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Cut-elimination in the theory of definable sets of natural numbers

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To the memory of Prof. V. A. Smirnov

Abstract. We prove a cut-elimination theorem for the second order arithmetic with the omega-rule and with the second order quantifiers running only on *definable* subsets of the set of natural numbers. Particulary, we get the consistency theorem for the semiformal axiomatic theory – the second order arithmetic of definable sets – with the Π^1_1 notion of deducibility.

The proof of the cut-elimination is produced by a synthese of the method of the completion of semivaluations and descriptive methods of axiomatic set theory.

1. We prove a cut-elimination theorem for the sequent calculus in the second order arithmetic. In this theory the second order variables (i.e. variables for *the sets* of natural numbers) are considered as variables for *definable* sets i.e. sets can be defined in the language considered. This situation is reflected in the rules of our theory, for example, in the rule introducing the second order universal quantifier. Our theory contains the following rule of inference with the infinite set of premises:

$$\frac{\Gamma \to \Delta\varphi(t) \quad \text{for all second-order } \textit{closed} \text{ terms } t}{\Gamma \to \Delta \quad \forall x\varphi(x)}$$

where x is a second-order variable and t is a second-order term.

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The proof of the cut-elimination is produced by a synthese of the method of the completion of semivaluations (cf. [10], [11]) and descriptive methods of axiomatic set theory ([7], [1]). Our exposition here is some strengthening and modification of the results which was announced firstly in [3] and published in [4].

2. Let us describe the language of our theory.

2.1. *Types* are, by definition, three natural numbers 0, 1 and 2. We fix some infinite countable set of symbols $\mathrm{Var}(\tau)$ for every type τ, the elements of the set $\mathrm{Var}(\tau)$ are *variables* of the type τ.

Intuitively, variables of the type 0 are considered as variables for natural numbers $0, 1, 2, \ldots$. The set of all natural numbers we denote by ω, so a variable $x \in \mathrm{Var}(0)$ *runs over* the set ω. Variables of type 1 are considered as variables for concrete statements of our language. And, finally, variables of type 2 are considered as variables for some subsets of the set ω. We do not suppose, in the general case, that a variable $x \in \mathrm{Var}(2)$ runs over the *whole* set $\mathcal{P}\omega$ of *all* subsets of ω.

If we consider variables of $\mathrm{Var}(2)$ as running over the whole set $\mathcal{P}\omega$ then we shall speak about the *natural interpretation* of variables.

2.2. Our language contains the constant 0 of type 0 and the one-place functional symbol S which corresponds to the familiar successor function (plus-one function).

Moreover, our language might contain some arbitrary list of functional symbols such that every symbol is of the type 0 and the argument places of these symbols are of type 0. We suppose that for any functional symbol f of our language some general recursive function $\tilde{f}$ is prescribed in some canonical way. Of course, $\tilde{f}$ is an n-argument place total function if f is an n-argument place functional symbol, particularly, $\tilde{S}(x) = x + 1$.

Our language might contain also some set of *predicate symbols* all argument places of which are of type 0. We suppose also that some general recursive predicate $\tilde{P}$ is prescribed in some canonical way to every predicate symbol P of our language with the same number of argument places. For example, the reader can suppose that our language contains the two-place familiar predicate symbol $=$ with the usual interpretation (the equality of two natural numbers).

2.3. Let us give now the inductive definition of the notion: an *expression of the type* τ; the set of all expressions of the type τ we denote by $\mathrm{Exp}(\tau)$, so we define, in an inductive way, the sets $\mathrm{Exp}(\tau)$.

1) $0 \in \mathrm{Exp}(0)$.

2) $x \in \mathrm{Var}(\tau) \implies x \in \mathrm{Exp}(\tau)$.

3) If $t_1, \ldots, t_n \in \mathrm{Exp}(0)$ and f is an n-places functional symbol of our language then $f(t_1, \ldots, t_n) \in \mathrm{Exp}(0)$.

4) If $t_1, \ldots, t_n \in \mathrm{Exp}(0)$ and P is an n-places predicate symbol of our language then $P(t_1, \ldots, t_n) \in \mathrm{Exp}(1)$.

Let us put in the items 5) – 7) below: $\varphi, \psi \in \mathrm{Exp}(1)$, $x \in \mathrm{Var}(\tau)$. Then

5)–7) $(\varphi \wedge \psi)$, $\neg\varphi$, $\forall x \varphi \in \mathrm{Exp}(1)$.

8) If $x \in \mathrm{Var}(0)$, $\varphi \in \mathrm{Exp}(1)$, then $\{x \mid \varphi\} \in \mathrm{Exp}(2)$.

9) $t \in \mathrm{Exp}(0)$, $T \in \mathrm{Exp}(2) \implies (t \varepsilon T) \in \mathrm{Exp}(1)$.

The definition of sets $\mathrm{Exp}(\tau)$ is complete. Expressions of the type *one* we shall call by *formulas*. The set of all formulas we denote by Fm, so $\mathrm{Fm} = \mathrm{Exp}(1)$. Formulas of the kind 4) we call *atomic* formulas and we denote the set of all such formulas by $\mathrm{At\,Fm}$.

2.4. We classify the occurrences of variables in an expression as *free* and *bound* in the usual way. The *quantifier prefixes* in our case are the parts like $\forall x$ and $\{x \mid \ldots\}$ which "bound" the corresponding occurrences of the variable x.

We identify systematically the expressions which differ only by a renaming of bound variables (*congruent* expressions in the terminology of [8]) and, in particular, change freely such expressions in our inferences.

An expression $(t)(x_1, \ldots, x_n \,\|\, t_1, \ldots, t_n)$ denotes the result of simultaneous substitutions instead of free occurrences of distinct variables $x_1, \ldots,$ x_n in t by the expressions $t_1, \ldots, t_n$ respectively. Here the variable x_i and the expression t_i have the same type. We suppose, moreover, that in the process of substitution some renaming of bound variables of t is produced avoiding the collision of variables. The expression considered we abbreviate sometimes as $t(x_1, \ldots, x_n \,\|\, t_1, \ldots, t_n)$ or even as $t(t_1, \ldots, t_n)$ if the mention of the variables $x_1, \ldots, x_n$ is inessential.

The variable x is *free* in an expression t if x occurs (at least once) freely in t, in such case x is a *parameter* of t. The expression t is called *closed* if it does not contain any free occurrence of variables, i.e. all variables in it are bound, there is no parameter in t. The set of all closed expressions of the type τ we denote by $\mathrm{Cl\,Exp}(\tau)$. The set of all closed formulas we denote by $\mathrm{Cl\,Fm}$, so $\mathrm{Cl\,Fm} = \mathrm{Cl\,Exp}(1)$, elements of $\mathrm{Cl\,Fm}$ are, by definition, *sentences* of our language. Similarly, the set of all closed atomic formulas we denote by $\mathrm{Cl\,At\,Fm}$.

2.5. Every closed formula φ of our language expresses some analytical statement if one takes into account that atomic closed formulas of our language have a natural *general recursive* interpretation and suppose that quantifiers of the type 2 run over *all* subsets of the set ω. From the usual set theoretical point of view (which we adopt in this article) such a φ is *true* or *false* in this *natural* interpretation. We shall write $\models \varphi$ if φ is true in the natural interpretation.

2.6. We use the usual theory of the sequent inference (see, for example, [8] §77 or [5] Part 1, §5) with minor changes which are connected mainly with the circumstance that we use rules of inference with *infinite* number of premises (so-called, *ω-rules*). So we are dealing not with the usual formal axiomatic theories but rather with so-called, *semiformal* theories.

The *sequent* is a figure of the kind $\Gamma \to \Delta$ where Γ and Δ are (maybe empty) lists of *closed* formulas. A *list* is, by definition, an *unordered* finite set of formulas with possible repetitions of formulas in it (a *multiset* of formulas). Note, that in sequents we use only lists of *closed* formulas.

3. Let us consider some semiformal axiomatic theory – *the second-order arithmetic of definable sets* – we denote this theory by $\mathrm{Def\,Ar}\,\omega 2$. In $\mathrm{Def\,Ar}\,\omega 2$ one can deduce some sequents of our language.

3.1. The *axioms* of our theory have one of the following forms:

1) $\varphi \to \varphi$ where φ is an arbitrary closed formula.
2) $\to \varphi$ where $\varphi \in \mathrm{Cl\,At\,Fm}$, $\models \varphi$.
3) $\varphi \to$ where $\varphi \in \mathrm{Cl\,At\,Fm}$, $\models \neg\varphi$.

3.2. The *structural rules of inference* of $\mathrm{Def\,Ar}\,\omega 2$ are the usual rules of *addition*, *shortening* and *cut*:

$$\frac{\Gamma \to \Delta}{\varphi\Gamma \to \Delta}\,(ad \to); \quad \frac{\Gamma \to \Delta}{\Gamma \to \Delta\varphi}\,(\to ad);$$

$$\frac{\varphi\varphi\Gamma \to \Delta}{\varphi\Gamma \to \Delta}\,(st \to); \quad \frac{\Gamma \to \Delta\varphi\varphi}{\Gamma \to \Delta\varphi}\,(\to st);$$

$$\frac{\Gamma \to \Delta\varphi\,;\ \varphi\Gamma \to \Delta}{\Gamma \to \Delta}\,(cut).$$

3.3. The *logical rules of inference* introduce logical connectives at right and left:

$$\frac{\varphi\Gamma \to \Delta}{(\varphi \wedge \psi)\Gamma \to \Delta}; \ \frac{\psi\Gamma \to \Delta}{(\varphi \wedge \psi)\Gamma \to \Delta} (\wedge \to);$$

$$\frac{\Gamma \to \Delta\varphi; \Gamma \to \Delta\psi}{\Gamma \to \Delta(\varphi \wedge \psi)} (\to \wedge);$$

$$\frac{\Gamma \to \Delta\varphi}{\neg\varphi\Gamma \to \Delta} (\neg \to); \quad \frac{\varphi\Gamma \to \Delta}{\Gamma \to \Delta\neg\varphi} (\to \neg);$$

$$\frac{\varphi(x\|t)\Gamma \to \Delta}{(t\varepsilon\{x \mid \varphi\})\Gamma \to \Delta} (\varepsilon \to); \quad \frac{\Gamma \to \Delta\varphi(x\|t)}{\Gamma \to \Delta(t\varepsilon\{x \mid \varphi\})} (\to \varepsilon);$$

$$\frac{\varphi(x\|t)\Gamma \to \Delta}{\forall x\varphi\Gamma \to \Delta} (\forall \to);$$

$$\frac{\Gamma \to \Delta\varphi(x\|t) \text{ for all } t \in \mathrm{Cl}\,\mathrm{Exp}(\tau)}{\Gamma \to \Delta\forall x\varphi} (\to \forall),$$

here $x \in \mathrm{Var}(\tau)$ and $t \in \mathrm{Cl}\,\mathrm{Exp}(\tau)$.

The formulation $\mathrm{Def}\,\mathrm{Ar}\,\omega 2$ is complete.

3.4. A sequent S is, by definition, *deducible* in the theory $\mathrm{Def}\,\mathrm{Ar}\,\omega 2$ (in symbols, $\vdash S$) if there exists an *inference* for S in $\mathrm{Def}\,\mathrm{Ar}\,\omega 2$.

A formula φ is *deducible* if the sequent $\to \varphi$ is deducible in $\mathrm{Def}\,\mathrm{Ar}\,\omega 2$.

An *inference* in $\mathrm{Def}\,\mathrm{Ar}\,\omega 2$ is an infinite branching tree-like figure such that some sequent is prescribed to every node of this tree and each relation between the parent-node and its sons forms some rule of inference of the theory $\mathrm{Def}\,\mathrm{Ar}\,\omega 2$. Axioms are prescribed to the leaves of the inference-tree. Moreover, we suppose that an ordinal (in the usual set-theoretical sense of this term) is prescribed to every node of the inference-tree and the following conditions are fulfilled:

– the ordinal 0 is prescribed to the leaves and
– the ordinal prescribed to the parent is the smallest ordinal which exceeds all ordinals prescribed to the sons.

Thus, the last node of the inference has some uniquely prescribed ordinal, it is the, so-called, *height* of the inference. Every inference has a height. Because of the infinitistic rule of inference $(\to \forall)$ the height might be an infinite ordinal, but it is always a *countable* ordinal. The prescribing of ordinals provides that all paths in inferences are finite so every inference-tree is well-founded.

As nodes of inference-trees some standard objects with the natural tree-like ordering could be chosen. For example, one can use as nodes finite corteges of natural numbers. The precise definition of the notion of inference in $\mathrm{Def\,Ar}\,\omega 2$ could be given by transfinite induction on its height. At each step of this transfinite induction the corresponding tree-like figure is defined by some fixed rule, say, in a minimal way. Such a definition provides a very important property: every inference in $\mathrm{Def\,Ar}\,\omega 2$ is a *constructible object* in a Gödel sense [7] and the notion of deducibility in $\mathrm{Def\,Ar}\,\omega 2$ is *absolute*. It means that every inference belongs to Gödel's constructible universe L and, moreover, some set in L is an inference if and only if it is true in the universe L. This follows easily from the form of the definition of inference in $\mathrm{Def\,Ar}\,\omega 2$ according to Gödel's theory of absoluteness [7].

3.4.1. A sequent S is deducible *without cuts* (in symbols, $\vdash^+ S$) if there exists an inference for S in $\mathrm{Def\,Ar}\,\omega 2$ such that the cut-rule (3.2 (*cut*)) does not occur in this inference. Now our main result could be formulated as

$$\vdash S \implies \vdash^+ S$$

for every sequent S.

3.5. Note than one can deduce in $\mathrm{Def\,Ar}\,\omega 2$ all usual laws of (classical) logic, the principle of arithmetical induction for all formulas of $\mathrm{Def\,Ar}\,\omega 2$ and the *full impredicative* comprehension axiom:

$$\tilde{\forall}\exists y \forall x (x \varepsilon y \equiv \varphi(x)),$$

where $\tilde{\forall}$ means bounding all parameters of the following formula by universal quantifiers. Besides that, $x \in \mathrm{Var}(0)$, $y \in \mathrm{Var}(2)$ and $\varphi(x)$ is an *arbitrary* formula such that y is not a parameter in $\varphi(x)$.

So the full classical second order arithmetic with the full impredicative comprehension axiom is imbedded in $\mathrm{Def\,Ar}\,\omega 2$.

Nevertheless, it is impossible to prove in the traditional Zermelo–Fraenkel system **ZFC** that every deducible formula is *true* in the natural interpretation:

$$\vdash \varphi \implies \models \varphi.$$

The difficulty arises in the rule $(\to \forall)$: if we have $\varphi(x \| t)$ for all *closed* expressions t, we cannot conclude that $\forall x \varphi(x)$. And what is more, it is not clear that our theory $\mathrm{Def\,Ar}\,\omega 2$ is *consistent*. Our cut-elimination result (3.4.1) just demonstrates that it is and, hence, our theory has some model.

4. A *constant semivaluation* is, by definition, a function V such that $\mathrm{dom} V \subseteq \mathrm{Cl\,Fm}$, $\mathrm{rng} V \subseteq \{0,1\}$ and the following conditions 1)–10) are fulfilled. Note that V is *not necessary* defined on the whole set $\mathrm{Cl\,Fm}$ so the denotation like $V(\varphi) = 1$ below means that $V(\varphi)$ *is defined and* $V(\varphi) = 1$.

1) $\varphi \in \mathrm{Cl\,At\,Fm}$, $V(\varphi) = 1 \implies \models \varphi$;
2) $\varphi \in \mathrm{Cl\,At\,Fm}$, $V(\varphi) = 0 \implies \models \neg\varphi$;
3) $V(\varphi \wedge \psi) = 1 \implies V(\varphi) = 1$ and $V(\psi) = 1$;
4) $V(\varphi \wedge \psi) = 0 \implies V(\varphi) = 0$ or $V(\psi) = 0$;
5) $V(\neg\varphi) = 1 \implies V(\varphi) = 0$;
6) $V(\neg\varphi) = 0 \implies V(\varphi) = 1$;
7) $x \in \mathrm{Var}(\tau)$, $V(\forall x\varphi) = 1 \implies V(\varphi(x\|t)) = 1$ for *all* $t \in \mathrm{Cl\,Exp}(\tau)$;
8) $x \in \mathrm{Var}(\tau)$, $V(\forall x\varphi) = 0 \implies$ there exists $t \in \mathrm{Cl\,Exp}(\tau)$, $V(\varphi(x\|t)) = 0$;
9) $V(t\varepsilon\{x \mid \varphi\}) = 1 \implies V(\varphi(x\|t)) = 1$;
10) $V(t\varepsilon\{x \mid \varphi\}) = 0 \implies V(\varphi(x\|t)) = 0$.

4.1. If some constant semivaluation V and a sequent $\Gamma \to \Delta$ is given then we define $V(\Gamma \to \Delta) = 1$ if there exists $\varphi \in \Delta$ such that $V(\varphi) = 1$, or,there exists $\psi \in \Gamma$ such that $V(\psi) = 0$. Further, we put $V(\Gamma \to \Delta) = 0$ if $V(\psi) = 1$ for all $\psi \in \Gamma$, and $V(\varphi) = 0$ for all $\varphi \in \Delta$. Particularly, $V(\to) = 0$. Note that in the definition $V(\Gamma \to \Delta) = 1$ we do not demand that V is defined on all members of Γ or Δ.

A constant semivaluation V is *total* if it is defined on the whole set $\mathrm{Cl\,Fm}$, i.e. if $\mathrm{dom}V = \mathrm{Cl\,Fm}$.

4.2. Our definition of a constant semivaluation only differs slightly from the definition of semivaluation in [10] and [11]. Namely, we use only *closed* formulas, so we are forced to change, for example, points 7) and 8) in the definition 4 and use there only closed expressions.

Theorem. *Let the sequent $\Gamma \to \Delta$ be not deducible in* $\mathrm{Def\,Ar}\,\omega 2$ *without cuts (i.e. it is not true that $\vdash^{+} \Gamma \to \Delta$). Then there exists a constant semivaluation V such that $V(\Gamma \to \Delta) = 0$.*

Proof. We construct the semivaluation V by systematic search of an inference without cuts for the sequent $\Gamma \to \Delta$. If the sequent $\Gamma \to \Delta$ is not deducible without cuts then there is some infinite path in this systematic searching tree. This path defines the semivaluation V. Namely, for all formulas on the left hand side of the infinite path we prescribe the value 1 in the semivaluation V and, correspondingly, for all formulas on the right hand side of our path we prescribe the value 0. A detailed construction for the systematic searching of the inference can be found in [10], [11]. For our theory this construction needs some obvious changes only.

5. Let us consider some constant semivaluation V. For every type τ we define the relation $g \approx t$ between some object g and expression $t \in \mathrm{Cl}\,\mathrm{Exp}(\tau)$.

$\tau = 0$. In this case $n \approx t$ if and only if $n \in \omega$ and the standard arithmetical value of t in the standard interpretation is just n.

$\tau = 1$. In this case $\varepsilon \approx \varphi$ if and only if ε is 1 or 0 (i.e. ε is a truth value) and $\varphi \in \mathrm{Cl}\,\mathrm{Fm}$ and

$$V(\varphi) = 1 \implies \varepsilon = 1, \quad V(\varphi) = 0 \implies \varepsilon = 0.$$

$\tau = 2$. In this case $g \approx t$ if and only if $g \subseteq \omega$, $t \in \mathrm{Cl}\,\mathrm{Exp}(2)$ and for every $n \in \omega$, $l \in \mathrm{Cl}\,\mathrm{Exp}(0)$ such that $n \approx l$, we have

$$V(l\varepsilon t) = 1 \implies n \in g, \quad V(l\varepsilon t) = 0 \implies n \notin g.$$

5.1. For every type τ we define a set – an *object domain* $B(\tau)$. Namely,

$$B(\tau) = \{g \mid (\exists t \in \mathrm{Cl}\,\mathrm{Exp}(\tau))(g \approx t)\}.$$

Now we define the set $\mathrm{Exp}^+(\tau)$ of expressions of type τ *with the constants from* B. The definition of $\mathrm{Exp}^+(\tau)$ goes quite parallel to the definition in Sect. 2.3 and only in point 1) of the new definition do we add a new inductive condition:

$$a \in B(\tau) \implies a \in \mathrm{Exp}^+(\tau).$$

One can imagine that an element $t^+ \in \mathrm{Exp}^+(\tau)$ is obtained from an element $t \in \mathrm{Exp}(\tau)$ by the *substitution* of some parameters of t by constants from B. Let us denote by $\mathrm{Cl}\,\mathrm{Exp}^+(\tau)$ the set of all *closed* elements of $\mathrm{Exp}^+(\tau)$, furthemore, $\mathrm{Fm}^+ = \mathrm{Exp}^+(1)$ and $\mathrm{Cl}\,\mathrm{Fm}^+ = \mathrm{Cl}\,\mathrm{Exp}^+(1)$.

5.2. For every $t \in \mathrm{Cl}\,\mathrm{Exp}^+(\tau)$ we define an object $W(t)$ by induction according to the definition of t (see the definition in 5.1):

1) $W0 = 0$;
2) $a \in B(\tau) \implies Wa = a$;
3) $W(f(t_1, \dots, t_n)) = \tilde{f}(Wt_1, \dots, Wt_n)$;
4) $W(P(t_1, \dots, t_n))$ is defined as 0 or 1 and $W(P(t_1, \dots, t_n)) = 1$ if and only if the statement $\tilde{P}(Wt_1, \dots, Wt_n)$ is *true*;
5) $W(\varphi \wedge \psi) = \min(W\varphi, W\psi)$;
6) $W(\neg\varphi) = 1 - W\varphi$;
7) $W(\forall x\varphi)$ is defined as 0 or 1 and $W(\forall x\varphi) = 1$ iff $W(\varphi(x\|a) = 1$ for all $a \in B(\tau)$ (here $x \in \mathrm{Var}(\tau)$, of course);
8) $W(\{x \mid \varphi\})$ is a set of natural numbers, $W(\{x \mid \varphi\}) \subseteq \omega$, namely, $n \in W(\{x \mid \varphi\}) \iff W(\varphi(x\|n)) = 1$;

9) $W(t\varepsilon T) = 1 \iff Wt \in WT$ and $W(t\varepsilon T) = 0 \iff Wt \notin WT$; Thus, Wt *is defined* for all $t \in \mathrm{Cl\,Exp}^+(\tau)$ for every $\tau = 0, 1, 2$ but, at this moment, we cannot assert $Wt \in B(\tau)$ (see 5.4 below).

5.3. Let t be an arbitrary expression, $x = (x_1, \dots, x_n)$ is a list of distinct variables such that all parameters of t are among the members of x. Let $x_i \in \mathrm{Var}(\tau_i)$. Let, further, $g = (g_1, \dots, g_n)$ be a list of objects, $g_i \in B(\tau_i)$, and $r = (r_1, \dots, r_n)$ a list of closed expressions, $r_i \in \mathrm{Cl\,Exp}(\tau_i)$ such that $g_i \approx r_i$. In this situation we shall say that $\theta = \langle x, g, r\rangle$ is a *valuation for the expression* t associated with the constant semivaluation V and we shall write $\theta \approx V$.

If we have some valuation $\theta = \langle x, g, r\rangle$ for the expression t then let us define $t'(\theta) = t(x\|r)$ and $t''(\theta) = t(x\|g)$. We abbreviate these denotations by $t' = t'(\theta)$, $t'' = t''(\theta)$ if it is clear what θ is. Thus, if $t \in \mathrm{Exp}(\tau)$ then $t' \in \mathrm{Cl\,Exp}(\tau)$ and $t'' \in \mathrm{Cl\,Exp}^+(\tau)$.

5.4. Theorem. *Let $\theta \approx V$ be a valuation for an expression t, $t \in \mathrm{Exp}(\tau)$. Then Wt'' is defined, $Wt'' \in B(\tau)$ and $Wt'' \approx t'$.*

Proof. It goes by induction on the construction of the term t. A similar construction (for a slightly different language and notations) could be found in [11], point 3.4.4.

Corollary. *If $\varphi \in \mathrm{Cl\,Fm}$ and $V(\varphi)$ is defined then $W\varphi = V\varphi$.*

So W is an *extension* of V on the whole set $\mathrm{Cl\,Fm}$.

6.1. Let V be a constant semivaluation which *is not total*, i.e. $\mathrm{dom}V$ is a proper subset of $\mathrm{Cl\,Fm}$. Then $B(2)$ is the set of *all* subsets of ω, $B(2) = \mathcal{P}\omega$. Hence, it is easy to see that in this case $W\varphi = 1 \iff \models \varphi$ for all $\varphi \in \mathrm{Cl\,Fm}^+$ and, moreover, $W(\{x \mid \varphi\}) = \{n \in \omega \mid \models \varphi(x\|n)\}$ for all $\{x \mid \varphi\} \in \mathrm{Cl\,Exp}^+(2)$.

Indeed, let φ_0 be a closed formula such that $V\varphi_0$ is not defined. Then $V(e\varepsilon\{x \mid \varphi_0\})$ is not defined for every $e \in \mathrm{Cl\,Exp}(0)$ (in view of 4, 9)–10)). Hence (in view of 5) $g \approx \{x \mid \varphi_0\}$ for every $g \subseteq \omega$. So $g \in B(2)$ for every $g \subseteq \omega$. But, in this situation, the definition of W in 5.2 coincides with the definition of classical truth in the natural interpretation.

6.2. Let us define now the domains $C(\tau)$ for any type $\tau = 0, 1, 2$ in the following way: $C(0) = B(0) = \omega$, $C(1) = B(1) = \{0, 1\}$, $C(2) = \{Wt \mid t \in \mathrm{Cl\,Exp}(2)\}$. It is evident that $C(\tau) \subseteq B(\tau)$.

For every closed expression t which contains constants from C only, we define an object Ut by induction on the construction of t. The definition goes parallel to 5.2 but, of course, we define now not Wt but Ut. The difference arises in the point 7) of 5.2. Namely, now we assert: $U(\forall x\varphi) = 1$ iff we have $U(\varphi(x\|a)) = 1$ for all $a \in C(\tau)$. So we appeal only to $a \in C(\tau)$ and do not use the whole domain $B(\tau)$.

6.3. Lemma. *Let V be a constant semivaluation, such that V is not total. Let us assume that the axiom of constructibility of Gödel* $\mathrm{V} = \mathrm{L}$ *takes place (see, for example, [7], [1]). Let $t \in \mathrm{Cl\,Exp}^{+}(\tau)$ and let all constants of t be from C. Then we have $Wt = Ut$.*

Proof. It goes by induction on the construction of t. The only nontrivial case arises when $t = \forall x\varphi$, $x \in \mathrm{Var}(2)$ and we need to prove $U(\forall x\varphi) = 1 \implies W(\forall x\varphi) = 1$.

Indeed, let us assume, that $U(\forall x\varphi) = 1$ and $W(\forall x\varphi) = 0$. Let $\varphi = \varphi(x, g_1, \dots, g_k)$ where all parameters and the domain of the two-type constants in φ are explicitely denoted. In view of the condition, we have $g_i \in C(2)$ so there are $t_i \in \mathrm{Cl\,Exp}(2)$ such that $g_i = Wt_i$. Let us denote $\psi(x) = \varphi(x, t_1, \dots, t_k)$. We have $\psi(x) \in \mathrm{Fm}$ and (see 5.4) $W(\varphi(g)) = W(\psi(g))$ for all $g \in B(2)$.

It follows from $\mathrm{V} = \mathrm{L}$ that there is an analytical formula $\gamma(u, v)$ with two parameters $u, v \in \mathrm{Var}(2)$ which gives the well-ordering of the set $\mathcal{P}\omega$ of all subsets of ω in the sense of the strict ordering $<$ in the universe L (see [1]).

Let us put $\xi(x) = (\neg\psi(x) \wedge \forall v(\gamma(v, x) \supset \psi(v))$. In view of our suppositions $W(\forall x\varphi(x)) = 0$ and 6.1 there exists $g \subseteq \omega$ such that $\models \neg\varphi(g)$. Then we have $\models \neg\psi(g)$. According to the definition of $\xi(x)$ there exists $g_0 \subseteq \omega$ such that $\models \xi(g_0)$ and, moreover, $\models \xi(g_1) \wedge \xi(g_2) \implies g_1 = g_2$. Let us define a closed expression $t_0 = \{y \mid \exists x(y \varepsilon x \wedge \xi(x))\}$ where $y \in \mathrm{Var}(0)$ and $x \in \mathrm{Var}(2)$. In view of the definition of $\xi(x)$ and 6.1 we get $Wt_0 = g_0$. Hence, $g_0 \in C(2)$, $\models \xi(g_0)$. Then $\models \neg\psi(g_0)$ and $\models \neg\varphi(g_0)$. So, by (6.1), $W(\varphi(g_0)) = 0$. Because of $g_0 \in C(2)$ (and the inductive supposition) we have $W(\varphi(g_0)) = U(\varphi(g_0)) = 0$. But this contradicts $U(\forall x\varphi(x)) = 1$.

6.4. Lemma. *Let us assume* $\mathrm{V} = \mathrm{L}$. *Let V be an arbitrary constant semivaluation. Then there exists a total constant semivaluation H which extends V, i.e. if $\varphi \in \mathrm{Cl\,Fm}$ and $V\varphi$ is defined then $V\varphi = H\varphi$.*

Proof. If V is total then simply put $H = V$. Let us suppose that V is not total. Let us consider the function U defined in 6.2 and let H be U – limited on the set $\mathrm{Cl\,Fm}$. We state that H is the wanted semivaluation. In view of 6.3 H coincides with W on $\mathrm{Cl\,Fm}$ so it is the prolongation of V as it follows from Corollary 5.4. We need only to check that H is really a semivaluation, i.e. that the conditions of Sect. 4 are fulfilled for H. The only nontrivial points in this checking are 7) and 8) when $x \in \mathrm{Var}(2)$.

Let $H(\forall x\varphi) = 1$ and $t \in \mathrm{Cl\,Exp}(2)$. We show $H(\varphi(t)) = 1$. Indeed, $H(\varphi(x\|t)) = W(\varphi(x\|t)) = W(\varphi(x\|Wt)) = 1$ in view of $W(\forall x\varphi(x)) = 1$ and the definition of W.

Conversely, let $H(\forall x\varphi) = 0$, then $W(\forall x\varphi) = U(\forall x\varphi) = 0$. Hence, there exists $g \in C(2)$, $U(\varphi(x\|g)) = 0$. By the definition of $C(2)$, there exists an expression $t \in \mathrm{Cl\,Exp}(2)$ such that $g = Wt = Ut$. But $U(\varphi(x\|g)) = U(\varphi(x\|Ut)) = U(\varphi(x\|t)) = W(\varphi(x\|t)) = H(\varphi(x\|t)) = 0$.

7. Theorem. *If H is a total constant semivaluation and the sequent S is deducible in* $\mathrm{Def\,Ar}\,\omega 2$, *i.e.* $\vdash S$, *then* $H(S) = 1$.

Proof. By a straightforward induction on the construction of the inference for the S in $\mathrm{Def\,Ar}\,\omega 2$.

7.1. Lemma. *Let us assume the axiom* $\mathrm{V} = \mathrm{L}$. *Then for every sequent S we have*

$$\vdash S \implies \vdash^{+} S.$$

Proof. Let us assume that $\vdash S$ and nevertheless, not $\vdash^{+} S$. According to 4.2 then there exists a constant semivaluation V such that $V(S) = 0$. According to 6.4 then there exists some *total* constant semivaluation H such that $H(S) = 0$. This contradicts Theorem 7.

8. Theorem. *If a sequent S is deducible in* $\mathrm{Def\,Ar}\,\omega 2$ *then it is deducible without cuts also.*

In other words, $\vdash S \implies \vdash^{+} S$ for every sequent S, and now this fact *does not depend* on the hypothesis $\mathrm{V} = \mathrm{L}$.

Proof. According to 7.1 this fact is true assuming $\mathrm{V} = \mathrm{L}$. But the deducibility of a sequent in $\mathrm{Def\,Ar}\,\omega 2$ is an *absolute* fact. Hence, we have a theorem. In a more detailed way one can argue as follows. If $\vdash S$ then there exists an inference D for S in $\mathrm{Def\,Ar}\,\omega 2$. Note that D is a constructible object (see 3.4) and so D is an element of L and D is an inference for S in the universe L too. Hence, in view of 7.1, in L there exists an inference D' for S containing no cuts. But in view of absoluteness, D' is an inference for S from an outer point of view too.

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