# The Pexider equation on $n$-semigroups and $n$-groups

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In this paper we describe a certain family of solutions of the Pexider equation on $n$-semigroups $S(\ )$ and $T[\ ]$ possessing invertible elements. If $T[\ ]$ is an $n$-group we give a general solution of the Pexider equation. Moreover, the above mentioned results are used to describe a certain family of solutions of another functional equation of the form:

$$F_1(F_2(\dots(F_n(x, s_n), \dots), s_2), s_1) = F_{n+1}(x, (s_1, s_2, \dots, s_n)).$$

The Pexider equation on $n$-semigroups ($n$-groupoids) is a straight-forward generalization of the Pexider equation on various algebraic structures with binary operations (cf. A. KRAPEŽ and M. A. TAYLOR [5]). Algebraically, the Pexider equation is related to the notions of homotopy and of isotopy.

We begin with some definitions.

A Pexider equation on binary groupoids $S$ and $T$ is a functional equation

$$\alpha_3(s_1 s_2) = \alpha_1(s_1)\alpha_2(s_2) \tag{1}$$

for arbitrary $s_1, s_2 \in S$, where $\alpha_1, \alpha_2, \alpha_3 : S \to T$ are unknown functions.

A triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ being a solution of equation (1) is called a homotopy from the groupoid $S$ into the groupoid $T$. If $\alpha_1, \alpha_2, \alpha_3$ are bijections, then the homotopy $(\alpha_1, \alpha_2, \alpha_3)$ is called an isotopy from the groupoid $S$ onto the groupoid $T$.

Let $a_1, a_2 \in T$ be arbitrary fixed elements of the groupoid $T$. Functions $L_{a_1}(t) = a_1 t$ and $R_{a_2}(t) = t a_2$ for $t \in T$ we call a left translation and a right translation on the groupoid $T$, respectively.

An element $t \in T$ is called an invertible element in the semigroup $T$ if $tT = Tt = T$. The symbol $R(T)$ will denote the set of all invertible elements in the semigroup $T$. It is known that if $R(T) \neq \emptyset$, then $R(T)$ is a subgroup of the semigroup $T$ and the identity of the subgroup $R(T)$ is an identity of the semigroup $T$ and so $T$ is a monoid (cf. [6]).

An element $t \in T$ is called a right-cancellative element in the semigroup $T$ if

$$\forall t_1, t_2 \in T [t_1 t = t_2 t \Rightarrow t_1 = t_2].$$

**Theorem 1.** *Let $S$ be a groupoid with identity, and let $T$ be a semigroup. If a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ is a solution of equation* (1) *on $S$ and $T$ such that $\alpha_1(1)$ and $\alpha_2(1)$*

*are elements from* $R(T)$, *then there exists a homomorphism* $\varphi\colon S\to T$ *and there exist elements* $a_1, a_2\in R(T)$ *such that*

$$\alpha_1 = L_{a_1}\varphi, \quad \alpha_2 = R_{a_2}\varphi, \quad \alpha_3 = L_{a_1}R_{a_2}\varphi. \tag{2}$$

*If* $\varphi\colon S\to T$ *is an arbitrary homomorphism and* $a_1, a_2\in T$ *are arbitrary elements, then a triple of functions* $(\alpha_1, \alpha_2, \alpha_3)$ *of form* (2) *is a solution of equation* (1).

PROOF. Denote $a_1:=\alpha_1(1)$ and $a_2:=\alpha_2(1)$. Let us put $\varphi(s):=a_1^{-1}\alpha_1(s)$ for every $s\in S$. Notice that $\alpha_3(s)=\alpha_1(1)\alpha_2(s)=a_1\alpha_2(s)$ and $\alpha_3(s)=\alpha_1(s)\alpha_2(1)=\alpha_1(s)a_2$ for $s\in S$. Hence, $\alpha_3(s)=a_1\alpha_2(s)=a_1a_1^{-1}(a_1\alpha_2(s))=a_1a_1^{-1}(\alpha_1(s)a_2)=a_1(a_1^{-1}\alpha_1(s))a_2=a_1\varphi(s)a_2$ for $s\in S$. Thus, $\alpha_1(s)=a_1\varphi(s)$, $\alpha_2(s)=\varphi(s)a_2$, $\alpha_3(s)=a_1\varphi(s)a_2$ for $s\in S$. It is easy to check that $\varphi$ is a homomorphism.
The proof of the second part of this theorem is obvious.

*Remark 1.* Theorem 1 remains true if instead of the assumption that $\alpha_2(1)\in R(T)$, the element $\alpha_2(1)$ is supposed to be a right-cancellative element in the semigroup $T$.

**Corollary 1.** *Let S be a groupoid with identity, and let T be a group. A triple of functions* $(\alpha_1, \alpha_2, \alpha_3)$ *of form* (2) *is a general solution of equation* (1) *if* $\varphi\colon S\to T$ *is an arbitrary homomorphism and* $a_1, a_2\in T$ *are arbitrary elements.*

**Corollary 2.** *Let* $(\alpha_1, \alpha_2, \alpha_3)$ *be a homotopy from a groupoid S with identity into a group T. If* $\alpha_i$ *is a bijection for a certain* $i\in\{1, 2, 3\}$, *then* $(\alpha_1, \alpha_2, \alpha_3)$ *is an isotopy from S onto T.*

Notice that formula (2) does not yield the general solution of equation (1). To show this, we consider the following

*Example.* Let $\{1, s\}$ be a group endowed with the operation:

| | 1 | $s$ |
|---|---|---|
| 1 | 1 | $s$ |
| $s$ | $s$ | 1 |

Let $T=\{1, t, 0\}$ be a monoid with the operation:

| | 1 | $t$ | 0 |
|---|---|---|---|
| 1 | 1 | $t$ | 0 |
| $t$ | $t$ | 1 | 0 |
| 0 | 0 | 0 | 0 |

Consider all the functions from the set $T^S$:

$$\begin{array}{lll} \alpha_1(1)=1 & \alpha_4(1)=t & \alpha_7(1)=0 \\ \alpha_1(s)=1 & \alpha_4(s)=1 & \alpha_7(s)=1 \\ \alpha_2(1)=1 & \alpha_5(1)=t & \alpha_8(1)=0 \\ \alpha_2(s)=t & \alpha_5(s)=t & \alpha_8(s)=t \\ \alpha_3(1)=1 & \alpha_6(1)=t & \alpha_9(1)=0 \\ \alpha_3(s)=0 & \alpha_6(s)=0 & \alpha_9(s)=0. \end{array}$$

The functions $\alpha_1, \alpha_2, \alpha_9$ are the only homomorphisms from $S$ into $T$. The triple of functions $(\alpha_3, \alpha_9, \alpha_9)$ satisfies equation (1). It is easy to check that the function $\alpha_3$ cannot be written in form (2) for any homomorphisms $\alpha_1, \alpha_2$, and $\alpha_9$.

Consider on groupoids $S$ and $T$ the following functional equation

$$\alpha_3(\mu(s_1)\lambda(s_2)) = \alpha_1(s_1)\alpha_2(s_2) \tag{3}$$

for arbitrary $s_1, s_2 \in S$, where $\mu, \lambda\colon S \to S$ are given bijections and $\alpha_1, \alpha_2, \alpha_3\colon S \to T$ are unknown functions.

Notice that equation (3) is equivalent to the equation

$$\alpha_3(s_1 s_2) = (\alpha_1\mu^{-1})(s_1)(\alpha_2\lambda^{-1})(s_2) \tag{4}$$

for arbitrary $s_1, s_2 \in S$.

The following lemma is an immediate consequence of equation (4) and Theorem 1.

**Lemma 1.** *Let $S$ be a groupoid with identity, and let $T$ be a semigroup. If a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ is a solution of equation (3) on $S$ and $T$ such that $\alpha_1\mu^{-1}(1)$ and $\alpha_2\lambda^{-1}(1)$ are elements from $R(T)$, then there exists a homomorphism $\varphi\colon S \to T$ and there exist elements $a_1, a_2 \in R(T)$ such that*

$$\alpha_1 = L_{a_1}\varphi\mu, \quad \alpha_2 = R_{a_2}\varphi\lambda, \quad \alpha_3 = L_{a_1}R_{a_2}\varphi. \tag{5}$$

*If the function $\varphi\colon S \to T$ is an arbitrary homomorphism and $a_1, a_2 \in T$ are arbitrary elements, then a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ of form (5) is a solution of equation (3).*

**Corollary 3.** *Let $S$ be a groupoid with identity, and let $T$ be a group. A triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ of form (5) is a general solution of equation (3) if $\varphi\colon S \to T$ is an arbitrary homomorphism and $a_1, a_2 \in T$ are arbitrary elements.*

Let $X$ be an arbitrary non-empty set, and let $S$ be an arbitrary groupoid. The set $X^X$ endowed with the composition of functions is a monoid.

Consider the following functional equation

$$F_1(F_2(x, s_2), s_1) = F_3(x, s_1 s_2) \tag{6}$$

for arbitrary $x \in X$ and $s_1, s_2 \in S$, where $F_i\colon X \times S \to X$ for $i=1, 2, 3$ are unknown functions.

**Lemma 2.** *Let $X$ be an arbitrary non-empty set, and let $S$ be an arbitrary groupoid. A triple of functions $(F_1, F_2, F_2)$ is a solution of equation* (6) *if and only if there exists a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ being a solution of equation* (1) *on the groupoid $S$ and the monoid $X^X$, and $\alpha_i(s)(x)=F_i(x, s)$ $( =1, 2, 3)$ for arbitrary $x\in X$ and $s\in S$.*

We omit the simple proof of this lemma.

We recall the following known facts.

A function $f\in X^X$ is an invertible element in the monoid $X^X$ iff the function $f$ is a bijection on the set $X$. A function $f\in X^X$ is a right-cancellative element in the monoid $X^X$ iff the function $f$ maps the set $X$ onto the set $X$.

**Theorem 2.** *Let $X$ be an arbitrary non-empty set, and let $S$ be an arbitrary groupoid with identity. If a triple of functions $(F_1, F_2, F_3)$ is a solution of equation* (6) *on $S$ and the functions $F_1(x, 1)$ and $F_2(x, 1)$ are bijections on the set $X$, then there exists a homomorphism and there exist functions $f_1, f_2\in X^X$ such that*

$$F_1(x, s) = L_{f_1}\varphi(s)(x), \quad F_2(x, s) = R_{f_2}\varphi(s)(x),$$
$$F_3(x, s) = L_{f_1}R_{f_2}\varphi(s)(x) \tag{7}$$

*for arbitrary $x\in X$ and $s\in S$. If $\varphi: S\to X^X$ is an arbitrary homomorphism and $f_1, f_2\in X^X$ are arbitrary functions, then a triple of functions $(F_1, F_2, F_3)$ of form* (7) *is a solution of equation* (6).

Proof. Since the triple of functions $(F_1, F_2, F_3)$ is a solution of equation (6), so according to Lemma 2, there exists a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ being a solution of equation (1) on the groupoid $S$ and the monoid $X^X$ such that

$$F_i(x, s) = \alpha_i(s)(x), \quad i = 1, 2, 3 \tag{8}$$

for arbitrary $x\in X$ and $s\in S$.

Notice that $\alpha_i(1)(x)=F_i(x, 1)$ are bijections for $i=1, 2$. Thus in virtue of Theorem 1, there exist functions $f_1, f_2\in X^X$ and there exists a homomorphism $\varphi: S\to X^X$ such that $\alpha_1=L_{f_1}\varphi$, $\alpha_2=R_{f_2}\varphi$, $\alpha_3=L_{f_1}R_{f_2}\varphi$. Hence and by (8) we get the functions $F_i$ $(i=1, 2, 3)$ in form (7).

The proof of the first part of this theorem is completed. The easy proof of the second part is omitted.

*Remark 2.* Theorem 2 remains true if instead of the assumption that $F_2(x, 1)$ is a bijection, the function $F_2(x, 1)$ is supposed to map the set $X$ onto the set $X$.

In the sequel we shall be concerned with the Pexider equation on $n$-semigroups. Definitions 2—5 and most of the notations are used according to papers [1] and [2].

Let $S(\circ)$ be a binary semigroup. The symbol $s_m^n$ denotes either the sequence $s_m, s_{m+1}, \dots, s_n$ or the element $s_m\circ s_{m+1}\circ\dots\circ s_n$ for arbitrary $s_m, s_{m+1}, \dots, s_n\in S$ if $m\leqq n$. The meaning of this symbol will uniquely result from the context. If $m>n$, then $s_m^n$ is an empty symbol. The $n$-termed sequence $s, s, \dots, s$ is denoted by $s^n$ and $x^0$ is an empty symbol.

*Definition 1.* A non-empty set $S$ endowed with an $n$-ary operation $(n\geqq 2)$ is called an $n$-groupoid. $n$-groupoids will be written as $S(\ )$ or $S[\ ]$.

*Definition 2.* An $n$-groupoid $S(\ )$ is said to be an $n$-quasigroup if for an arbitrary $i\in\{1, \dots, n\}$ the equation

$$(s_1^{i-1}, x, s_{i+1}^n) = s$$

has the unique solution for arbitrary $s_1, \dots, s_n \in S$.

*Definition 3.* An $n$-groupoid $S(\ )$ is said to be an $n$-semigroup if for arbitrary $i,j\in\{1, \dots, n\}$ the following equality is satisfied

$$\left(s_1^{i-1}, (s_i^{i+n-1}), s_{i+n}^{2n-1}\right) = \left(s_1^{j-1}, (s_j^{j+n-1}), s_{j+n}^{2n-1}\right)$$

for arbitrary $s_1, \dots, s_{2n-1} \in S$.

An element $e\in S$ is said to be an identity of the $n$-groupoid $S(\ )$ if $(e^{i-1}, s, e^{n-i}) = s$ for every $s\in S$ and for every $i\in\{1, \dots, n\}$.

If there exists an identity of an $n$-semigroup $S(\ )$, then $S(\ )$ is called an $n$-monoid. The identity of a binary monoid will be denoted by 1.

*Definition 4.* If an $n$-quasigroup $S(\ )$ is an $n$-semigroup, then $S(\ )$ is called an $n$-group.

Let $S_i$ $(i=1, \dots, n)$ be non-empty subsets of an $n$-groupoid $S(\ )$. We define the following set:

$$(S_1, S_2, \dots, S_n) := \{(s_1, s_2, \dots, s_n)\in S\colon\ s_i\in S \text{ for } i = 1, \dots, n\}.$$

*Definition 5.* An element $s$ of an $n$-groupoid $S(\ )$ is called $k$-invertible if $(S^{k-1}, s, S^{n-k}) = S$. If an element $s\in S$ is 1-invertible and $n$-invertible in an $n$-groupoid $S(\ )$, then $s$ is called a bilateral invertible element. If for an arbitrary $k\in\{1,\dots, n\}$ an element $s\in S$ is $k$-invertible in an $n$-groupoid $S(\ )$, then $s$ is called an invertible element. The set of all invertible elements in an $n$-groupoid $S(\ )$ i s denoted by $R(S)$.

**Theorem 3.** (GLUSKIN [2]) *Every bilateral invertible element of an $n$-semigroup is an invertible element.*

**Theorem 4.** (GLUSKIN [2]) *Let $S(\ )$ be an arbitrary $n$-semigroup for which $R(S)\neq \emptyset$.*

On the set $S$ one can define the binary operation $\circ$ such that:

1) $(s_1^n) = s_1\circ\lambda(s_2)\circ\lambda^2(s_3)\circ\dots\circ\lambda^{n-1}(s_n)\circ a$ for arbitrary $s_1, s_2, s_3, \dots, s_n\in S$;
2) $S(\circ)$ is a binary monoid with the same set $R(S)$ of invertible elements;
3) $\lambda$ is an automorphism of the monoid $S(\circ)$;
4) $a\in R(S)$ and $\lambda(a)=a$;
5) $\lambda^{n-1}(x) = a\circ x\circ a^{-1}$ *for every* $x\in S$.

The binary monoid $S(\circ)$ will be called a monoid associated with the $n$-semigroup $S(\ )$ and it will be denoted by $(S, \circ, \lambda, a)$. A monoid associated with an $n$-group is a binary group. A monoid associated with a binary monoid $S(\ \cdot\ )$ is the same binary monoid $S(\ \cdot\ )$.

**Corollary 4** (GLUSKIN [2]) *Let $S(\ )$ be an $n$-monoid. On the set $S$ one can define the binary operation $\circ$ such that:*

1) $(s_1^n) = s_1\circ s_2\circ\dots\circ s_n$ *for arbitrary* $s_1, s_2, \dots, s_n\in S$;
2) $S(\circ)$ *is a binary monoid.*

For simplicity, the monoid associated with the $n$-monoid $S(\ )$ will be denoted by $S(\circ)$ or $\bar{S}$.

*Definition 6.* A Pexider equation on $n$-groupoids $S(\ )$ and $T[\ ]$ is the functional equation

$$\alpha_{n+1}((s_1, s_2, \ldots, s_n)) = [\alpha_1(s_1), \alpha_2(s_2), \ldots, \alpha_n(s_n)] \tag{9}$$

for arbitrary $s_1, s_2, \ldots, s_n \in S$, where $\alpha_i\colon S \to T$ (for $i=1, \ldots, n+1$) are unknown functions.

A sequence of functions $(\alpha_1, \ldots, \alpha_{n+1})$ being a solution of equation (9) is called a homotopy from an $n$-groupoid $S(\ )$ into an $n$-groupoid $T[\ ]$.

If $\alpha_1, \ldots, \alpha_{n+1}$ are bijections, then a homotopy $(\alpha_1, \ldots, \alpha_{n+1})$ is called an isotopy from an $n$-groupoid $S(\ )$ onto an $n$-groupoid $T[\ ]$.

If a sequence $(\alpha, \alpha, \ldots, \alpha)$ is a homotopy (an isotopy) from an $n$-groupoid $S(\ )$ into (onto) an $n$-groupoid $T[\ ]$, then the function $\alpha$ is called a homomorphism (an isomorphism) from an $n$-groupoid $S(\ )$ into (onto) an $n$-groupoid $T[\ ]$.

**Theorem 5.** *If two n-monoids are isotopic, then they are isomorphic.*

PROOF. Let $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ be an isotopy from an $n$-monoid $S(\ )$ onto an $n$-monoid $T[\ ]$, and let $S(\circ)$ and $T[\cdot]$ be monoids associated with $S(\ )$ and $T[\ ]$ respectively. By the definition of the isotopy we have $\alpha_{n+1}((s_1, s_2, \ldots s_n)) = [\alpha_1(s_1), \alpha_2(s_2), \ldots, \alpha_n(s_n)]$ for arbitrary $s_1, s_2, \ldots, s_n \in S$. In virtue of Corollary 4 we obtain (10) $\alpha_{n+1}(s_1 \circ s_2 \circ \ldots \circ s_n) = \alpha_1(s_1) \cdot \alpha_2(s_2) \cdot \ldots \cdot \alpha_n(s_n)$ for arbitrary $s_1, s_2, \ldots, s_n \in S$. Let us put $a_i := \alpha_i(1)$ for $i=1, \ldots, n$. By (10) we have $\alpha_{n+1}(s_1 \circ s_2) = \alpha_1(s_1) \cdot (\alpha_2(s_2) \cdot a_3^n)$ for arbitrary $s_1, s_2 \in S$. Put $\beta(s_2) := \alpha_2(s_2) \cdot a_3^n$ for every $s_2 \in S$, and so $\beta = R_{a_3^n} \alpha_2$. Besides, $\alpha_{n+1}(s_2) = a_1 \cdot \alpha_2(s_2) \cdot a_3^n$ for every $s_2 \in S$, i.e. $\alpha_{n+1} = L_{a_1} R_{a_3^n} \alpha_2$. Since $\alpha_2$ and $\alpha_{n+1}$ are bijections and $L_{a_1} R_{a_3^n} = R_{a_3^n} L_{a_1}$ so $L_{a_1}$ and $R_{a_3^n}$ are bijections. Thus, the function $\beta$ is a bijection. Whence $\alpha_{n+1}(s_1 \circ s_2) = \alpha_1(s_1) \cdot \beta(s_2)$ for arbitrary $s_1, s_2 \in S$, and so the monoids $S(\circ)$ and $T[\cdot]$ are isotopic. It is the well known fact that isotopic binary monoids are isomorphic (cf. [4]). Thus there exists an isomorphism $\varphi\colon S \to T$ of the monoids $S(\circ)$ and $T[\cdot]$. It is easy to see that $\varphi$ is also an isomorphism of the $n$-monoids $S(\ )$ and $T[\ ]$.

*Remark 3.* Let us notice that two isotopic $n$-groups $(n > 2)$ need not be isomorphic (a suitable example can be found in Belousov [1]).

It is easy to prove the following two propositions.

**Proposition 1.** *Every n-groupoid isotopic to an n-quasigroup is an n-quasigroup.*

**Proposition 2.** *Every n-semigroup isotopic to an n-quasigroup is an n-group.*

An $n$-groupoid isotopic to an $n$-group need not be an $n$-group (a suitable example can be found in the paper [4], p. 101).

**Theorem 6.** *Let $S(\ )$ and $T[\ ]$ be n-semigroups for which $R(S) \neq \emptyset$ and $R(T) \neq \emptyset$. Let $(S, \circ, \lambda, a)$ and $(T, \cdot, \mu, b)$ be monoids associated with $S(\ )$ and $T[\ ]$, respectively. If a sequence of functions $(\alpha_1, \ldots, \alpha_{n+1})$ is a solution of equation (9) on the n-semigroups $S(\ )$ and $T[\ ]$, and $\alpha_i(1)$ (for $i=1, \ldots, n-1$), $\alpha_n(a^{-1})$ are elements from*

*$R(T)$, then there exists a homomorphism $\varphi: S\to T$ of monoids $(s, \circ, \lambda, a)$ and $(T, \cdot, \mu, b)$, and there exist elements $a_1, \dots, a_n \in R(T)$ such that*

$$(11)\qquad \begin{cases} \alpha_1 = L_{a_1}\varphi, \\ \alpha_k = \mu^{1-k} L^{-1}_{a_2^{k-1}} R_{a_2^k}\varphi\lambda^{k-1} \quad \text{for} \quad k = 2, \dots, n-1, \\ \alpha_n = \mu^{1-n} L^{-1}_{a_2^{n-1}} R_{a_2^n}\varphi R_a \lambda^{n-1}, \\ \alpha_{n+1} = L_{a_1} R_{a_2^n\cdot b}\varphi. \end{cases}$$

*If $\varphi: S\to T$ is an arbitrary homomorphism of monoids $(S, \circ, \lambda, a)$ and $(T, \cdot, \mu, b)$ and $a_2, \dots, a_{n-1}\in R(T)$ while $a_1, a_n$ are arbitrary elements from $T$, then a sequence of functions of form* (11) *is a solution of equation* (9) *on n-semigroups $A(\ )$ and $T[\ ]$.*

PROOF. *If* $n=2$ then this theorem is an immediate consequence of Theorem 1. Suppose that $n\geqq 3$. It follows from the assumptions of this theorem that $\alpha_{n+1}((s_1, s_2, \dots, s_n))=[\alpha_1(s_1), \alpha_2(s_2), \dots, \alpha_n(s_n)]$ for arbitrary $s_1, s_2, \dots, s_n\in S$. In virtue of Theorem 4 we have $\alpha_{n+1}(s_1\circ\lambda(s_2)\circ\dots\circ\lambda^{n-1}(s_n)\circ a)=\alpha_1(s_1)\cdot\mu\alpha_2(s_2)\cdot\dots\cdot\mu^{n-1}\alpha_n(s_n)\cdot b$. Put $a_i:=\mu^{i-1}\alpha_i(1)$ (for $i=1, \dots, n-1$) and $a_n:=\mu^{n-1}\alpha_n(a^{-1})$. Whence $\alpha_{n+1}(s_1\circ\lambda(s_2))=\alpha_1(s_1)\cdot(\mu\alpha_2(s_2)\cdot a_3^n\cdot b)$ for arbitrary $s_1, s_2\in S$. Putting $\beta_2:=R_{a_3^n\cdot b}\mu\alpha_2$ we have $\alpha_{n+1}(s_1\circ\lambda(s_2))=\alpha_1(s_1)\cdot\beta_2(s_2)$ for arbitrary $s_1, s_2\in S$. Since $\alpha_1(1)=a_1$ and $\beta_2\lambda^{-1}(1)=\beta_2(1)=\mu\alpha_2(1)\cdot a_3^n\cdot b=a_2^n\cdot b$ are elements from $R(T)$, then according to Lemma 1 there exists a homomorphism $\varphi: S\to T$ of monoids $(S, \circ, \lambda, a)$ and $(T, \cdot, \mu, b)$ such that $\alpha_1=L_{a_1}\varphi$, $\beta_2=R_{a_2^n\cdot b}\varphi\lambda$, $\alpha_{n+1}=L_{a_1}R_{a_2^n\cdot b}\varphi$. Hence, $R_{a_3^n\cdot b}\mu\alpha_2=R_{a_2^n\cdot b}\varphi\lambda$, and so $\alpha_2=\mu^{-1}R_{a_2}\varphi\lambda$. Thus, $\alpha_1=L_{a_1}\varphi$, $\alpha_2=\mu^{-1}R_{a_2}\varphi\lambda$, $\alpha_{n+1}=L_{a_1}R_{a_2^n\cdot b}\varphi$.

Let us assume that $3\leqq k\leqq n-1$. Putting $s_1=1, \dots, s_{k-2}=1$, $s_{k+1}=1, \dots, s_{n-1}=1, s_n=a^{-1}$ we have $\alpha_{n+1}(\lambda^{k-2}(s_{k-1})\circ\lambda^{k-1}(s_k))=(a_1^{k-2}\cdot\mu^{k-2}\alpha_{k-1}(s_{k-1}))\cdot(\mu^{k-1}\alpha_k(s_k)\cdot a_{k+1}^n\cdot b)$ for arbitrary $s_{k-1}, s_k\in S$. Let us put $\beta_{k-1}:=L_{a_1^{k-2}}\mu^{k-2}\alpha_{k-1}$, $\beta_k:=R_{a_{k+1}^n\cdot b}\mu^{k-1}\alpha_k$ and so $\alpha_{n+1}(\lambda^{k-2}(s_{k-1})\circ\lambda^{k-1}(s_k))=\beta_{k-1}(s_{k-1})\cdot\beta_k(s_k)$ for arbitrary $s_{k-1}, s_k\in S$. Let us notice that $\beta_{k-1}\lambda^{2-k}(1)=\beta_{k-1}(1)=a_1^{k-2}\cdot\mu^{k-2}\alpha_{k-1}(1)=a_1^{k-1}$ and $\beta_k\lambda^{1-k}(1)=\beta_k(1)=\mu^{k-1}\alpha_k(1)\cdot a_{k+1}^n\cdot b=a_k^n\cdot b$ are elements from $R(T)$. It follows from Lemma 1 that there exists a homomorphism $\psi: S\to T$ of monoids $(S, \circ, \lambda, a)$ and $(T, \cdot, \mu, b)$ such that $\beta_{k-1}=L_{a_1^{k-1}}\psi\lambda^{k-2}$, $\beta_k=R_{a_k^n\cdot b}\psi\lambda^{k-1}$, $\alpha_{n+1}=L_{a_1^{k-1}}R_{a_k^n\cdot b}\psi$. Thus, we obtain $L_{a_1^{k-1}}R_{a_k^n\cdot b}\psi=L_{a_1}R_{a_2^n\cdot b}\varphi$, whence $L_{a_2^{k-1}}\psi=R_{a_2^{k-1}}\varphi$, and so $\psi=L^{-1}_{a_2^{k-1}}R_{a_2^{k-1}}\varphi$. Since $R_{a_{k+1}^n\cdot b}\mu^{k-1}\alpha_k=R_{a_k^n\cdot b}\psi\lambda^{k-1}$, then $\alpha_k=\mu^{1-k}R_{a_k}\psi\lambda^{k-1}=\mu^{1-k}R_{a_k}L^{-1}_{a_2^{k-1}}R_{a_2^{k-1}}\varphi\lambda^{k-1}=\mu^{1-k}L^{-1}_{a_2^{k-1}}R_{a_2^k}\varphi\lambda^{k-1}$ for $k=3, \dots, n-1$. Let us notice that the function $\alpha_2$ can also be written in the above form.

To determine the function $\alpha_n$ put $s_1=1, \dots, s_{n-2}=1$ and consider the following equation $\alpha_{n+2}(\lambda^{n-2}(s_{n-1})\circ\lambda^{n-1}(s_n)\circ a)=(a_1^{n-2}\mu^{n-2}\alpha_{n-1}(s_{n-1}))\cdot(\mu^{n-1}\alpha_n(s_n)\cdot b)$ for arbitrary $s_{n-1}$, $s_n\in S$. Set $\beta_{n-1}:=L_{a_1^{n-2}}\mu^{n-2}\alpha_{n-1}$ and $\beta_2:=R_b\mu^{n-1}\alpha_n$. Whence, $\alpha_{n+1}(\lambda^{n-2}(s_{n-1})\circ R_a\lambda^{n-1}(s_n))=\beta_{n-1}(s_{n-1})\cdot\beta_n(s_n)$ for arbitrary $s_{n-1}$, $s_n\in S$. Notice that $\beta_{n-1}\lambda^{2-n}(1)=\beta_{n-1}(1)=a_1^{n-2}\cdot\mu^{n-2}\alpha_{n-1}(1)=a_1^{n-1}$ and $\beta_n(R_a\lambda^{n-1})^{-1}(1)=\beta_n\lambda^{1-n}R_{a^{-1}}(1)=\beta_n\lambda^{1-n}(a^{-1})=\beta_n(a^{-1})=\mu^{n-1}\alpha_n(a^{-1})\cdot b=a_n\cdot b$. According to Lemma 1 there exists a homomorphism $\chi\colon S\to T$ of monoids $(S,\circ,\lambda,a)$ and $(T,\cdot,\mu,b)$ such that $\beta_{n-1}=L_{a_1^{n-1}}\chi\lambda^{n-2}$, $\beta_n=R_{a_n\cdot b}\chi R_a\lambda^{n-1}$, $\alpha_{n+1}=L_{a_1^{n-1}}R_{a_n\cdot b}\chi$. Hence, $L_{a_1^{n-1}}R_{a_n\cdot b}\chi=L_{a_1}R_{a_2^n\cdot b}\varphi$, and so $\chi=L^{-1}_{a_2^{n-1}}R_{a_2^{n-1}}\varphi$. Thus, $R_b\mu^{n-1}\alpha_n= =R_{a_n\cdot b}\chi R_a\lambda^{n-1}=R_{a_n\cdot b}L^{-1}_{a_2^{n-1}}R_{a_2^{n-1}}\varphi R_a\lambda^{n-1}$, and so $\alpha_n=\mu^{1-n}L^{-1}_{a_2^{n-1}}R_{a_2^n}\varphi R_a\lambda^{n-1}$. Finally, we obtain the formulas of form (11).

We shall prove the second part of this theorem. Let us notice that

$$\begin{aligned}\alpha_{n+1}((s_1,s_2,s_3,\ldots,s_n)) &= \alpha_{n+1}(s_1\circ\lambda(s_2)\circ\lambda^2(s_3)\circ\ldots\circ\lambda^{n-1}(s_n)\circ a) =\\ &= L_{a_1}R_{a_2^n\cdot b}\varphi(s_1\circ\lambda(s_2)\circ\lambda^2(s_3)\circ\ldots\circ\lambda^{n-1}(s_n)\circ a) =\\ &= a_1\cdot\varphi(s_1)\cdot\varphi\lambda(s_2)\cdot\varphi\lambda^2(s_3)\cdot\ldots\cdot\varphi\lambda^{n-1}(s_n)\cdot\varphi(a)\cdot a_2^n\cdot b =\\ &= (a_1\cdot\varphi(s_1))\cdot(\varphi\lambda(s_2)\cdot a_2)\cdot((a_2)^{-1}\cdot\varphi\lambda^2(s_3)\cdot a_2^3)\cdot\ldots\cdot((a_2^{n-1})^{-1}\cdot\\ &\cdot\varphi\lambda^{n-1}(s_n)\cdot\varphi(a)\cdot a_2^n\cdot b) = \alpha_1(s_1)\cdot\mu\alpha_2(s_2)\cdot\mu^2\alpha_3(s_3)\cdot\ldots\cdot\mu^{n-1}\alpha_n(s_n)\cdot b =\\ &= [\alpha_1(s_1),\alpha_2(s_2),\alpha_3(s_3),\ldots,\alpha_n(s_n)]\end{aligned}$$

for arbitrary $s_1,\ldots,s_n\in S$.

The proof of this theorem is completed.

**Corollary 5.** *Let $S(\ )$ be an n-semigroup such that $R(S)\neq\emptyset$, and let $(S,\circ,\lambda,a)$ be a monoid associated with $S(\ )$. Let $T[\ ]$ be an n-group, and let $(T,\cdot,\mu,b)$ be a group associated with $T[\ ]$. A sequence of functions $(\alpha_1,\ldots,\alpha_{n+1})$ of form (11) is a general solution of equation (9) if $\varphi\colon S\to T$ is an arbitrary homomorphism from the monoid $(S,\circ,\lambda,a)$ into the group $(T,\cdot.\mu,b)$ and $a_1,\ldots,a_n\in T$ are arbitrary elements.*

**Corollary 6.** *Let $S(\ )$ be an n-semigroup such that $R(S)\neq\emptyset$, and let $T[\ ]$ be an n-group. Let $(\alpha_1,\ldots,\alpha_{n+1})$ be a homotopy from the n-semigroup $S(\ )$ into the n-group $T[\ ]$. If $\alpha_i$ is a bijection for a certain $i\in\{1,\ldots,n+1\}$, then $(\alpha_1,\ldots,\alpha_{n+1})$ is an isotopy. Furthermore, the monoid $(S,\circ,\lambda,a)$ is isomorphic to the group $(T,\cdot,\mu,b)$.*

Let $X$ be an arbitrary non-empty set, and let $S(\ )$ be an arbitrary $n$-groupoid $(n\geqq 3)$.

Let us consider the following functional equation

$$F_1(F_2(\ldots(F_n(x,s_n),\ldots),s_2),s_1)=F_{n+1}(x,(s_1,s_2,\ldots,s_n)) \tag{12}$$

for arbitrary $x\in X$ and $s_1,s_2,\ldots,s_n\in S$, where $F_i\colon X\times S\to X$ (for $i=1,\ldots,n+1$) are unknown functions. Equation (12) is an analogue for $n$-groupoids of the functional equation considered by Grząślewicz (cf. [3]).

We introduce an $n$-ary operation $[\ ]$ on $X^X$ by defining

$$[f_1,f_2,\ldots,f_n]:=f_1f_2\ldots f_n$$

for arbitrary functions $f_1,f_2,\ldots,f_n\in X^X$.

The expression on the right-side of the above equality is the $n$-fold composition of functions.

Then $X^X[\ ]$ is an $n$-monoid.

**Theorem 7.** *Let $X$ be an arbitrary non-empty set, and let $S(\ )$ be an arbitrary n-groupoid. A sequence of functions $(F_1,\ldots,F_{n+1})$ is a solution of equation (12) on the n-groupoid $S(\ )$ if and only if there exists a sequence of functions $(\alpha_1,\ldots,\alpha_{n+1})$ being* *a solution of equation* (9) *on the n-groupoid* $S(\ )$ *and the n-monoid* $X^X[\ ]$, *and* $F_i(x,s)= =\alpha_i(s)(x)$ *for arbitrary* $x\in X$, $s\in S$, $i\in\{1, \dots, n+1\}$.

PROOF. (i) Let us assume that the sequence $(F_1, F_2, \dots, F_{n+1})$ is a solution of equation (12). Then,

$$\begin{aligned}\alpha_{n+1}\big((s_1, s_2, \dots, s_n)\big)(x) &= F_{n+1}\big(x, (s_1, s_2, \dots, s_n)\big) = \\ = F_1\big(F_2(\dots(F_n(x, s_n), \dots), s_2), s_1\big) &= \alpha_1(s_1)\big(\alpha_2(s_2)(\dots(\alpha_n(s_n)(x))\dots\big) = \\ = \big(\alpha_1(s_1)\alpha_2(s_2)\dots\alpha_n(s_n)\big)(x) &= [\alpha_1(s_1), \alpha_2(s_2), \dots, \alpha_n(s_n)](x)\end{aligned}$$

for arbitrary $x\in X$ and $s_1, s_2, \dots, s_n\in S$.

(ii) Let the sequence $(\alpha_1, \alpha_2, \dots, \alpha_{n+1})$ be a solution of equation (9) on the $n$-groupoid $S(\ )$ and the $n$-monoid $X^X[\ ]$. Then, $F_1\big(F_2(\dots(F_n(x, s_n), \dots), s_2), s_1\big)= =\big(\alpha_1(s_1)\alpha_2(s_2)\dots\alpha_n(s_n)\big)(x)=[\alpha_1(s_1), \alpha_2(s_2), \dots, \alpha_n(s_n)](x)=\alpha_{n+1}\big((s_1, s_2, \dots, s_n)\big)(x)= =\big(F_{n+1}x, (s_1, s_2, \dots, s_n)\big)$ for arbitrary $x\in X$ and $s_1, s_2, \dots, s_n\in S$.

**Theorem 8.** *Let* $X$ *be an arbitrary non-empty set. Let* $S(\ )$ *be an n-semigroup such that* $R(S)\neq\emptyset$, *and let* $(S, \circ, \lambda, a)$ *be a monoid associated with* $S(\ )$. *If a sequence of functions* $(F_1, \dots, F_{n+1})$ *is a solution of equation* (12) *on the n-semigroup* $S(\ )$ *and the functions* $F_i(x, 1)$ $\big($*for* $i=1, \dots, n-1\big)$, $F_n(x, a^{-1})$ *are bijections on the set* $X$, *then there exists a homomorphism* $\varphi\colon S\to X^X$ *of binary monoids* $(S, \circ, \lambda, a)$ *and* $X^X$, *and there exist functions* $f_1, \dots, f_n\in X^X$ *such that*

$$(13)\qquad \begin{cases} F_1(x, s) = L_{f_1}\varphi(s)(x), \\ F_k(x, s) = L^{-1}_{f_2^{k-1}}R_{f_2^k}\varphi\lambda^{k-1}(s)(x) \quad \textit{for} \quad k = 2, \dots, n-1, \\ F_n(x, s) = L^{-1}_{f_2^{n-1}}R_{f_2^n}\varphi R_a\lambda^{n-1}(s)(x), \\ F_{n+1}(x, s) = L_{f_1}R_{f_2^n}\varphi(s)(x), \end{cases}$$

*for arbitrary* $x\in X$ *and* $s\in S$.

*If* $\varphi\colon S\to X^X$ *is an arbitrary homomorphism of the binary monoids* $(S, \circ, \lambda, a)$ *and* $X^X$, *and* $f_2, \dots, f_{n-1}$ *are arbitrary bijections on the set* $X$ *while* $f_1, f_n\in X^X$ *are arbitrary functions, then a sequence of functions* $(F_1, \dots, F_{n+1})$ *of form* (13) *is a solution of equation* (12) *on the n-semigroup* $S(\ )$.

PROOF. Let the sequence of functions $(F_1, \dots, F_{n+1})$ be a solution of equation (12) satisfying the assumptions of this theorem. It follows from Theorem 7 that there exists a sequence of functions $(\alpha_1, \dots, \alpha_{n+1})$ being a solution of equation (9) on the $n$-semigroup $S(\ )$ and the $n$-monoid $X^X[\ ]$, and furthermore $F_i(x, s)=\alpha_i(s)(x)$ for arbitrary $x\in X$, $s\in S$, $i\in\{1, \dots, n+1\}$. The functions $f_i(x):=\alpha_i(1)(x)=F_i(x, 1)$ (for $i=1, \dots, n-1$) and $f_n(x):=\alpha_n(a^{-1})(x)=F_n(x, a^{-1})$ for $x\in X$ are bijections on the set $X$. It folows from Theorem 6 that there exists a homomorphism $\varphi\colon S\to X^X$ of monoids $(S, \circ, \lambda, a)$ and $X^X$ such that

$$(14)\qquad \begin{aligned} \alpha_1 &= L_{f_1}\varphi, \\ \alpha_k &= L^{-1}_{f_2^{k-1}}R_{f_2^k}\varphi\lambda^{k-1} \quad \text{for} \quad k = 2, \dots, n-1, \\ \alpha_n &= L^{-1}_{f_2^{n-1}}R_{f_2^n}\varphi R_a\lambda^{n-1}, \\ \alpha_{n+1} &= L_{f_1}R_{f_2^n}\varphi. \end{aligned}$$

Thus, applying the equalities $F_i(x, s)=\alpha_i(s)(x)$ we obtain formulas (13).

We shall prove the second part of the theorem. It follows from Theorem 6 that the sequence of functions $(\alpha_1, \ldots, \alpha_{n+1})$ of form (14) is a solution of equation (9) on the $n$-semigroup $S(\ )$ and the $n$-monoid $X^X[\ ]$. Notice that for the functions $F_i$ of form (13) we have $F_i(x, s)=\alpha_i(s)(x)$ for arbitrary $x\in X$, $s\in S$, $i\in\{1, \ldots, n+1\}$. Thus according to Theorem 7, the sequence of functions $(F_1, \ldots, F_{n+1})$ of form (13) is a solution of equation (12) on the $n$-semigroup $S(\ )$.

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