On a problem of G. O. H. Katona

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Dedicated to Prof. Béla Gyires on his 80th birthday

Abstract. Let H be a finite ordered set (say, different real numbers, $|H|=n$) however the ordering is unknown for us.

G. O. H. Katona raised the following problem: If $A=\{p, q\}$ $(1\leqq p<q\leqq n)$ and x, y are arbitrary elements of H then find the minimal number of comparisons needed to decide whether the indexes of the elements x, y are in A or not (say in decreasing order). In the general case the answer is unknown. In this paper we prove that if $A=\{1, q\}$ $(1<q\leqq n)$ then the number of comparisons is at least

$$\min\left(n+q-3, 2n-\left\lceil\frac{q}{2}\right\rceil-2\right).$$

1. Introduction

Let H be a finite ordered set (say, different real numbers), however the ordering is unknown for us. There are many situations when we want to obtain certain information concerning H using pairwise comparisons of the elements. The simplest question of this type: Which is the largest (smallest) element in H? It is easy to prove that any strategy for finding the largest (smallest) element needs at least $n-1$ comparisons.

It we want to determine the two largest elements then $n-2+\lceil\log_2 n\rceil$ comparisons are needed ([3], [4. 211—212 p.]). Moreover it is proved ([8], [9]) that it is impossible to find a pair of consecutive elements with a smaller number of comparisons. ($\lceil x\rceil$($\lfloor x\rfloor$) denotes the smallest (largest) integer $\geqq x(\leqq x)$.) IRA POHL ([1]) has proved that at least $n+\left\lceil\frac{n}{2}\right\rceil-2$ comparisons are needed if we want to determine the largest and smallest elements simultaneously (see also [5], [7], [8]). If we want to decide only whether x_1 and x_2 are the largest and smallest elements in H then we need at least $n+\left\lceil\frac{n-1}{2}\right\rceil-2$ comparisons ([5]) and to decide whether x_1, x_2 are neighboring, element in H we need at least $2(n-2)$ $(n\geqq 3)$ comparisons ([11]).

AMS (MOS) subject classifications (1980)
Primary: 05A05
Secondary: 05A15
Key words and phrases: Finite ordered set, searching, optimal strategy.

G. O. H. KATONA suggested the following problem:

Let $A=\{r_1, r_2, \ldots, r_k\}$ $(1 \leqq r_1 < r_1 < \ldots < r_k \leqq n)$ and let x be an arbitrary element of H. The question is: how many comparisons are needed if we want to decide whether the index of the element x is in A or not?

KATONA proved ([2]) that the minimal number of comparisons is $n-1$ (see also [6], [8]).

G. O. H. KATONA suggested the following problem, too:

If $A=\{p, q\}$ and x, y are arbitrary elements of H and we want to decide whether the indexes of elements x, y are in A or not then how many comparisons are needed?

He found the strategy ([2]) which gives the answer using $n+q-3$ comparisons if $1 \leqq p < q \leqq \left[\frac{n}{2}\right]$ and he conjuctured that this strategy is optimal.

In the recent paper [6] it is proved that the conjecture of G. O. H. Katona is true (see also [8]).

In a recent paper [10] the following cases were proved: If $A=\{r_1, r_2, \ldots, r_k\}$ $(1 \leqq r_1 < r_2 < \ldots < r_k = n)$ and $r_k = k$ $(r_k \neq n)$ then we need at least $n-1$ comparisons and if $r_k \neq k$, $r_k \leqq \left[\frac{n}{2}\right]$ then we need at least $n+r_k-3$ comparisons.

In this paper we prove the following:

If $A=\{1, q\}$ $(1 < q \leqq n)$ then the number of comparisons is at least

$$\min\left(n+q-3,\ 2n-\left[\frac{q}{2}\right]-2\right) \quad (n > 2).$$

2. Notations, definitions

Let x, y be arbitrary e'ements of the set H and let $A=\{1, q\}$ $(1 < q \leqq n)$.

Suppose that we want to decide whether the indexes of elements x, y are in the set A or not (that is the element x or y is the largest and the other the q-th)? The first pair to be compared is denoted by $S_0=(a, b)$. If the result of the comparison is $a > b$ then the value of the variable ε_1 is 1. In the opposite case $(a < b)$ $\varepsilon_1 = 0$.

The choice of the next pair $S_1(\varepsilon_1)$ depends on ε_1. Suppose $S_1(\varepsilon_1) = (c(\varepsilon_1), d(\varepsilon_1))$. Define ε_2 to be 1 if $c(\varepsilon_1) > d(\varepsilon_1)$ and to be 0 otherwise. Continuing this procedure

$$S_{i-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-1}) \tag{1}$$

is defined for some 0, 1 sequences $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-1}$ with the restriction that if $S_{i-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-1})$ is defined then $S_{i-2}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-2})$ is defined, too. The value of ε_i is 1 or 0 according to whether the first or the second member is larger. A set of questions given in this way will be called a strategy suitable for deciding the questions "whether the indexes of x, y are in A or not" iff for all sequences $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ when

(2) $S_{l-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{l-1})$ is determined, but

(3) $S_l(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$ is not

(4) $\begin{cases} \text{then the answers } \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l \text{ (together with the questions } S_0, S_1(\varepsilon_1), \ldots \\ \ldots, S_{l-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{l-1}) \text{ give a unique reply to the problem: the indexes of} \\ x, y \text{ are in } A \text{ or not.} \end{cases}$

We use the notation $\mathcal{S}(A)$ for such a strategy. We say that the strategy $\mathcal{S}(A)$ is finished for the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ if the conditions (2)—(4) are satisfied. The maximum length of the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ finishing the strategy is called its length. It will be denoted by $L(\mathcal{S}(A))$.

Denote by $T_i(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ the inequality set up from the pair $S_{i-1}(\varepsilon_1, \varepsilon_2, \ldots \ldots, \varepsilon_{i-1})$ on the basis of the answer ε_i. Now we can express condition (4) in a modified way: The inequalities

(5) $$T_1(\varepsilon_1), T_2(\varepsilon_1, \varepsilon_2), \ldots, T_l(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$$

uniquely decide whether the indexes of x, y are in A or not. The situation $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ of $\mathcal{S}(A)$ is the situation after answering the question $S_{i-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-1})$. That is, we have the inequalities $T_1(\varepsilon_1), T_2(\varepsilon_1, \varepsilon_2), \ldots, T_i(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ and denote them by E_i.

We will use the following fact several times.

If a system of inequalities E_i is given, then only those inequalities are consequences of them which are deducible by transitivity.

That is let

$$H = \{h_1, h_2, \ldots, h_n\}$$

be an ordered set:

$$h_1 < h_2 < \ldots < h_n.$$

A set of inequalities

(*) $$x_i < x_j \quad (i \neq j,\ i,\ j \in \{1, 2, \ldots, n\})$$

is given and we are looking for a solution of (*) in the set H. If the inequalities

$$x_{i_1} < x_{i_2}, x_{i_2} < x_{i_3}, \ldots, x_{i_{k-1}} < x_{i_k} \quad (1 < k)$$

are all in (*), then $x_{i_1} < x_{i_k}$ is deducible from (*). The set of inequalities deducible from (*) is called the extension of (*). (*) is non-contradictory if there is no inequality $x_i < x_i$ in the extension of (*).

Lemma 0. *If (*) is non-contradictory and the inequalities $x_v < x_w$ and $x_o > x_w$ are not in the extension of (*), then there is a solution of (*) in H satisfying $x_v > x_w$ and such a solution too for which $x_v < x_w$.*

In this paper we dont prove Lemma 0. The proof of Lemma 0, can be found in [11].

We now introduce the concept of graph — realization. Let the elements of a set H correspond to the vertices of a graph $\vec{G}$. Let a comparison be an edge of $\vec{G}$ between the corresponding vertices. Let the answer be the orientation of this edge in the following way: if we compare two elements — say c and d — in some state and the result of the comparison is $c > d$ then we direct the edge from c to d, conversely, when $c < d$ we direct the edge from d to c.

In the state $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i)$ let $\vec{G}^i$ denote the graph derived in this way. By the above correspondence we uniquely associate an oriented graph to all states of $\mathcal{S}(A)$. It follows from the correspondence that in an arbitrary state $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i)$ of $\mathcal{S}(A)$ the relation $a>b$ is realised if and only if an oriented path leads in $\vec{G}^i$ from a to b. Denote by G^i the graph obtained from $\vec{G}^i$ by cancelling the direction of edges.

3. The result

Theorem. *If $A=\{1, q\}$ and the set $\mathcal{C}$ is the strategy which is able to decide whether the indexes of elements x, y are in a set A or not (that is the element x or y is the largest and the other is the q-th in H), then*

$$\min_{\mathcal{S}(A)\in\mathcal{C}} L(\mathcal{S}(A)) = \min\left\{n+q-3,\ 2n-\left[\frac{q}{2}\right]-2\right\} \tag{6}$$

if $n>2$.

Proof of the theorem.

We distinguish two cases: 1. $n>\left[\frac{3q}{2}\right]-1$, 2. $n\leqq\left[\frac{3q}{2}\right]-1$.

1. Suppose that $n>\left[\frac{3q}{2}\right]-1$.

We can easily find a strategy $\mathcal{S}(A)$ which in at most $n+q-3$ steps gives the answer to the question. This strategy is due to G. O. H. Katona. Let $S_0=(x, y)$. We can suppose that $x>y$. We compare all the elements $H\setminus\{x, y\}$ with y.

If the element y is larger in $n-q$ cases and smaller in $q-2$ cases then y is the q-th element in H.

After this we compare the element x with the elements which are larger than y. If the element x is larger in these comparisons, then the element x is the largest in H. The number of comparisons is $n+q-3$.

2. Suppose that $n\leqq\left[\frac{3q}{2}\right]-1$.

We determine the strategy $\mathcal{S}(A)$ which gives the answer to the question in at most $2n-\left[\frac{q}{2}\right]-2$ steps. We compare the elements x, y with the elements from $H\setminus\{x, y\}$ until for some element from $H\setminus\{x, y\}$ — say a — $x>a>y$ or $x<a<y$ holds. Such element obviously exists if the indexes of x, y are in A.

We can suppose that $x>a>y$.

If $x>a>y$ holds and in this situation the number of those elements which are smaller than y is $n-q$, then from the elements not occurring until now we determine the largest and the smallest element and finally we compare x with the largest, y with the smallest element and we get the answer to our question.

Suppose that whenever $x>a>y$, the number of those elements which are smaller than y is smaller than $n-q$. If there exist two elements which have not occurred yet — say c, d — then we compare these two elements. We can suppose that $c>d$. We compare the element y with the element d. If $y>d$ then we compare c with the element not yet occurred — say e — and the smaller element with y etc. If there is no such an element e then we compare c with y and in case $y<c$ we compare c with x, too. If $y<d$, then we compare c with x. We can suppose that $x>c$.

If there exist two elements which have not occurred yet — say f, g — then we compare these two elements. We can suppose that $f<g$. Compare y with f. If $y>f$, then we compare g with the element not yet occurred and the smaller with y etc. If $y<f$ then we compare x with g. If there exist two elements which have not occurred until.... In this way we give the answer to our question and we have done at most

$$2(n-q)+q-1+\left\lceil\frac{q-3}{2}\right\rceil = 2n-q+\left\lceil\frac{q-1}{2}\right\rceil-2 = 2n-\left\lceil\frac{q}{2}\right\rceil-2$$

comparisons.
We have proved by this that

$$\min_{\mathcal{S}(A)\in\mathcal{C}} L(\mathcal{S}(A)) \leqq \min\left(n+q-3,\ 2n-\left\lceil\frac{q}{2}\right\rceil-2\right).$$

It remains to prove that

$$L(\mathcal{S}(A)) \geqq \min\left(n+q-3,\ 2n-\left\lceil\frac{q}{2}\right\rceil-2\right) \tag{7}$$

holds for any strategy $\mathcal{S}(A)$.

This will be done in the following way.

An algorithm will be given which determines a branch of the strategy, that is, a sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ finishing it.
This branch will have a length

$$\geqq \min\left(n+q-3,\ 2n-\left\lceil\frac{q}{2}\right\rceil-2\right).$$

The algorithm determines the ε's recursively.
Partitions of $H-\{x, y\}$ will be used. These partitions will also be defined recursively, for any situation $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)$ along the indicated branch. The branch and the partitions will be determined simultaneously. A partition has 5 classes: $B_1^i, B_2^i, B_3^i, B^i, C^i$.

The heuristic meaning of the classes is:

$B_1^i\cup B_2^i\cup B^i$: the set of elements which will be greater than the element $y(x)$ and smaller than $x(y)$;

B_3^i: the set of elements which will be smaller than both x and y;

C^i: the set of elements yielding essential information in the given situation.

At the beginning $C^0=H\setminus\{x, y\}$, $B_1^0, B_2^0, B_3^0, B^0=\emptyset$. Suppose that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i$ and $B_1^i, B_2^i, B_3^i, B^i, C^i$ are already defined.

The next description determines ε_{i+1} and $B_1^{i+1}, B_2^{i+1}, B_3^{i+1}, B^{i+1}, C^{i+1}$.

Let $S_i(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)=(g, h)$.

The new values of ε_{i+1}, B_1^{i+1}, etc. will depend on the classes containing g and h, resp. The classes which can be obtained by interchanging the role of g and h will not be treated separately. As regards B_1^{i+1}, etc. we will indicate the new class only for an element.

Then it will be obviously omitted from its old class. We can suppose that the element y has not occurred as x before.

Suppose that the element x first occurs in $S_j(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j)$. Let

$$S_j(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j) = (x, c).$$

We distinguish two cases:

Case I. $c \in B_1^j \cup B_2^j \cup B^j$. In this case $x < c$ and if $c \in B^j$ then $c \in B_1^{j+1}$.

Case II. $c \in \{y\} \cup B_3^j \cup C^j$. In this case $x > c$ and if $|B_3^j| < n-q$, $c \in C^j$ then $c \in B_3^{j+1}$ and if $|B_3^j| = n-q$, $c \in C^j$ then $c \in B_2^{j+1}$.

We define ε_{i+1}.

1. $g, h \in C^i$	$\varepsilon_{i+1} = 1$ and if $\lvert B_1^i \cup B_2^i \cup B^i \rvert < q-2$, $\lvert B_3^i \rvert < n-q$ then $h \in B_3^{i+1}, g \in B^{i+1}$; if $\lvert B_3^i \rvert = n-q$ then $g \in B_1^{i+1}, h \in B_2^{i+1}$; if $\lvert B_1^i \cup B_2^i \cup B^i \rvert = q-2$ then $g, h \in B_3^{i+1}$.
2. $g \in B^i, h \in C^i$	$\varepsilon_{i+1} = 1$ and if $\lvert B_3^i \rvert < n-q$ then $h \in B_3^{i+1}$; if $\lvert B_3^i \rvert = n-q$ then $h \in B_2^{i+1}, g \in B_1^{i+1}$.
3. $g, h \in B^i$	$\varepsilon_{i+1} = 1, g \in B_1^{i+1}, h \in B_2^{i+1}$.
4. $g \in B_r^i, h \in B_s^i$ $(r < s;\ r, s \in \{1, 2, 3\})$	$\varepsilon_{i+1} = 1$.
5. $g \in B_3^i, h \notin B_3^i$	$\varepsilon_{i+1} = 0$ and if $h \in C^i, \lvert B_1^i \cup B_2^i \cup B^i \rvert < q-2$ then $h \in B^{i+1}$; if $h \in C^i, \lvert B_1^i \cup B_2^i \cup B^i \rvert = q-2$ then $h \in B_3^{i+1}$.
6. $g \in B_1^i, h \in B^i \cup C^i$	$\varepsilon_{i+1} = 1$, and if $\lvert B_3^i \rvert < n-q, h \in C^i$ then $h \in B_3^{i+1}$; if $\lvert B_3^i \rvert = n-q, h \in C^i$ then $h \in B_2^{i+1}$; if $h \in B^i$ then $h \in B_2^{i+1}$.
7. $g \in B_2^i, h \in B^i \cup C^i$	if $\lvert B_3^i \rvert < n-q, h \in C^i$ then $\varepsilon_{i+1} = 1, h \in B_3^{i+1}$; if $\lvert B_3^i \rvert = n-q, h \in C^i$ then $\varepsilon_{i+1} = 0, h \in B_1^{i+1}$; if $h \in B^i$ then $\varepsilon_{i+1} = 0, h \in B_1^{i+1}$.
8. $g, h \in B_r^i$ $(r \in \{1, 2, 3\})$	$\varepsilon_{i+1} =$ arbitrary, except if it is determined by the extension of ε_i.
9. $g = x, h = y$	in the case I.: $x < y$; in the case II.: $x > y$.

10. $g = x,\ h \in C^i$ in the case I.:

if $|B_1^i \cup B_2^i \cup B^i| < q-2$ then $\varepsilon_{i+1} = 0,\ h \in B_1^{i+1}$;

if $|B_1^i \cup B_2^i \cup B^i| = q-2$ then $\varepsilon_{i+1} = 1,\ h \in B_3^{i+1}$;

in the case II.: $\varepsilon_{i+1} = 1$ and

if $|B_3^i| < n-q$ then $h \in B_3^{i+1}$;

if $|B_3^i| = n-q$ then $h \in B_2^{i+1}$.

11. $g = x,\ h \in B_1^i \cup B_2^i \cup B^i$ in the case I.: $\varepsilon_{i+1} = 0$ and if $h \in B^i$ then $h \in B_1^{i+1}$;

in the case II.: $\varepsilon_{i+1} = 1$ and if $h \in B^i$ then $h \in B_2^{i+1}$.

12. $g = y,\ h \in C^i$ in the case I.: $\varepsilon_{i+1} = 1$ and

if $|B_3^i| < n-q$ then $h \in B_3^{i+1}$;

if $|B_3^i| = n-q$ then $h \in B_2^{i+1}$.

in the case II.:

if $|B_1^i \cup B_2^i \cup B^i| < q-2$ then $\varepsilon_{i+1} = 0,\ h \in B_1^{i+1}$;

if $|B_1^i \cup B_2^i \cup B^i| = q-2$ then $\varepsilon_{i+1} = 1,\ h \in B_3^{i+1}$.

13. $g=y,\ h \in B_1^i \cup B_2^i \cup B^i$ in the case I.: $\varepsilon_{i+1}=1$ and if $h \in B^i$ then $h \in B_2^{i+1}$ in the case II.: $\varepsilon_{i+1}=0$ and if $h \in B^i$ then $h \in B_1^{i+1}$.

In this way we have defined the value of ε_{i+1} and the sets $B_1^{i+1}, B_2^{i+1}, B_3^{i+1}, B^{i+1}, C^{i+1}$.

It is worth-while to show in a figure the possible changes of elements among the classes (see Fig. 1).

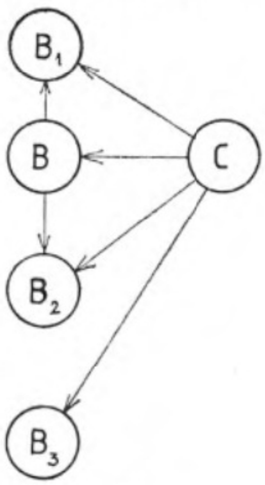

Fig. 1.

Suppose that the strategy $\mathcal{S}(A)$ is finished for the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$. It will be denoted by $P(A)$. The length $|P(A)|$ of $P(A)$ is l. We shall prove

$$l \geqq \min\left(n+q-3,\ 2n-\left[\frac{q}{2}\right]-2\right).$$

We can easily see that if the strategy $\mathcal{S}(A)$ is finished for the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$ then case I. or case II. holds. If case I. has resulted then largest element in H is y and the q-th is x, and if case II. has resulted the largest element in H is x and the q-th is y.

From definition $P(A)$ it follows that if some element — say b — $\in B_r^i$ $(r\in\{1,2,3\})$ then $b\in B_r^j$ $(j>i)$ and if $b\in B^i$ then $b\in B_1^l\cup B_2^l$.

From definition $P(A)$ it follows, too, that if $b>a\in\mathcal{E}_l$ and $b\in B_r^l$, $a\in B_s^l$ $(r\neq s)$ then $r<s$ $(r,s\in\{1,2,3\})$ and the elements of $B_1^l\cup B_2^l$ are smaller than $y(x)$ and larger than $x(y)$ and the elements of B_3^l are smaller than x, y.

Suppose that $|B_1^i\cup B_2^i\cup B^i|<q-2$, $|B_3^i|<n-q$ and that either $|B_1^{i+1}\cup B_2^{i+1}\cup B^{i+1}|=q-2$ or $|B_3^{i+1}|=n-q$ holds. We prove the following Lemma:

Lemma 1. *If $|B_3^{i+1}|=n-q$, then the elements of B_3^{i+1} occur at least twice as smaller ones in $\mathcal{E}_l$.*

If $|B_1^{i+1}\cup B_2^{i+1}\cup B^{i+1}|=q-2$ then in case I. *(in case* II.*) the elements of $B_1^l\cup B_2^l$ occur with at least one element from $\{x\}\cup B_3^l$ $(\{y\}\cup B_3^l)$ in $\mathcal{E}_l$.*

PROOF. We prove the first half of the statement. Suppose that $|B_3^i|<n-q$, $|B_1^i\cup B_2^i\cup B^i|<q-2$ and $|B_3^{i+1}|=n-q$ hold.

Let — say — a be an arbitrary element in B_3^{i+1}. Suppose that the element a first occurs in the $S_j(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j)$. Let

$$S_j(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j) = (a, b) \quad (j < i+1).$$

From definition $P(A)$ it follows that $a\in C^j$, $\varepsilon_{j+1}=0$ (because $a\in B_3^{j+1}$). Thus we used 1., 2., 6., 7., 10. or 12.

If we used 1., 2., 6. or 7. then $b\in B_1^{j+1}\cup B_2^{j+1}\cup B^{j+1}$ and $b\in B_1^l\cup B_2^l$. It follows from this that the element a occurs in $\mathcal{E}_l$ as a smaller one with an element from $\{x\}\cup B_3^l$ (in case I.) or with an element from $\{y\}\cup B_3^l$ (in case II.), that is the statement holds. If we use 10,, then x has already occured and in its first occurance with some element, say e, x is greater than e. On the basis of $\mathcal{E}_l$ $a<y$ because the element a occurs in $\mathcal{E}_l$ with an element from $\{y\}\cup B_3^l$ and the element a is smaller in this inequality.

The statement follows from this.

If we use 12. then — as we can easily see — the statement again holds.

With this we proved the first half of the Lemma.

Now we prove the other half.

Suppose that $|B_3^i|<n-q$, $|B_1^i\cup B_2^i\cup B^i|<q-2$ and $|B_1^{i+1}\cup B_2^{i+1}\cup B^{i+1}|=q-2$.

Let — say — a be an arbitrary element in $B_1^l\cup B_2^l$. Suppose that the element a first occurs in the $S_j(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j)$. Let

$$S_j(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j) = (a, b).$$

From the definition of $P(A)$ it follows that $a\in C^j$, $\varepsilon_{j+1}=1$ (because $a\in B_1^{i+1}\cup \cup B_2^{i+1}\cup B^{i+1}$). Now we use 1., 2., 5., 6., 7., 10. or 12.

If we use 1. then $a>b$, $b\in B_3^{j+1}$, $b\in B_3^l$. We can easily see that the use of cases 2., 6., 7. is impossible. If we use 5. then $b\in B_3^j$, $b\in B_3^l$. If we use 10. then $b=x$, $\varepsilon_{j+1}=1$ (because $a\in B_1^l\cup B_2^l$).

This implies that the case I. holds and $a\in B_1^{j+1}, a\in B_1^l$. We can similarly prove the statement when we use 12. With this the Lemma is proved.

Finally we prove (7).

We distinguish two cases: 1. $|B_3^{i+1}|=n-q$; 2. $|B_1^{i+1}\cup B_2^{i+1}\cup B^{i+1}|=q-2$.

1. If $|B_3^{i+1}|=n-q$ ($|B_3^i|<n-q$, $|B_1^i\cup B_2^i\cup B^i|<q-2$), then $B_3^{i+1}=B_3^l$ and — according to Lemma 1. — the elements of B_3^l occur in $\mathscr{E}_l$ at least $2(n-q)$ times as a smaller one.

We can suppose that the element x is the largest and y is the q-th in H. (If the element y is the largest and x is the q-th then we can prove the statement similarly.) From definition $P(A)$ it follows that $B^l\cup C^l=\emptyset$. Consider the inequalities between the elements of the set $\{x, y\}\cup B_1^l\cup B_2^l$.

We can easily see that if $a\in B_1^l(B_2^l)$ then there exists an inequality $a>b$ $(a<b)$ in $\mathscr{E}_l$ with $b\in\{y\}\cup B_2^l$ $(b\in\{x\}\cup B_1^l)$.

Suppose that $a\in B_1^l$ and the element a first occurs in $S_j(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j)$. Let $S_j(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j)=(a, b)$. By the definition of $P(A)$ we have $a\in C^j$, $\varepsilon_{j+1}=1$. If $j<i+1$, $b\neq y$ then we use 1., 5. (supposing $x>y$ and $a\in B^{j+1}$, $b\in B_3^{j+1}$.

Later the element a occurs with the element e from $\{y\}\cup B\cup B_2$: in S_j as a larger one because $a\in B_1^l$ and $e\in\{y\}\cup B_2^l$.

If $j<i+1$ and $b=y$ then we use 12. and $a\in B_1^{j+1}$, $a\in B_1^l$.

If $j\geqq i+1$ then we use 1., 5., 7. or 12. If we used 1., 7., 12. then $a\in B_1^{j+1}$, $b\in\{y\}\cup B_2^{j+1}$. If we used 5. then $b\in B_3^j$, $a\in C^j$, $a\in B^{j+1}$ and later the element a occurs with the element e from $\{y\}\cup B_2\cup B$ and $a>e$ because we supposed that $a\in B_1^l$ therefore the element e is in $\{y\}\cup B_2^l$.

With this we have proved that if $a\in B_1^l$ then there exists an inequality $a>b$ in $\mathscr{E}_l$ with $b\in\{y\}\cup B_2^l$. We can prove similarly that if $a\in B_2^l$ then there exists an inequality $a<b$ in $\mathscr{E}_l$ with $b\in\{x\}\cup B_1^l$. With this the proof of the statement is finished. We prove that the element x occurs with an element from $\{y\}\cup B_2^l$ or the element y occurs with an element from $\{x\}\cup B_1^l$ in $\mathscr{E}_l$ (we supposed $x>y$). Suppose the element y does not occur with an element from $\{x\}\cup B_1^l$. We prove that in this case the element x occurs with an element from $\{y\}\cup B_2^l$ in $\mathscr{E}_l$. Let $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{j+1})$ be the first statement in which the element x occurs with an element from $B_1^{j+1}\cup B_2^{j+1}\cup B^{j+1}\cup\{y\}$ in $\mathscr{E}_{j+1}$. Let

$$S_j(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j) = (x, b)$$

and — because we supposed the element x to be the largest — $x>b$. From the definition of $P(A)$ it follows that

$$b\in B_1^j\cup B_2^j\cup B^j\cup C^j\cup\{y\}, \; b\in B_2^{j+1}\cup\{y\}.$$

With this the statement is proved.

From these it follows that the number of inequalities $a>b$ in $\mathscr{E}_l$ in which $a\in B_1^l\cup\{x\}$, $b\in B_2^l\cup\{y\}$ is at least $\left[\frac{q-1}{2}\right]$ because $|B_1^l\cup\{x\}|+|B_2^l|=q-1$.

Consider the graph G^l. We can easily see that the subgraph induced by $\{x\}\cup B_1^l$ $(\{y\}\cup B_2^l)$ is connected.

Consequently there are at least $|B_1^l|$ $(|B_2^l|)$ edges among the vertices in $\{x\}\cup B_1^l$ $(\{y\}\cup B_2^l)$.

From these it follows that the number of inequalities in $\mathcal{E}_l$ is at least

$$2(n-q)+\left[\frac{q-1}{2}\right]+q-2 = 2n-\left[\frac{q}{2}\right]-2.$$

2. Suppose that $|B_1^i\cup B_2^i\cup B^i|<q-2$, $|B_3^i|<n-q$ and $|B_1^{i+1}\cup B_2^{i+1}\cup B^{i+1}|=$ $=q-2$. (We can suppose that x is the largest element in H.)

Suppose that the element x first occurs in $S_j(\varepsilon_1, \varepsilon_2, ..., \varepsilon_j)$. Let

$$S_j(\varepsilon_1, \varepsilon_2, ..., \varepsilon_j) = (x, c).$$

Because $x>c$ it follows that the case II. holds and $c\in\{y\}\cup B_3^{j+1}$.

It follows from Lemma 1. that if $a\in B_1^l\cup B_2^l$ then there exists an inequality $a>b$ in $\mathcal{E}_l$ with $b\in B_3^l\cup\{y\}$. This implies that the number of inequalities $a>b$ in $\mathcal{E}_l$ with $a\in\{x\}\cup B_1^l\cup B_2^l$, $b\in\{y\}\cup B_3^l$ is at least $q-1$. We can easily see that the subgraph induced by $\{x\}\cup B_1^l\cup B_2^l$ $(\{y\}\cup B_3^l)$ is connected.

Consequently there are at least $|B_1^l\cup B_2^l|$ $(|B_3^l|)$ edges among the vertices in $\{x\}\cup B_1^l\cup B_2^l$ $(\{y\}\cup B_3^l)$. From these it follows that the number of inequalities in $\mathcal{E}_l$ is at least

$$q-1+|B_1^l\cup B_2^l|+|B_3^l| = n+q-3$$

since $|B_1^l\cup B_2^l|+|B_3^l|=n-2$. From the cases 1. and 2. it follows, that

$$l \geqq \min\left(n+q-3,\ 2n-\left[\frac{q}{2}\right]-2\right).$$

With this the proof of our theorem complete.

Acknowledgement. Finally the author would like to thank professor G. O. H. Katona for his valuable remarks.

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(Received June 2, 1984, in a reviewed form on February 20, 1989.)