Embedding partially ordered topological spaces in hyperspaces

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0. Introduction

Partially ordered topological spaces (in short POTS) are topological spaces endowed with a closed partial order and a hyperspace is the set of all non-empty closed subsets of a topological space topologised by the Vietoris topology. Subspaces of Hausdorff hyperspaces are POTS under the partial order of set-inclusion. The converse question — can all POTS, or at least members of some special class, be realised as subspaces of hyperspaces in some systematic way? — is naturally interesting, but apparently not simply answered, like the corresponding purely order-theoretic question.

In this paper, this question is addressed and some sufficient conditions for the natural order-isomorphism of a POTS X into the corresponding hyperspace $CL(X)$, to be topological also, are noted. Such results are likely to throw more light on both classes of spaces, specially in respect of arc-theorems in KOCH [2], WARD [10, 11], McWATERS [5] and MISRA [7].

1. Preliminaries

We recall some key terms and simple facts.

1.1. A *partial order* $\leqq$ on a non-empty set X is a reflexive, anti-symmetric and transitive binary relation on X and is viewed as a subset of $X\times X$. For each x in X, the subset $\{y\colon y\leqq x\}$ is denoted by $L(x, \leqq)$ or, when there is no confusion, simply by $L(x)$. If $\mathcal{I}$ is a topology on X such that $\leqq$ is a closed subset of the product space $X\times X$, then $\leqq$ is said to be *closed*[1] (or more specifically $\mathcal{I}$*-closed*) and the triple $(X, \leqq, \mathcal{I})$ is called a *partially ordered topological space* (or POTS in short).

1.2. For a topological space $(X, \mathcal{I})$ the collection of all non-empty closed subsets of X is denoted by $CL(X)$. For any finite collection $\{V_1, V_2, \dots, V_m\}$ of non-empty open subsets of X, $\langle V_1, \dots, V_m\rangle$ stands for the subcollection of $CL(X)$ consisting of all F contained in $V_1\cup\dots\cup V_m$ such that $F\cap V_i\neq\emptyset$ for $i=1, 2, \dots, m$. The family of all collections of the form $\langle V_1, \dots, V_m\rangle$ is a base for a topology on $CL(X)$,

[1] The less natural term "Continuous partial order" is also often used (e.g., [9]), perhaps due to the connection between a continuous function and its graph ([3], p. 142, Theorem 2).

called the *Vietoris topology*. The *hyperspace* $CL(X)$ is the set $CL(X)$ provided with the Vietoris topology.

Our main references for POTS and hyperspaces are [1], [8], [9] and [3], [6] respectively.

1.3. Let $(X, \leqq, \mathscr{I})$ be a POTS, $\mathscr{U}$ any Hausdorff topology on X, and $I_X = \{(x, x) \colon x \in X\}$ the identity partial order on X. Following are some easily observed facts relevant to our discussion.

1.3.1. If $\mathscr{I} \subseteq \mathscr{U}$, then $\leqq$ is $\mathscr{U}$-closed.

1.3.2. I_X is $\mathscr{U}$-closed (see e.g., [3], p. 140, Theorem 2).

1.3.3. $\mathscr{I}$ is necessarily a Hausdorff topology.

(One may argue directly or note, using convergent nets from $I_X \subseteq \leqq$, and anti-symmetry of $\leqq$, that I_X is also $\mathscr{I}$-closed and then appeal to ([3], p. 141, Theorem 4)).

1.3.4. The function $(x_1, x_2) \rightsquigarrow (x_2, x_1)$ being a homeomorphism on $X \times X$, $(X, \leqq^{-1}, \mathscr{I})$ is also a POTS.

1.3.5. For each x in X, $L(x, \leqq)$ is a closed subset of $(X, \mathscr{I})$ and so is $L(x, \leqq^{-1})$.

1.3.6. For any regular T_1-space X and any non-empty subcollection **S** of the hyperspace $CL(X)$, $\subseteq$ is a closed partial order on **S** (see e.g., [3], p. 167, Theorem 1).

1.3.7. The function $x \rightsquigarrow L(x)$ is an order-isomorphism from $(X, \leqq)$ into $(CL(X), \subseteq)$.

2. Some embedding theorems

The hyperspace $CL(X)$ of a discrete space $(X, \mathscr{D})$ is discrete and for any Hausdorff space $(X, \mathscr{U})$ the function $x \rightsquigarrow \{x\}$ is a topological embedding (see e.g., [6], p. 153). Therefore, it is obvious that for the POTS $(X, \leqq, \mathscr{D})$ and $(X, I_X, \mathscr{U})$ the order-isomorphism $x \rightsquigarrow L(x)$ of 1.3.7 is in fact a topological isomorphism, i.e., an embedding of POTS. In general, it is not so — even for reasonably nice spaces (see Example 2.6). Thus the question

2.1. For what POTS $(X, \leqq, \mathscr{I})$ the order-isomorphism $L \colon X \to L[X] \subseteq CL(X)$ is a topological isomorphism? — is non-trivial, and its answers are of interest as noted earlier.

Some partial answers to this question are given in this section. The lemma below, provides a sufficient condition for the continuity of the function L, and is the main tool used in the proofs of Theorems 2.4 and 2.5 following it.

2.2 Lemma. *Let $(X, \leqq, \mathscr{I})$ be a POTS and x_0 a point in X. If for each net (x_λ) converging to x_0,*

(1) Every net (y_λ) in $X - L(x_0)$, such that $y_\lambda \leqq x_\lambda$ for all λ, has a cluster point in X; and

(2) For each $x < x_0$, there is a net (y_λ) clustering at x and such that $y_\lambda \leqq x_\lambda$ for all λ.

Then $L \colon X \to CL(X)$ is continuous at x_0.

Proof. We prove the contrapositive. Suppose L is not continuous at x_0. Then, there exists a neighbourhood $\langle V_1, \ldots, V_m \rangle$ of $L(x_0)$ such that for each neighbourhood U of x_0, there is a point x_U in U such that $L(x_U) \notin \langle V_1, \ldots, V_m \rangle$. If it so happens that for each neighbourhood U of x_0, an x_U can be picked up for which $L(x_U)$ is not contained in $V_1\cup\ldots\cup V_m$ then there is a point $y_U\leqq x_U$, that does not belong to $V_1\cup\ldots\cup V_m$. The net (y_U), being outside the open set $V_1\cup\ldots\cup V_m$, cannot have any cluster point in the set $L(x_0)\subseteqq V_1\cup\ldots\cup V_m$. But, as the net (x_U) converges to x_0 and $\leqq$ is closed all cluster points of (y_U) have to be in $L(x_0)$. So, the net (y_U) has no cluster points in X and condition (1) is contradicted.

On the other hand, if a choice of (x_U) as above cannot be made, then there exists a neighbourhood W of x_0 such that for each point x in W, $L(x)$ is a subset of $V_1\cup\ldots\cup V_m$, but is disjoint from some member of $\{V_1, \ldots, V_m\}$. Now, for each neighbourhood U of x_0, let x_U be a point in $U\cap W$. Clearly, the net (x_U) converges to x_0 and since $\{V_1, \ldots, V_m\}$ is a finite family there is some member V_i and a subnet (x_{U_γ}) of (x_U) such that for each γ, $L(x_{U_\gamma})$ is disjoint from V_i. Thus, we are led to a net (x_{U_γ}) converging to x_0 such that no net of the form (y_γ), with $y_\gamma\leqq y_{U_\gamma}$, can cluster to any point of the non-empty set $L(x_0)\cap V_i$, and so condition (2) is contradicted.

2.3 Theorem. *If $\mathscr{I}$ is a minimal topology on X with respect to which the partial order $\leqq$ on X is closed and $(X, \mathscr{I})$ is a regular space then the function L is an embedding of POTS $(X, \leqq, \mathscr{I})$ in $CL(X)$, whenever it is continuous.*

PROOF. Let $\mathscr{V}$ be the topology on $L[X]$ relativized from $CL(X)$ and $\mathscr{U}=\{L^{-1}[V]\colon V\in\mathscr{V}\}$. Then, in view of statements 1.3.6 and 1.3.7, $(L[X], \subseteqq, \mathscr{V})$ is a POTS and $\mathscr{U}$ is a topology on X with respect to which $\leqq$ is closed. So, if L is continuous, then $\mathscr{U}\subseteqq\mathscr{I}$ and hence due to the minimality of $\mathscr{I}$, $\mathscr{U}=\mathscr{I}$. This means that $L\colon(X, \leqq, \mathscr{I})\to(L[X], \subseteqq, \mathscr{V})$ is a topological isomorphism.

2.4 Theorem. *For any linearly ordered space $(X, \leqq, \mathscr{I})$ where $\mathscr{I}$ is the interval topology, $L\colon X\to CL(X)$ is an embedding of POTS $(X, \leqq, \mathscr{I})$ into $CL(X)$.*

PROOF. Let x_0 be any point of X and (x_λ) any net in X converging to x_0. For any net (y_λ) in $X-L(x_0)$ such that $y_\lambda\leqq x_\lambda$ for all λ, we have, $x_0<y_\lambda\leqq x_\lambda$ for all λ and hence (y_λ) also converges to x_0. Thus, condition (1) of Lemma 2.2 is satisfied. If $x<x_0$ is any point in X, then the net (x_λ) is eventually in the neighbourhood $X-L(x)$ of x_0 and so the net (y_λ) given by: $y_\lambda=x_\lambda$ if $x_\lambda\in L(x)$ and $y_\lambda=x$ otherwise, converges to x, and $y_\lambda\leqq x_\lambda$ for all λ. Thus, condition (2) of Lemma 2.2 is also satisfied, and L is therefore continuous. As the interval topology $\mathscr{I}$ of a linearly ordered space is known to be the smallest topology with respect which the linear order is closed (see e.g., [9], p. 148), Theorem 2.3 completes the proof.

2.5 Theorem. *Let $(X, \leqq, \mathscr{I})$ be a compact POTS such that*

(a) no point in X has an immediate predecessor, that is there do not exist distinct points x_1, x_2 in X with $L(x_2, \leqq)\cap L(x_1, \leqq^{-1})=\{x_1, x_2\}$; and

(b) each point x in X has a neighbourhood W such that under $\leqq$, x is comparable to all members of W and $W\cap L(x)$ is linearly ordered.

Then $L\colon X\to CL(X)$ is an embedding of POTS $(X, \leqq, \mathscr{I})$ into $CL(X)$.

PROOF. The topology $\mathscr{I}$, being compact Hausdorff, is minimal Hausdorff and so is also a minimal topology with respect to which $\leqq$ is closed (see statement 1.3.3). Therefore, in view of Theorem 2.3, it is enough to show that L is continuous. We aim to use Lemma 2.2 for this.

Let x_0 be any point in X and (x_λ) any net in X converging to x_0. Since each net in X has cluster points, condition (1) of Lemma 2.2 is trivially satisfied. For checking the other condition, let $x < x_0$ be any point in X and W a neighbourhood of x_0 of the type described in condition (b) above. The net (x_λ) is eventually in W. If it is frequently in the set $W - L(x_0)$, then the net (y_λ) given by: $y_\lambda = x$ if $x_\lambda \in W - L(x_0)$ and $y_\lambda = x_\lambda$ otherwise, clusters at x (it takes the value x frequently) and satisfies $y_\lambda \leqq x_\lambda$ for all λ, due to the comparability property of W. If instead, (x_λ) is eventually in $W \cap L(x_0)$, then also there is a net (y_λ) clustering at x and satisfying $y_\lambda \leqq x_\lambda$ for all λ, as shown below.

Let C be a maximal chain from x to x_0. Then C is compact (see e.g., [9], Lemma 4) and its subchain $C - \{x_0\}$ has x_0 as an accumulation point. For, otherwise, $C - \{x_0\}$ is itself a compact space with a closed linear order $\leqq$, and its maximal element (see [1], p. 63, Theorem 16 or [9], Theorem 1) is an immediate predecessor of x_0, violating condition (a) of the theorem. Thus, $W \cap (C - \{x_0\})$ has infinitely many points. Let y_1, y_2 be points in this set such that $x < y_1 < y_2 < x_0$. Since the net (x_λ) is assumed to be eventually in $W \cap L(x_0)$ and is also eventually in the neighbourhood $X - L(y_1)$ of x_0, it is in fact eventually in the set $(W \cap L(x_0)) \cap (X - L(y_1))$. The point y_2 is a member of this set, which being a subset of $W \cap L(x_0)$, is linearly ordered under $\leqq$, and as the net (x_λ) is eventually also in the neighbourhood $X - L(y_2)$ of x_0, it follows that eventually $y_2 < x_\lambda$. But then, the net (y_λ) given by: $y_\lambda = x$ whenever $y_2 < x_\lambda$ and $y_\lambda = x_\lambda$ otherwise, converges to x and satisfies the condition $y_\lambda \geqq x_\lambda$ for all λ.

Thus, condition (2) of the Lemma 2.2 is also satisfied and the proof is complete.

2.6 *Example*. Let $X = \{(r, r): r \in \mathbf{R},\ 0 \leqq r \leqq 1\} \cup \{(0, r): r \in \mathbf{R}$ and $0 \leqq r \leqq 1\}$ be given the subspace topology from $\mathbf{R}^2$ and a partial order α defined by: $(p_1, q_1)\alpha(p_2, q_2)$ if and only if any of the following hold: $p_1 = p_2$ and $q_1 = q_2$; $p_1 = q_1$, $p_2 = q_2$ and $p_1 \leqq p_2$; $p_1 = 0 = p_2$ and $q_1 \leqq q_2$; $p_1 = 0$ and $q_1 = 1 = p_2 = q_2$. Then, α is a closed partial order, the space X is compact, Hausdorff, connected and locally connected but $L: X \to CL(X)$ is not continuous at the point (1, 1).

3. *Remarks*

3.1. In Lemma 2.2, we may, in condition (1), replace "any net (x_λ)" by "any net (x_λ) in $X - L(x_0)$" and in condition (2), replace "any $x < x_0$" by "any $x < x_0$ in some neighborhood V of x_0". This leads to a less elegant and only apparently stronger statement (with almost the same proof as given for Lemma 2.2).

3.2. For compact Hausdorff POTS condition (1) of Lemma 2.2 is trivially satisfied. So for such spaces condition (2) of the lemma provides a suficient condition for L to be a topological isomorphism (see Theorem 2.3). This observation can be used to deduce some known embedding results for topological semi-lattices and semigroups. For example, given a compact Hausdorff semi-lattice $(X, \wedge, \mathscr{I})$, any x_0 in X and $x < x_0$, if a net (x_λ) converges to x_0 then the net $(x_\lambda \wedge x)$ converges to $x_0 \wedge \wedge x = x$. Thus we are immediately led to ([4], Theorem 1.2).

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