological vector space and C is a convex cone in X then $P_C\in\mathscr{R}$ whenever* [6, 8]:

(a) *C is closed (a particular case of* (i));

(b) *there exists a continuous C-positive linear functional f on X, i.e. $f(x)\geqq 0$ for $x\in C$ and $f(x)>0$ if $x\in C\setminus(-C)$ (a particular case of* (ii));

(c) *the vector subspace $\bar{C}\cap(-\bar{C})$ is Hausdorff and finite dimensional; in particular if C is contained in a finite dimensional Hausdorff vector space or if $\bar{C}\cap(-\bar{C})=\{0\}$ (the latter is a particular case of* (iii)).

PROOF. The proof of (i) is straightforward using Lemma 1.2 (iv), since then $T(P)=P\cap\ell(\tilde{P})=P\cap\ell(P)=\ell(P)$.

To demonstrate (ii) assume that f is a continuous utility function for a preorder P. Define the preorder R on X by xRy if and only if $f(x)\leqq f(y)$ for $x, y\in X$. Observe that $P\subseteqq R$, $(x, y)\in\ell(R)$ implies $f(x)=f(y)$ and since f is continuous the preorder R is closed. Thus $\tilde{P}\subseteqq R$ by Lemma 1.2 (iii) and (ii). Since $P\cap\ell(R)\subseteqq\ell(P)$ we have

$$l(P)\subseteqq P\cap l(\tilde{P})\subseteqq P\cap\ell(R)\subseteqq\ell(P).$$

Hence, by Lemma 2.1 (ii),

$$\ell(P) \subseteq k(P) \subseteq P \cap \ell(\tilde{P}) = \ell(P),$$

which proves that $P \in \mathscr{R}$.

The proofs of (iii) and (iv) follow from Proposition 2.5 and Corollary 3.3 respectively.

3.5 Proposition. *The class $\mathscr{R}$ is closed under inverses and arbitrary intersections.*

Proof. The closedness under inverses follows from Proposition 3.1 and from the fact that $(P^{-1} \cap L)^{\sim} = ((P \cap L)^{\sim})^{-1}$ for every preorder P and every equivalence relation L (Lemma 1.2 (ii)).

Let $\{P_i\}_{i \in I}$ be a family of preorders in $\mathscr{R}$, i.e. $k(P_i)(\equiv T^{\alpha(P_i)}(P_i)) = \ell(P_i)$ for every $i \in I$. Put $P \equiv \bigcap_{i \in I} P_i$. By the definition of the operation k and since $P \subseteq P_i$ we obtain that

$$k(P) \subseteq T^{\alpha(P_i)}(P) \subseteq T^{\alpha(P_i)}(P_i) \equiv k(P_i)$$

for every $i \in I$. Hence

$$k(P) \subseteq \bigcap_{i \in I} k(P_i) = \bigcap_{i \in I} \ell(P_i).$$

On the other hand $\bigcap_{i \in I} \ell(P_i)$ is an equivalence contained in every P_i so in P and $\ell(P)$ is the largest equivalence relation contained in P. Thus $\bigcap_{i \in I} \ell(P_i) \subseteq \ell(P)$. By Lemma 2.1 (i) $\ell(P) \subseteq k(P)$ so $k(P) = \ell(P)$, which proves that $P \in \mathscr{R}$. □

From Remark 3.2 and Proposition 3.5 we obtain

3.6 Corollary. *The class $\mathscr{C}$ is closed under arbitrary intersections.*

4. On the existence of maximal elements

Let P be a transitive relation on X and let B be a nonempty subset of X. Observe first that the relation $P_= = P \cup \Delta$ is a preorder on X.

4.1 Definition. We say that $e \in B$ is maximal up to indifference (also called efficient) in B, and write $e \in E_P(B)$, if ePx and $x \in B$ implies xPe (see [6] and the references therein).

Observe that $E_{P_=}(B) = E_P(B)$ for every subset B of X and every transitive relation P.

We say that a subset B of X is P-compact if there exists $x \in X$ such that either $B \cap P_=(x)$ or $B \cap (P_=)^{\sim}(x)$ is nonempty and compact. Of course, every nonempty and compact subset is P-compact for every transitive relation P. It is easy to show that $E_P(B \cap (P_=)^{\sim}(x)) \subseteq E_P(B \cap P_=(x)) \subseteq E_P(B)$ for every subset B of X, $x \in X$ and every transitive relation P on X.

The following proposition was proved in [3] (Theorem 2.3 and Corollary 2.8). We present another proof below which is shorter and straightforward, without making use of the quotient structure.

4.2 Proposition. *Let P be a transitive relation such that $P_=(x)$ is closed for every $x \in X$. Let B be a P-compact subset of X. Then $E_P(B) \neq \Phi$.*

Proof. Without loss of generality we may assume that B is nonempty and compact. Consider the family $\mathscr{F} = \{F_x\}_{x \in B}$ of nonempty closed subset of B, where $F_x = B \cap P_=(x)$ for $x \in B$, ordered by inclusion. Since any chain $\{F_{x_i}\}_{i \in I}$ in $\mathscr{F}$ has a finite intersection property we must have that some $z \in \bigcap_{i \in I} F_{x_i}$. Using transitivity of P we have that $F_z \subseteq F_{x_i}$ for every $i \in I$. Thus F_z is a lower bound for the chain. By Kuratowski—Zorn Lemma there exists a minimal element F_e in $\mathscr{F}$ for some $e \in B$. It follows from transitivity of P that $e \in E_{P_=}(B) = E_P(B)$. □

4.3 Theorem. *Let P be a transitive relation on X and let $P_= \in \mathscr{R}$. Then $E_P(B) \neq \Phi$ for every P-compact subset B of X.*

Proof. Without loss of generality we may assume that P is a preorder and that B is nonempty and compact. If α is an ordinal we denote by $[x]_\alpha$ the equivalence class of $x \in X$ with respect to the equivalence relation $\ell((T^\alpha(P))^{\sim})$. Define a transfinite sequence of nonempty compact sets B_α as follows

(i) $B_0 = B$;

(ii) $B_{\alpha+1} = B \cap [e_{\alpha+1}]_{\alpha+1}$ for some $e_{\alpha+1} \in E_{(T^\alpha(P))^{\sim}}(B)$, which existence is guaranteed by Proposition 4.2.

(iii) $B_\alpha = \bigcap_{\beta < \alpha} B_\beta$ if α is a limit ordinal.

The sets B_α are nonempty and compact, because equivalence classes of closed equivalence relations are closed subsets in X and $\{B_\alpha\}$ has a finite intersection property since it is a chain.

The proofs of the following inclusions are rather technical so we omit them (compare the proof of Theorem 2.2 in [6]). Put $P_\alpha = T^\alpha(P)$ for every ordinal α.

(a) $E_{P_{\alpha+1}}(B_{\alpha+1}) \subseteq E_{P_\alpha}(B_{\alpha+1}) \subseteq E_{P_\alpha}(B_\alpha)$

(b) $E_{P_\alpha}(B_\alpha) \subseteq E_P(B)$ for α a limit ordinal.

Thus $E_{P_\alpha}(B_\alpha) \subseteq E_P(B)$ for every α.

Since $P \in \mathscr{R}$ we must have that

$$P_{\alpha(P)} \equiv \hbar(P) = \ell(P)$$

is an equivalence relation. Hence we obtain

$$\Phi \neq B_{\alpha(P)} = E_{P_{\alpha(P)}}(B_{\alpha(P)}) \subseteq E_P(B),$$

which completes the proof of the theorem. □

The proof of the following theorem uses Theorem 2.6, Lemma 2.1 (ii) and (iii) and it is similar to that of Theorem 3.1 in [8] so we omit it.

4.4 Theorem. *If P is a transitive relation then for every subset B of X*

$$E_P(B) = E_{\varrho(P_=)}(B) \cap E_{\hbar(P_=)}(B).$$

In particular, if $P_= \in \mathscr{R}$ then

$$E_P(B) = E_{\varrho(P_=)}(B).$$

Theorem 4.3 states that $P_= \in \mathscr{R}$ is a sufficient condition for the existence of maximal points in every compact (in fact, P-compact) set. It is obvious that this condition is not a necessary one. One can find a preorder P with closed upper sections (by Proposition 4.2 $E_P(B) \neq \Phi$ for a nonempty compact set B) yet not a member of $\mathscr{R}$. For instance the preorder P on the space of real numbers $\mathbf{R}$ defined by $P(x)=\mathbf{R}$ if $x \notin Z$ and $P(x)=Z \cap \{y : x \leqq y\}$ if $x \in Z$, where Z is the set of integers, has closed upper sections but P does not belong to $\mathscr{R}$ (P is not an equivalence relation whilst $\tilde{P}=\mathbf{R}\times\mathbf{R}$ is an equivalence relation). However, if X is a topological vector space and C is a convex cone in X such that $C \notin \mathscr{C}$ and $(\ell(P_C))^{\sim}$ is a metrizable vector space (see Remark 3.2) then Proposition 2.1 in [6] applied to $(\ell(P_C))^{\sim}$ and $\ell(P_C)$ says that we can find a compact set $B \subseteqq (\ell(P_C))^{\sim}$ such that $E_{\ell(P_C)}(B)=\Phi$ (see also Theorem 3.2 (ii) in [8]). The metrizability assumption is essential as shown by Example 1.1 in [7].

$\mathscr{K}_0 \subseteqq \mathscr{K}$ be the class of preorders P which are determined by convex cones C such that $(\ell(P_C))^{\sim}$ is metrizable and $\mathscr{K}_m \subseteqq \mathscr{K}_0$ be the sublcass of preorders defined on metrizable topological vector spaces. Then $P \in \mathscr{K}_0 \cap \mathscr{R}$ is also a necessary condition for $E_{P_C}(B) \neq \Phi$ for every nonempty compact set B in the domain of P.

4.5 Problem. Find other classes $\mathscr{P}_0$ (containing $\mathscr{K}_m$) of transitive relations on topological spaces for which the condition $P \in \mathscr{P}_0 \cap \mathscr{R}$ is a necessary one for the existence of maximal elements in compact sets.

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