On predictive deconvolution of a seismic signal

By PHAN DANG CAU (Hanoi)

Abstract. Robinson's statistical minimum-delay model (or the method of predictive deconvolution) has been effectively used in seismic prospecting for oil and gas. It is used to eliminate multiple reflections from surface layers and reverberations in a water layer. However, in our opinion, this model is not clear in some respects.

In this paper we try to give a new interpretation and a more general condition for this model, which are possibly more suitable in practice. We also point out that with the new conditions, the computation process based on observations is just the same as in the case of Robinson's model.

§ 1. Robinson's model

In order to fix ideas, let us consider a specific physical situation, namely the problem of seismic exploration for oil and gas in the earth's sedimentary strata. The source is an explosion or another form of energy which is introduced into the ground at the surface. The reflection response x_n is the seismic reflection record (time series) which is digitally recorded at the surface. The reflection coefficient sequence ε_n is a digitized reprezentation of the reflectivity of the Earth as a function of depth. As a result, knowledge of the ε_n sequences for various geographic locations on the surface allows the seismic interpreter to make contour maps of the earth's sedimentary structure at depth.

Under certain assumptions (see [8], p. 457) Robinson introduced the following equation:

$$x_n + a_1 x_{n-1} + \ldots + a_p x_{n-p} = \varepsilon_n, \quad n = 0, 1, 2, \ldots \tag{1}$$

where a_s, $s=1, 2, \ldots, p$ are the unknown parameters, which depend on the geological structure of the prospected area.

In the case of a noise appearing, the reflection response has the form

$$y_n = x_n + v_n \tag{2}$$

Where v_n is a noise. Here we suppose that the noise is eliminated. The predictive deconvolution problem is to compute the ε_n-s from the x_n-s. However, (1) implies a system of equations having more unknown variables than the number of equations, so it is impossible to find the ε_n-s.

Robinson proposed the statistical method as follows:

He supposes that the sequence ε_n is a random white noise, i.e.

(3) $$E\varepsilon_n = 0, \quad \text{var}\,\varepsilon_n = \sigma^2 > 0, \quad E\varepsilon_n\varepsilon_s = 0 \quad n \neq s.$$

He supposes further that

(4) $$1 + a_1 z + a_2 z^2 + \ldots + a_p z^p \neq 0 \quad \text{for} \quad |z| \leqq 1.$$

Thus x_n is a stationary auto-regressive process. As well known, then the coefficients $a_1, a_2, \ldots, a_p$ can be estimated from the observed x_n-s, and the ε_n-s are estimated by

(5) $$\hat{\varepsilon}_n = x_n + \hat{a}_1 x_{n-1} + \ldots + \hat{a}_p x_{n-p}.$$

§ 2. Some remarks on Robinson's model

In his model, Robinson identifies the random variables ε_n with the reflection coefficients. However this identification contradicts the fact that the reflection coefficients are deterministic physical quantities. Thus if we consider the model (1) as a stochastic model and denote the reflection coefficients by ε_n, then it is better to write the model (1) in an other form:

$$x_n + a_1 x_{n-1} + a_2 x_{n-2} + \ldots + a_p x_{n-p} = u_n$$

where u_n is a random variable depending on ε_n in some way.

We have recourse to the irregularity of the sequence u_n to obtain information on the earth's sedimentary structure at depth. The assumption that u_n is a stationary process seems not always suitable in practice.

§ 3. The modified model

In order to modify Robinson's model so that it be more suitable to practice, let us first consider the simplest case:

Suppose after explosion the input signal $f(t)$ propagates to the earth's crust. When meeting an interface having reflection coefficient ε it reflects to the surface with reflected wave $g(t) = \varepsilon f(t)$. Since the elastic wave $f(t)$ represents the motion of a particle about its equilibrium point, $f(t)$ always has a damped sinusoidal form (in the case of an explosion it is relatively narrow with great frequency). Now let us consider some observed value u of $g(t)$. In geophysics the arrival time of a reflected wave is usually considered as an uniformly distributed random variable (i.e. we do not know exactly when the reflected wave appears). Thus the observed value u can also be considered as such a random variable that

$$u = g(\tau)$$

where τ is a uniform random variable on some interval $[a, b]$. We have

$$Eu = \frac{1}{b-a}\int_a^b \varepsilon f(t)\,dt \approx 0$$

$$Eu^2 = \frac{\varepsilon^2}{b-a}\int_a^b f^2(t)\,dt = c\varepsilon^2,$$

or more exactly speaking, Eu is negligible at Eu^2. Thus although ε is some fixed value, in the case $\varepsilon > 0$ the measured value u may be an arbitrary value in $[-\varepsilon, \varepsilon]$. Therefore u cannot be considered as an approximation of ε.

For the above reason we propose to modify (1) by the new model

(6) $$x_n + a_1 x_{n-1} + \ldots + a_p x_{n-p} = u_n \quad n = 0, 1, 2, \ldots,$$

where

(7) $$Eu_n = 0, \quad Eu_n^2 = c\varepsilon_n^2.$$

Because the reflection coefficients are different, Eu_n^2 cannot be constant. However, by measuring the reflection coefficients at a used oil well, WHITE and OBRIEND (1974) of British Petroleum, SCHOENBERGER and LEVIN (1974) of Exxon see that for N large enough

(8) $$\frac{1}{N}\sum_{n=0}^{N-1} \varepsilon_n^2 \approx \gamma^2 > 0$$

(see [9]. p. 490).

By (7) and (8) we have

(9) $$\frac{1}{N}\sum_{n-0}^{N-1} Eu_n^2 = c\frac{1}{N}\sum_{n-1}^{N-1} \varepsilon_n^2 \approx c\gamma^2 = \sigma^2 > 0.$$

We suppose that in the case of a complicated geological phenomenon, the u_n-s are independent variables.

In summary, we suppose that the reflection response x_n satisfies (6) and the following conditions:

(a) The variables $u_0, u_1, u_2, \ldots$ are independent with mean 0 and

(10) $$E|u_n|^{2+\varepsilon_0} < k < \infty \quad \text{for some} \quad k \quad \text{and} \quad \varepsilon_0 > 0$$

(11) (b) $$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} Eu_n^2 = \sigma^2 > 0$$

(12) (c) $$1 + a_1 z + a_2 z^2 + \ldots + a_p z^p \neq 0 \quad \text{for} \quad |z| \leqq 1$$

We would like to remark that condition (10) is obvious because the variables u_n are bounded. For the validity of condition (12) see [8] and [9].

§ 4. Predictive deconvolution of a long-run stationary auto-regressive process

Definition 1. We call a process x_n satisfying (6) and the conditions (10), (11), (12) a long-run stationary auto-regressive process.

Definition 2. We call the process y_n a stationary auto-regressive process corresponding to the above defined long-run stationary process x_n if y_n satisfies $y_n+a_1y_{n-1}+ +\dots+a_py_{n-p}=v_n$ where v_n is white noise with $Ev_n^2=\sigma^2$.

Theorem 1. *Let x_n be a long-run stationary process satisfying* (6), (10), (11), (12). *Then there exist the limits*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}x_{n+s}x_n=\varphi_s \quad \text{a.s.} \quad s=\dots 0,1,2,\dots \tag{13}$$

where φ_s is the correlation function of the corresponding stationary process y_n.

PROOF. By (12) we can take the reciprocal $B(z)$ of the Z-transform $A(z)$

$$B(z)=\frac{1}{1+a_1z+\dots+a_pz^p}=b_0+b_1z+b_2z^2+\dots \tag{14}$$

and the process x_n can be written in the form

$$x_n=\sum_{s=0}^{\infty}b_su_{n-s} \quad \text{where} \quad u_n=0 \quad \text{for} \quad n<0.$$

By (10) $Eu_n^2<d$ for some $d>0$.
Let

$$\xi_n=u_n^2-Eu_n^2, \quad \delta_0=\frac{\varepsilon_0}{2}.$$

Using Minkowski's inequality, we have

$$(E|\xi_n|^{1+\delta_0})^{\frac{1}{1+\delta_0}}=(E|u_n^2-Eu_n^2|^{1+\delta_0})^{\frac{1}{1+\delta_0}}\leqq$$

$$\leqq(E|u_n|^{2+\varepsilon_0})^{\frac{1}{1+\delta_0}}+Eu_n^2<K^{\frac{1}{1+\delta_0}}+d$$

from which

$$E|\xi_n|^{1+\delta_0}<(K^{\frac{1}{1+\delta_0}}+d)^{1+\delta_0}<\infty.$$

The sequence ξ_n satisfies the conditions of Markov's theorem (see [4], p. 287) therefore

$$P\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\xi_n=0.$$

Thus we have

$$P\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}(u_n^2-Eu_n^2)=P\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=0}^{N-1}u_n^2-\frac{1}{N}\sum_{n=0}^{N-1}Eu_n^2\right)=0. \tag{15}$$

By (11) and (15)

$$P\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}u_n^2=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}Eu_n^2=\sigma^2.$$

Since the u_n^2-s are independent variables, by the theorem 3.2 in [1], p. 159 we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}u_n^2=\sigma^2 \quad \text{a.s.} \tag{16}$$

For $s=1, 2, 3, \dots$ we can write

$$\frac{1}{N}\sum_{n=0}^{N-1}u_n u_{n+s}=\sum_{l=0}^{s}\frac{1}{N}\sum_{\tau=0}^{\left[\frac{N-l-1}{s+1}\right]}u_{\tau s+l+\tau}u_{(\tau+1)s+l+\tau}. \tag{17}$$

Now consider

$$E\left(\frac{1}{N}\sum_{\tau=0}^{M}\eta_\tau\right)^2=\frac{1}{N^2}\sum_{\tau=0}^{M}Eu^2_{\tau s+l+\tau}Eu^2_{(\tau+1)s+l+\tau}\leqq\frac{Nd^2}{N^2}=\frac{d^2}{N}\to 0$$

where

$$M=\left[\frac{N-l-1}{s+1}\right],\quad \eta_\tau=u_{\tau s+l+\tau}u_{(\tau+1)s+l+\tau}.$$

Here we have used the independence of the variables u_n. Using Tchebychef's inequality we get

$$P\lim_{N\to\infty}\frac{1}{N}\sum_{\tau=0}^{M}\eta_\tau=0.$$

We can see that the variables η_τ-s are independent, and using the theorem 3.2 again we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\tau=0}^{M}\eta_\tau=0 \quad \text{a.s.}$$

From and (17) we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}u_n u_{n+s}=0 \quad \text{a.s.} \tag{18}$$

From (17) and (18) we get

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}x_{n+s}x_n=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}b_l b_r u_{n+s-l}u_{n-r}=$$

$$=\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}b_l b_r\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}u_{n+s-l}u_{n-r}=\sigma^2\sum_{r=0}^{\infty}b_{r+s}b_r=Ey_{n+s}y_n=\varphi_s.$$

Thus the proof is complete.

Remark. If we know φ_s, $s=0, 1, \ldots, p$ then $a_1, a_2, \ldots, ap$ are determined by Yule—Walker-type equations

(19)
$$\begin{pmatrix} \varphi_0 & \varphi_1 & \cdots \varphi_{p-1} \\ \varphi_1 & \varphi_0 & \cdots \varphi_{p-2} \\ \vdots & & \\ \varphi_{p-1} & \varphi_{p-2} & \cdots \varphi_0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix} = - \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_p \end{pmatrix}.$$

Therefore the u_n- are determined by

(20)
$$u_n = x_n + a_1 x_{n-1} + \ldots + a_p x_{n-p}.$$

In practice we can estimate φ_s by

$$r_s = \frac{1}{N} \sum_{n=0}^{N-1} x_{n+s} x_n$$

and then $a_1, a_2, \ldots, a_p$ are estimated by

(21)
$$R\hat{a} = -r$$

where

$$\hat{a} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_p)', \quad r = (r_1, r_2, \ldots, r_p)'$$

$$R = \begin{pmatrix} r_0 & r_1 & \cdots r_{p-1} \\ r_1 & r_0 & \cdots r_{p-2} \\ \vdots & & \\ r_{p-1} & r_{p-2} & \cdots r_0 \end{pmatrix}.$$

Since the matrix on the left hand side of (19) is positive definite, for N large enough the matrix R is invertible and we have

$$\hat{a} = -R^{-1} r.$$

Using the above theorem we get

$$\lim_{N \to \infty} \hat{a} = \lim_{N \to \infty} -Rr = - \begin{pmatrix} \varphi_0 & \varphi_1 & \cdots \varphi_{p-1} \\ \varphi_1 & \varphi_0 & \cdots \varphi_{p-2} \\ \vdots & & \\ \varphi_{p-1} & \varphi_{p-2} & \cdots \varphi_0 \end{pmatrix}^{-1} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_p \end{pmatrix} = a \quad \text{a.s.}$$

therefore the estimate $\hat{u}_n$ of u_n is obtained by

$$\hat{u}_n = x_n + \hat{a}_1 x_{n-1} + \ldots + \hat{a}_p x_{n-p}$$

and we have

(22)
$$\lim_{N \to \infty} \hat{u}_n = u_n \quad \text{a.s.}$$

§ 5. Limiting distribution of the estimate

In this section we shall show that it is difficult to test hypotheses for a long-run statitonary process.

For simplicity, let us consider the first order process

$$x_n + a x_{n-1} = u_n \quad n = 0, 1, 2, \ldots \tag{23}$$

Where the sequence u_n satisfies (10). The condition (11) is replaced by a more special condition:

Let L be a subset of the set $\mathcal{N}$ of non-negative integers such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_L(n) = \lambda \quad 0 < \lambda \leqq 1. \tag{24}$$

Where

$$\chi_L(n) = \begin{cases} 1 & n \in L \\ 0 & \text{otherwise.} \end{cases}$$

Let $Eu_n^2 = \sigma^2 \chi_L(n)$ that is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} Eu_n^2 = \lambda\sigma^2. \tag{25}$$

Let

$$y_n = x_{n-1} u_n, \quad S = \sum_{n=0}^{N-1} y_n, \quad D = \sqrt{ES^2}.$$

Theorem 2. *For the above mentioned process x_n, if $\lambda > \frac{1}{2}$ then $\frac{N(\hat{a}-a)}{D}$ has a limiting normal distribution $\mathcal{N}\left(0, \frac{1}{\varphi_0^2}\right)$.*

Proof. It is easily seen that

$$\frac{N(\hat{a}-a)}{D} = -r_0^{-1} \frac{S}{D}.$$

Because $P \lim_{N\to\infty} r_c^{-1} = \varphi_0^{-1} > 0$ we have to prove that $\frac{S}{D} \to \mathcal{N}(0, 1)$. Let $H = \mathcal{N} \setminus L$ then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_H(n) = 1 - \lambda < \frac{1}{2}.$$

Hence we can choose N_0 such that for every $N > N_0$

$$N - 2\sum_{n=1}^{N} \chi_H(n) - 1 > \delta_0 N \quad \text{for some} \quad \delta_0 > 0. \tag{26}$$

We can see

$$\chi_L(n)\chi_L(n-1) \geqq 1 - [\chi_H(n) - \chi_H(n-1)]. \tag{27}$$

Hence

$$ES^2 = \sum_{n=1}^{N} Eu_n^2 Ex_{n-1}^2 = \sum_{n=1}^{N} \sum_{s=0}^{n-1} a^{2(n-s-1)} \chi_L(n)\chi_L(s) \geqq \sum_{n=1}^{N} \chi_L(n)\chi_L(n-1) \geqq$$

$$\geqq \sum_{n=1}^{N} \left[1-(\chi_H(n)+\chi_H(n-1))\right] \geqq N-2\sum_{n=1}^{N} \chi_H(n)-1 > \delta_0 N.$$

Thus for $N > N_0$

$$D^2 = ES^2 > \delta_0 N \tag{28}$$

Now for $m \in \mathcal{N}$ let

$$x_n^{(m)} = \sum_{s=0}^{m} (-a)^s u_{n-s}$$

$$z^{(m)} = x_n - x_n^{(m)} = \begin{cases} \sum_{s=m+1}^{n} (-a)^s u_{n-s} & n > m \\ 0 & n \leqq m \end{cases}$$

$$y_n^{(m)} = x_{n-1}^{(m)} u_n$$

$$S^{(m)} = \sum_{n=1}^{N} y_n^{(m)}$$

$$Z^{(m)} = S - S^{(m)} = \sum_{n=1}^{n} \sum_{s=m+1}^{n-m} (-a)^s u_n u_{n-1-s}.$$

Then

$$EZ^{2(m)} = \sum_{n=1}^{N} \sum_{s=m+1}^{n-1} Eu_n^2 u_{n-1-s}^2 a^{2s} \leqq \frac{Na^{2(m+1)}}{1-a^2}. \tag{29}$$

For $N > N_0$ using (28) and (29) we have

$$\frac{EZ^{2(m)}}{D^2} \leqq \frac{Na^{2(m+1)}}{(1-a^2)D} \leqq \frac{Na^{2(m+1)}}{(1-a^2)N\delta_0} = \frac{a^{2(m+1)}}{(1-a^2)\delta_0} \to 0 \quad \text{as} \quad m \to \infty.$$

Now for given k, $2m < k < N$, let

$$N = Mk + r \quad r < k.$$

$$z_s = y_{(s-1)k+1}^{(m)} + y_{(s-1)k+2}^{(m)} + \dots + y_{sk-m}^{(m)} \quad s = 1, 2, \dots, M$$

$$z_{M+1} = y_{Mk+1}^{(m)} + \dots + y_N^{(m)}$$

$$v_s = y_{sk-m+1}^{(m)} + y_{sk-m+2}^{(m)} + \dots + y_{sk}^{(m)} \quad s = 1, 2, \dots, M.$$

If we put

$$Z_{kN} = \sum_{s=1}^{M+1} z_s, \quad X_{kN} = \begin{cases} \sum_{s=1}^{M} v_s & k < N \\ 0 & k \geqq N. \end{cases}$$

Then $S^{(m)} = X_{kN} + Z_{kN}$.

It is easy to see that

$$EX_{kN}^2 < MC_m, \quad E|z_k|^{2+\varepsilon_0} < G_k.$$

Hence

$$\frac{EX_{kN}^2}{N} \leqq \frac{M}{N} C_m \leqq \frac{1}{k} C_m.$$

We can see that $S^{(m)}$ too has the property (28), i.e. for $N > N_0$

$$ES^{2(m)} = D_m^2 > \delta_0 N.$$

Thus for $N > N_0$

$$0 < \delta_0 \leqq \frac{ES^{2(m)}}{N} \leqq \frac{EZ_{kN}^2}{N} + \frac{1}{k} C_m.$$

Therefore there exist $\gamma_0 > 0$ and k_0 such that for every $k > k_0$

$$D_{kN}^2 = EZ_{kN}^2 > \gamma_0 N.$$

We can see that the variables $z_1, z_2, \dots \; z_{M+1}$ are idependent and

$$\frac{1}{D_{kN}^{2+\varepsilon_0}} \sum_{n=1}^{M+1} E|z_n|^{2+\varepsilon_0} \leqq \frac{(M+1)G_k}{N^{1+(\varepsilon_0/2)}\gamma_0^{1+(\varepsilon_0/2)}} \leqq \frac{NG_k}{N^{1+(\varepsilon_0/2)}\gamma_0^{1+(\varepsilon_0/2)}} = \frac{1}{N^{\varepsilon_0/2}} \frac{G_k}{\gamma_0^{1+(\varepsilon_0/2)}} \to 0.$$

Hence by Ljapounov's theorem (see [4], p. 374), for fixed k

$$\frac{Z_{kN}}{D_{kN}} \to \mathcal{N}(0, 1) \quad \text{as} \quad N \to \infty.$$

Now let us consider

$$\frac{EX_{kN}^2}{D_{kN}^2} = \frac{EX_{kN}^2}{N} \frac{N}{D_{kN}^2} \leqq \frac{1}{k} c_m \frac{N}{\gamma_0 N} = \frac{c_m}{k}.$$

Then

$$\text{(30)} \qquad \frac{S}{D} = \frac{Z_{kN}}{D_{kN}\sqrt{1+\dfrac{EX_{kN}^2}{D_{kN}^2}}} + \frac{X_{kN}}{D_{kN}\sqrt{1+\dfrac{EX_{kN}^2}{D_{kN}^2}}} = U_{kN} + V_{kN}$$

uniformly in N as $k \to \infty$. By (30)

$$P \lim_{k \to \infty} V_{kN} = 0 \quad \text{uniformly in} \quad N.$$

Applying Anderson's theorem (see [2], p. 415) we have

$$\frac{S}{D} \to \mathcal{N}(0, 1) \quad \text{as} \quad N \to \infty.$$

Thus the theorem is proved.

Remark. It is well known, that if x_n is a stationary autoregressive process, then

$$E(\sqrt{N}(\hat{a}-a))^2 \to \frac{\sigma^2}{\varphi_0}.$$

Therefore var $\hat{a}$ can be estimated and we can test hypotheses. For the process x_n satisfying (23) we can see $D^2>0$ for $\lambda>\frac{1}{2}$, but D^2 may be 0 for $\lambda\leqq\frac{1}{2}$. Thus in the general case it is difficult to estimate the variance of $\hat{a}$ and therefore it is difficult to test hypotheses.

§ 6. On the rate of convergence

As we have seen in the previous section, for the long-run stationary process, we cannot always give the asymptotic variance of the estimates. Here we show that under certain conditions, we can give a rough bound for this quantity. Let us write

$$R^{-1} = (c_{ik}) \quad i, k = 1, 2, \dots, p.$$

Theorem 3.

(31) (a) *If* $E|c_{ik}|^2<K_1<\infty$ *for* $i, k=1, 2, \dots, p$ *then there exists* $L_1>0$ *such that*

$$E|a_i-\hat{a}_i| \leqq \frac{L_1}{N^{1/2}} \quad i = 1, 2, \dots, p. \tag{32}$$

(33) (b) *If* $E|u_n|^8<K_2<\infty$, $E|c_{ik}|^4<K_3<\infty$ *then there exists* $L_2>0$ *such that*

$$E|u_n-\hat{u}_n| \leqq \frac{L_2}{N^{1/4}}. \tag{34}$$

PROOF. For an arbitrary random variable x having q-th moment, let us write

$$\|x\|_q = (E|x|^q)^{1/q}.$$

Let further $\tilde{x}_n=(x_n, x_{n-1}, \dots, x_{n-p+1})'$ where $x_n=0$ for $n<0$. Then (7) can be written in the form

$$x_n+\tilde{x}'_{n-1}a = u_n.$$

We can see that

$$R = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}_{n-1}\tilde{x}'_{n-1}, \quad r = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}_{n-1}x_n$$

$$a-\hat{a} = R^{-1}\left(\frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}_{n-1}u_n\right).$$

Hence

$$a_i-\hat{a}_i = \sum_{k=1}^{p} c_{ik}\left(\frac{1}{N}\sum_{n=0}^{N-1} x_{n-k}u_n\right) \quad i = 1, 2, \dots, p. \tag{35}$$

(a) Suppose (31) is satisfied.

Notice that

$$\left\|\frac{1}{N}\sum_{n=0}^{N-1} x_{n-k}u_n\right\|_2^2 = \frac{1}{N^2}\sum_{n=0}^{N-1} Eu_n^2 Ex_{n-k}^2 \leqq \frac{cd}{N}$$

where $Eu_n^2 < d$, $Ex_n^2 < c$. Hence

$$E|a_i - \hat{a}_i| = \|a_i - \hat{a}_i\|_1 = \left\|\sum_{k=1}^{P} c_{ik}\left(\frac{1}{N}\sum_{n=0}^{N-1} x_{n-k}u_n\right)\right\|_1 \leqq$$

$$\leqq \sum_{k=1}^{P}\left\|c_{ik}\left(\frac{1}{N}\sum_{n=0}^{N-1} x_{n-k}u_n\right)\right\|_1 \leqq \sum_{k=1}^{P}\|c_{ik}\|_2\left\|\frac{1}{N}\sum_{n=0}^{N-1} x_{n-k}u_n\right\|_2 \leqq$$

$$\leqq \frac{PK_1^{1/2}c^{1/2}d^{1/2}}{N^{1/2}} = \frac{L_1}{N^{1/2}}$$

which proves (32).

(b) Suppose (33) is satisfied. Then we can see

$$Ex_n^8 < K_4 \quad \text{for some} \quad K_4 > 0.$$

Notice that for arbitrary random variables x and y having $2k$-th moment

$$\|xy\|_k \leqq \|x\|_{2k}\|y\|_{2k}. \tag{36}$$

Suppose n is the largest value between different n, s, l and h. Then u_n is independent of u_s, u_l, u_h, x_{n-k}, x_{s-k}, x_{l-k}, x_{h-k} and hence

$$Eu_n u_l u_s u_h x_{n-k}x_{l-k}x_{s-k}x_{h-k} = 0. \tag{37}$$

Using (36) we can show that

$$E|u_n^2 u_s u_l x_{n-k}^2 x_{s-k} x_{l-k}| \leqq K_2^{1/2}K_4^{1/2}. \tag{38}$$

By (36) and (38) we have

$$\left\|\frac{1}{N}\sum_{n=0}^{N-1} x_{n-k}u_n\right\|_4^4 = \frac{1}{N^4}\sum_{s=0}^{N-1}\sum_{n=0}^{N-1}\sum_{l=0}^{N-1} Eu_n^2 u_s u_l x_{n-k}^2 x_{s-k}x_{l-k} \leqq \frac{K_2^{1/2}K_4^{1/2}}{N}.$$

Thus

$$\left\|\frac{1}{N}\sum_{n=0}^{N-1} x_{n-k}u_n\right\|_4 \leqq \frac{K_2^{1/8}K_4^{1/8}}{N^{1/4}}.$$

We now consider

$$\|a_i - \hat{a}_i\|_2 = \left\|\sum_{k=1}^{P} c_{ik}\left(\frac{1}{N}\sum_{n=0}^{N-1} x_{n-k}u_n\right)\right\|_2 \leqq \sum_{k=1}^{P}\left\|c_{ik}\left(\frac{1}{N}\sum_{n=0}^{N-1} x_{n-k}u_n\right)\right\|_2 \leqq \tag{39}$$

$$\leqq \sum_{k=1}^{P}\|c_{ik}\|_4\left\|\frac{1}{N}\sum_{n=0}^{N-1} x_{n-k}u_n\right\|_4 \leqq \frac{PK_2^{1/8}K_3^{1/4}K_4^{1/8}}{N^{1/4}}.$$

Hence

$$E|u_n-\hat{u}_n| = \Big\|\sum_{i=1}^{P}(a_i-\hat{a}_i)x_{n-i}\Big\|_1 \leqq \sum_{i=1}^{P}\|(a_i-\hat{a}_i)x_{n-i}\|_1 \leqq$$

$$\leqq \sum_{i=1}^{P}\|a_i-\hat{a}_i\|_2\|x_{n-i}\|_2 \leqq \frac{P^2K_2^{1/8}K_3^{1/4}K_4^{1/8}C^{1/2}}{N^{1/4}} = \frac{L_2}{N^{1/4}}.$$

Therefore (34) and thus *theorem* 3 *is proved.*

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INSTITUTE OF COMPUTER SCIENCES
AND CYBERNETICS, HA NOI

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