Almost additive functions on Ehresmann groupoids

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ROMAN GER has proved in [1] the theorem on almost additive functions on semigroups. In this paper we generalize this result to some substructures of an Ehresmann groupoid. A generalization of the notion of a group which is an Ehresmann groupoid, resulted from geometrical considerations, but it is also used in other branches of mathematics (see [3], [4]).

Let X be an arbitrary set and "+" an arbitrary operation on X, i.e. $+: D\to X$, $D\subset X\times X$[1]. A pair $(A; +)$ is called an Ehresmann groupoid iff the following conditions are satisfied

G1. If in the equation $x+(y+z)=(x+y)+z$ one of its sides is defined (i.e. $(x, y+z)\in D$ or $(x+y, z)\in D$) or both the products $y+z$ and $x+y$ are defined (i.e. (y, z), $(x, y)\in D$) then both sides of the equation are defined and the equality holds.

G2. To every element x there exist elements e_x^-, e_x^+, called the left unit and the right unit of x, respectively, such that (e_x^-, x), $(x, e_x^+)\in D$ and equalities $e_x^-+x=x$ and $x+e_x^+=x$ hold.

G3. To every element x there exists an inverse element $-x$ such that $(-x, x)$, $(x, -x)\in D$, $-x+x=e_x^+$ and $x+(-x)=e_x^-$.

An Ehresmann groupoid is called a Brandt groupoid iff the following condition holds

G4. To every two elements x, y there exists an element z such that $(x, z)\in D$ and $(z, y)\in D$.

Example 1. Let A be an arbitrary set and G an arbitrary group. Let in the sets $A\times A\times G$ and $A\times A$ an operation "+" be defined as follows: $((x, y, \alpha), (u, v, \beta))\in D$ and $((x, y), (u, v))\in D$ iff $y=u$. Then $(x, y, \alpha)+(u, v, \beta)=(x, v, \alpha\beta)$ and $(x, y)+$ $+(u, v)=(x, v)$. Groupoids $(A\times A\times G; +)$ and $(A\times A; +)$ are called the product groupoid and the pair groupoid, respectively. They are Brandt groupoids.

The following two theorems are known Every Brandt groupoid is isomorphic to some product groupoid (see [4], p. 11).

Every Ehresmann groupoid is a disjoint union of Brandt groupoids (see [5], p. 11).

Let $(X, +)$ and $(Y, +)$ be arbitrary Ehresmann groupoids. Let D_X and D_Y be the domains of operations "+" in X and Y, respectively. A function $f\colon X\to Y$ is said to

[1] We shall use a symbol "+" instead of "·" as usual in groupoid because of the tradition of the problem.

be a homomorphism iff for every $x, y \in X$ if $(x, y) \in D_X$ then $(f(x), f(y)) \in D_Y$ and $f(x+y)=f(x)+f(y)$. We shall prove the lemma

Lemma 1. *Let X, Y be Ehresmann groupoids and let a function $f: X \to Y$ be a homomorphism. If X and Y are disjoint unions families A_i, $i \in I$ and B_t, $t \in T$ of Brand's groupoids, respectively then for every $i \in I$ there exists $t \in T$ such that $f(A_i) \subset B_t$.*

PROOF. Let $x, y \in A_i$. If $(x, y) \in D_X$ then, because f is a homomorphism, $(f(x), f(y)) \in D_Y$ whence there exists $t \in T$ such that $f(x), f(y) \in B_t$. If $(x, y) \notin D_X$ there exists $z \subset A_i$ such that (x, z), $(z, y) \in D_X$. So $(f(x), f(z))$, $(f(z), f(y)) \in D_Y$ whence there exist s, $t \in T$ such that $f(x), f(z) \in B_t$ and $f(z), f(y) \in B_s$. Thus, because $B_t \cap B_s \neq \emptyset$, we have $s=t$. So there exists $t \in T$ such that $f(x)$, $f(y) \in B_t$.

Due to Lemma 1 the problem of the almost additive function on the Ehresmann groupoid can be reduced to the problem of almost additive function on the Brandt groupoid.

We shall use also the following properties of groupoid

G5. $e_x^- = e_{-x}^+$ and $e_x^+ = e_{-x}^-$.

G6. If $(x, z) \in D$ then $(-z, -x) \in D$ and $-(x+z) = -z+(-x) = -z-x$.

G7. If $e_x^+ = e_y^- = e_z^-$ then $(x, z) \in D$ and $x+z = x+e_x^+ + z = x+(y-y)+z = (x+y)-$ $-(z+y)$.

G8. If $x=y+z$ then $x-z=y$ and $-y+x=z$.

Put moreover $W_x^- = \{y \in X: e_x^- = e_y^-\}$ and $W_x^+ = \{y \in X: e_x^+ = e_y^+\}$.

Let $(X, +)$ be a group. A non-empty family $\mathscr{I}$ of subsets of X is called a proper linearly invariant ideal (p.l.i. ideal) in X iff satisfying the following conditions

I1. $\mathscr{I}$ is a proper ideal in X (i.e. if $A \in \mathscr{I}$ and $B \subset A$ then $B \in \mathscr{I}$; and if $A, B \in \mathscr{I}$ then $A \cup B \in \mathscr{I}$ and $X \notin \mathscr{I}$).

I2. If $A \in \mathscr{I}$ then $-A \in \mathscr{I}$.

I3. If $A \in \mathscr{I}$ and $x \in X$ then $x+A \in \mathscr{I}$.

Instead of a conjunction of conditions I2. and I3. it is used the equivalent condition

I4. If $A \in \mathscr{I}$ and $x \in X$ then $x-A \in \mathscr{I}$

(see [2], p. 437). If $(X, +)$ is a Brandt groupoid then it is easy to prove that the condition I4. is necessary for this conjunction but it is not sufficient. It can be seen from the following example

Example 2. Let K be an infinite set, let a be an arbitrary element of K and let a family $\mathscr{I}_1$ be an ideal in K containing infinite sets. Consider the pair groupoid $X=K \times K$ the family $\mathscr{I}_2$ of finite subsets of X and a set $\{a\} \times A$ for an infinite $A \in \mathscr{I}_1$. The smallest (in the inclusion sense) ideal $\mathscr{I}$ containing the family $\mathscr{I}_2$ satisfies the condition I4. but does not satisfy neither I2. nor I3.

By a p.l.i. ideal in a groupoid X we shall mean a non-empty family I of subsets X satisfying conditions I1.—I3.

We say that a condition is satisfied $\mathscr{I}$-almost everywhere in X (written $\mathscr{I}$-(a.e.) in X) where $\mathscr{I}$ is a p.l.i. ideal in X iff there exists a set $A \in \mathscr{I}$ such that the condition in question is satisfied for every $x \in X \setminus A$.

Ideals $\mathscr{I}_1$ in X and $\mathscr{I}_2$ in $X \times X$ are called the conjugate ideals iff for every set $A \in \mathscr{I}_2$ we have $A[x] := \{y \in X: (x, y) \in A\} \in \mathscr{I}_1$, $\mathscr{I}_1$-(a.e.) in X, i.e., iff there exists a set $U \in \mathscr{I}_1$ such that $A[x] \in \mathscr{I}_1$ for $x \in X \setminus U$.

Consider now a Brandt groupoid $(X, +)$, a p.l.i. ideal $\mathscr{I}$ and a subset $S\subset X$ satisfying the following conditions

S1. $S+S\subset S$.
S2. $S-S=X$.
S3. For every $s, t\in X$ if $e_s^-=e_t^-$ then $(s+S)\cap(t+S)\notin\mathscr{I}$.

Example 3. Let $X=R\times R$ and $\mathscr{I}$ be a family of bounded sets. The set $S=\{(x, y)\colon y\leqq x\}$ satisfies the conditions S1.—S3.

Example 4. Let $X=K\times K$ be an infinite pair groupoid, A be an infinite subset of K and $\mathscr{I}$ be a family of finite subsets. The set $S=K\times A$ satisfies the Conditions S1.—S3.

Now, we shall prove three lemmas analogous to those proved in [1] reasoning in the similar way.

Lemma 2. *Let $(X, +)$ be a Brandt groupoid, $\mathscr{I}$ be a p.l.i. ideal in X and S be a subset of X fulfilling the conditions* S1.—S3. *Then, for every sets $U, V\in\mathscr{I}$, we have*

$$(S\setminus U)-(S\setminus V) = X.$$

Proof. Take an $x\in X$ and sets $U, V\in\mathscr{I}$. By S2. there exist $s, t\in S$ such that $x=s-t$. Then $e_s^+=e_t^+$, whence $e_{-s}^-=e_{-t}^-$ and, by S3., we have

$$A = [-s+(S\setminus U)]\cap[-t+(S\setminus V)] = [(-s+S)\cap(-t+S)]\setminus[(-s+U)\cup(-t+V)]\notin\mathscr{I}.$$

Thus, in particular, $A\neq\emptyset$. Consequently there exists $y\in A$ which means that $y\in -s+(S\setminus U)$ and $y\in -t+(S\setminus V)$ whence $e_y^-=e_s^+=e_t^+$, $(s, y)\in D$, $(t, y)\in D$, $s+y\in S\setminus U$ and $t+y\in S\setminus V$. We have

$$x = s-t = s+e_y^- -t = s+(y-y)-t = (s+y)-(t+y)\in(S\setminus U)-(S\setminus V),$$

which completes the proof, because the converse inclusion is evident.

Lemma 3. *Under the conditions of Lemma 2 for every $U\in\mathscr{I}$ and every $u, x, y\in S\setminus U$ if $e_x^+=e_y^+$ and $e_u^-=e_y^-$ then there exist $s, t\in S\setminus U$ such that $t\in u+S$ and $s-t=x-y$.*

Proof. Taking in Lemma 2. $V=-u+U$ we obtain $(S\setminus U)-[S\setminus(-u+U)]=X$. This means that for every $z\in X$ there exist $w\in S\setminus U$ and $v\in S\setminus(-u+U)$ such that $z=w-v$. Consider an expression $x-y+u$. By our conditions it is sensible. Take $z:=x-y+u$ and choose a suitable v. Since $(z, v)\in D$, we can put $s:=z+v=x-y+u+v$ and $t:=u+v$. It is evident that $s\in S\setminus U$, $t\in u+S$ and $s-t=x-y$. On the other hand for arbitrary $V\subset X$ we have $u+V\subset W_u^-$ and $W_u^-\cap U=u-u+U$ whence

$$t = u+v\in u+[S\setminus(-u+U)] = \big(u+[S\setminus(-u+U)]\big)\cap W_u^- =$$
$$= [(u+S)\setminus(u-u+U)]\cap W_u^- = [(u+S)\cap W_u^-]\setminus(U\cap W_u^-)\subset(S\cap W_u^-)\setminus(U\cap W_u^-) =$$
$$= (S\setminus U)\cap W_u^- \subset S\setminus U.$$

Thus $t\in S\setminus U$.

Let $\mathscr{I}_1$ and $\mathscr{I}_2$ be two conjugate p.l.i. ideals in X and in $X\times X$, respectively. Let, moreover, a set $S\subset X$ fulfill conditions S1.—S3. and let $f\colon S\to Y$ be a function such that

$$f(x+y)=f(x)+f(y) \quad \mathscr{I}_2\text{-(a.e.) in } (S\times S)\cap D \tag{1}$$

i.e. $M:=\{(x,y)\in(S\times S)\cap D\colon f(x+y)\neq f(x)+f(y)\}\in\mathscr{I}_2$. Whence, because $\mathscr{I}_1$ and $\mathscr{I}_2$ are conjugate, there exists a set $U\in\mathscr{I}_1$ such that $M[x]=\{y\in X\colon (x,y)\in M\}\in\mathscr{I}_1$ for $x\notin U$.

Lemma 4. *For every* $x,y,u,v\in S\setminus U$ *the equality* $x-y=u-v$ *implies* $f(x)-f(y)=f(u)-f(v)$.

Proof. Take $x,y,u,v\in S\setminus U$. If $x-y=u-v$ then $e_y^-=e_v^-$ whence $(-v,y)\in D$. We have also $-v+v+S\subset S$ and by S3.

$$(-v+y+S)\cap S\supset(-v+y+S)\cap(-v+v+S)\notin\mathscr{I}_1.$$

On the other hand, because $x,y,u,v\notin U$, $M[x]$, $M[y]$, $M[u]$, $M[v]\in\mathscr{I}_1$, whence $((-v+y+M[x])\cup M[u]\in\mathscr{I}_1$ and $M[v]\cup(-v+y+M[y])\in\mathscr{I}_1$. So

$$A=(-v+y+(S\setminus M[x]))\cap(S\setminus M[u])\setminus(M[v]\cup(-v+y+M[y]))=$$

$$=([(-v+y+S)\cap S]\setminus[(-v+y+M[x])\cup M[u]])\setminus[M[v]\cup(-v+y+M[y])]\notin\mathscr{I}_1$$

and, in particular, $a\neq\emptyset$. Thus there exists $s\in A$ such that the expression $-y+v+s$ is sensible so we can define $z:=-y+v+s$. The elements z and s fulfill the following conditions

$$s\in S,\ s\notin M[u],\ s\notin M[v],\ z\in S,\ z\notin M[x],\ z\notin M[y],$$

whence $(u,s)\notin M$, $(v,s)\notin M$, $(x,z)\notin M$, $(y,z)\notin M$, thus because

$$x+z=x-y+v+s=u-v+v+s=u+s,$$

we have

$$f(x)+f(z)=f(x+z)=f(u+s)=f(u)+f(s). \tag{2}$$

Similarly, because $y+z=v+s$, we have

$$f(y)+f(z)=f(v)+f(s). \tag{3}$$

Hence by (2) and (3) $f(x)-f(y)=f(u)-f(v)$.

Now we shall formulate the main theorem, analogous also to that proved in [1]. We shall not prove it because due to Lemmas 2—4 and the sensibility of expressions guaranteed in the assumptions this proof is not different from the Ger's one.

Theorem. *Let* $(X,+)$ *and* $(Y,+)$ *be arbitrary groupoids and let* $\mathscr{I}_1$ *and* $\mathscr{I}_2$ *be conjugate p.l.i. ideals in* X *and in* $X\times X$, *respectively. Further on, let a set* $S\subset X$ *fulfill conditions* S1.—S3. *If* $f\colon S\to Y$ *is a function fulfilling* (1) *then there exists the unique homomorphism* $g\colon X\to Y$ *such that*

$$g|_S=f \quad \mathscr{I}_1\text{-(a.e.) in } S. \tag{4}$$

In a group condition S3. and condition $S\notin\mathscr{I}$ are equivalent (see [2] p. 441). However in a groupoid the condition $S\notin\mathscr{I}$ is necessary for the Condition S3. but it is not sufficient. We can show it taking an infinite groupoid X, an ideal of finite subsets of X and a subset $S=W_x^+$ for a certain $x\in X$. Condition S3. cannot be replaced in the theorem by the condition $S \notin \mathcal{I}$. It can be seen from the following examples

Example 5. Let $X=R^+ \times R^+$ $(R^+=\{x\in R\colon x>0\})$. Let an ideal $\mathcal{I}$ be generated by the family $\{W_x^+ \cup W_x^-\}_{x\in X}$. Further on, let $S=\{(x, y)\colon y\leqq x\}$. Then $S\notin\mathcal{I}$ but condition S3. does not hold. Let $a, b\in R^+$ and $a\neq b$. An identity $f((x, y))= =(x, y)$ for $x, y\in S$ such that $x, y\neq a$ fulfils the condition (1). Of course the identity on X is an extension of f, but also the homomorphism $g\colon X\to X$ defined by

$$g((x, y)) = \begin{cases} (x, y) & \text{for} \quad x, y \neq a, \\ (b, y) & \text{for} \quad x = a, \\ (x, b) & \text{for} \quad y = a, \end{cases}$$

is an extension of f.

Example 6. Let $X=R\times R$, let $\mathcal{I}_1$ be the family of bounded sets and let $S=\{(x, y)\colon x, y\langle 0\vee x, y\rangle 0\vee y=0\}$. It is evident that $S\notin\mathcal{I}_1$. Condition S3. is not fulfilled because if we take $s=(0, x)$ and $t=(0, y)$ for $x<0$ and $y>0$ then $(s+S)\cap(t+S)=\{(0, z)\colon z\leqq 0\}\cap\{(0, z)\colon z\geqq 0\}=\{(0, 0)\}\in\mathcal{I}_1$, Condition S1. holds because for (x, y), $(y, z)\in S$

$$\text{if} \quad x, y < 0 \quad \text{then} \quad y, z < 0 \quad \text{or} \quad z = 0,$$

$$\text{if} \quad x, y > 0 \quad \text{then} \quad y, z > 0 \quad \text{or} \quad z = 0,$$

$$\text{if} \quad y = 0 \quad \text{then} \quad z = 0,$$

so in every case $(x, z)\in S$. Condition S2. holds because $(x, y)=(x, 0)+(0, y)= =(x, 0)-(y, 0)\in S-S$ for every $x, y\in R$. Let a function $f\colon S\to X$ be defined as follows

$$f((x, y)) = \begin{cases} (x+1, y+1) & \text{for} \quad x \leqq 0, \\ (x, y), & \text{for} \quad x > 0. \end{cases}$$

Consider the set

$$M = \{((x, y), (u, z))\in(S\times S)\cap D\colon f((x, y)+(u, z)) \neq f((x, y))+f((u, z))\} =$$

$$= \{((x, y), (y, z))\in S\times S\colon f((x, z)) \neq f((x, y))+f((y, z))\}.$$

We have $M[(x, y)]=\{(y, z)\in S\colon f((x, z))+f((x, y))+f((y, z))\}$, so

$$M[(x, y)] = \begin{cases} \emptyset & \text{for} \quad x \leqq 0 \quad \text{or} \quad y \neq 0, \\ \{(0, 0)\} & \text{for} \quad x > 0 \quad \text{and} \quad y = 0. \end{cases}$$

Thus for every $(x, y)\in S$ $M[(x, y)]\in\mathcal{I}_1$. So f is a homomorphism $\mathcal{I}_2$-(a.e.) in $(S\times S)\cap D$ for every p.l.i. ideal $\mathcal{I}_2$ in $X\times X$ conjugate with $\mathcal{I}_1$. Suppose now that a homomorphism $g\colon X\to X$ is an extension of f. Then for every $x, z\in R$ $g((x, 0))+ +g((0, z))=g((x, z))$. Two cases are possible: $x\leqq 0$ or $x>0$. If $x\leqq 0$ then $(u, v)=g((x, z))=g((x, 0))+g((0, z))=(x+1, 1)+g((0, z))$ whence

$$g((0, z)) = (1, v). \tag{5}$$

If $x>0$ then $(u, v)=g((x, z))=g((x, 0))+g((0, z))=(x, 0)+g((0, z))$ and consequently $g((0, z))=g(0, v)$ which contradicts (5). Thus the homomorphism g is not equal to f in the set $\{(x, 0)\colon x\leqq 0\}\notin \mathcal{I}_1$ or in the set $\{(x, 0)\colon x>0\}\notin \mathcal{I}_1$. It is contradictory to (4).

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