Lattice ordered rings

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Our article "Lattice Ordered Rings" has been published in "Publicationes Mathematice", Debrecen, v. 26, 1979, 51—53, ([2]) with which we considered we have solved the problem No 34 in the monograph of Fuchs, "Partially Ordered Algebraic Systems", 1963, ([1]). Unfortunately it appeared that the theorem exposed so far is true only as regards orders of the following type: Let in the ring R exist an order P such that for every two elements $g, g' \in R$, $g \neq g'$, there exists an element $s \in R$, so that $s-g \in P$, $s-g' \in P$ and if t is such an element of R, that t and s are comparable with respect to P and $t-g \in P$, $t-g' \in P$, then it follows that $t-s \in P$. The theorem is true for such orders.

The present paper solves that problem completely.

Before exposing the theorem for completeness we are going to formulate some basic definitions.

Definition. We shall say that the partially ordered ring R is lattice ordered if there exists an order P such that for each pair $g, g' \in R$, $g \neq g'$ there exists on element $s \in R$ such that $s-g \geqq 0$, $s-g' \geqq 0$ and for any element $t \in R$, with $t-g \geqq 0$, $t-g' \geqq 0$ one has $t-s \geqq 0$.

If R is a lattice ordered ring, then for each pair of elements $g, g' \in R$, $g \neq g'$ there exists a nonvoid set of elements satisfying the above-mentioned condition. It is denoted by $S(g, g')$. Obviously then $H(t-g, t-g')$ is a conic semiring for every $t \in S(g, g')$.

Let $\bar{R}$ denote the set of all ordered pairs of different elements from R, i.e.

$$\bar{R} = R^2 \setminus \operatorname{diag} R$$

and Φ the set of all functions defined on $\bar{R}$ with values in R, i.e. $\Phi = \{f \colon \bar{R} \to R / f(g, g') = s, \ \forall (g, g') \in \bar{R}, \ s \in R\}$.

When the ring R is such that $S(g, g') \neq \emptyset$ then also $\Phi \neq \emptyset$.

Theorem. *The partial order P in the ring R can be extended to a lattice order if and only if there exists a function $f \in \Phi$ satisfying the following conditions:*

(A) *For any set of elements $(g_i, g_i') \in \bar{R}$, $i = 1, 2, \dots, n$, the semiring*

$$H(P, s_1-g_1, s_1-g_1', \dots, s_n-g_n, s_n-g_n', t_1-s_1, \dots, t_n-s_n)$$

is conic, where $s_i = f(g_i, g_i')$, $i = 1, \dots, n$, and $t_1, \dots, t_n$ are elements from R for which

$$t_i-g_i, \ t_i-g_i' \in H(P, \{s_k-g_k, s_k-g_k' / k = 1, \dots, i\})$$

in each $i = 1, \dots, n$.

(B) *If for the element* $(a, b)\in\overline{R}$ *there is a set of elements* $(l_i, l'_i)\in\overline{R}$ *and elements* $\overline{V}_i\in R$, $i=1, \ldots, m$, *such that*

$$v_i-l_i,\ v_i-l'_i\in H\big(P, \{w_k-l_k, w_k-l'_k/w_k = f(l_k, l'_k),\ k = 1, \ldots, i\}\big)$$

for each $i=1, \ldots, m$, *and if*

$$a, b\in H\big(-P, \{l_k-w_k, l'_k-w_k, w_k-v_k/k = 1, \ldots, m\}\big)$$

then

$$f\big(0, f(a, b)\big) = 0.$$

PROOF. *Necessity:* Let R be an l-ring and P an l-order in R. Then for every pair $(a, b)\in\overline{R}$ there exist such an element $s\in R$ that $s-a\in P$, $s-b\in P$ and if the element t in R is such that, $t-a\in P$, $t-b\in P$, then $t-s\in P$, i.e., $a\vee b=s$.

Now let the function $f\in\Phi$ satisfy $f(a, b)=a\vee b=s$ for each pair $(a, b)\in\overline{R}$. We shall show that this function satisfies the conditions (A) and (B).

By the definition of the function f it follows that for any set of elements $(g_i, g'_i)\in\overline{R}$ and any set of elements $t_i\in R$, $i=1, \ldots, n$, for which

$$t_i-g_i,\ t_i-g'_i\in H\big(P, \{s_k-g_k, s_k-g'_k/k = 1, \ldots, i\}\big)$$

$i=1, \ldots, n$, it is true that the semiring

$$H\big(P, \{s_k-g_k, s_k-g'_k, t_k-s_k/k = 1, \ldots, n\}\big)$$

is a conic one, because $H(s_i-g_i, s_i-g'_i, t_i-s_i)\subseteqq P$ for every $i=1, \ldots, n$. It follows that condition (A) is satisfied.

Because

$$H\big(-P, \{l_k-w_k, l'_k-w_k, w_k-v_k/w_k = f(l_k, l'_k),\ k = 1, \ldots, m\}\big) \subseteqq -P$$

for every pair $(l_k, l'_k)\in\overline{R}$ and every element $v_k\in R$, $k=1, \ldots, m$, for which the condition

$$v_k-l_k,\ v_k-l'_k\in H\big(P, \{w_j-l_j, w_j-l'_j/j = 1, \ldots, k\}\big), \quad k = 1, \ldots, m,$$

has been satisfied,

$$a, b\in H\big(-P, \{l_k-w_k, l'_k-w_k, w_k-v_k/w_k = f(l_k, l'_k),\ k = 1, \ldots, m\}\big)$$

implies that $a<0$, $b<0$ with respect to P. Thus for $f(a, b)=a\vee b$ $f(a, b)<0$ is satisfied. Hence,

$$f\big(0, f(a, b)\big) = 0.$$

This shows that condition (B) is satisfied.

Sufficiency. Let P be a ring with partial order in R for which there exists a function $f\in\Phi$, satisfying conditions (A) and (B). We shall indicate that the order P is extended to an l-order in R.

Let us designate by Δ the set of those extensions of P, in which the same function f satisfies (A) and (B). Let

$$\Delta_1 = \{P_\alpha \supset P/\alpha\in I, P_{\alpha'} \subset P_{\alpha''} \Leftrightarrow \alpha' \leqq \alpha''\} \subset \Delta.$$

We shall indicate that $\overline{P}=\bigcup_{\alpha\in I} P_\alpha\in\Delta$.

We admit that $\bar{P}$ does not satisfy condition (A). Then for a given set of elements $(g_i, g_i')\in\bar{R}$ there exist elements $t_i\in R$, $i=1, \dots, n$, for which the condition is satisfied

$$t_i-g_i,\ t_i-g_i'\in H(\bar{P}, \{s_k-g_k, s_k-g_k'/s_k=f(g_k, g_k'),\ k=1, \dots, i\})$$

for every $i=1, \dots, n$, but the semiring

$$H_1=H(\bar{P}, \{s_i-g_i, s_i-g_i', t_i-s_i/i=1, \dots, n\})$$

is not conic.

Let $a, -a\in H_1$. For any given $t_1, \dots, t_n$, when expressing a and $-a$ a finite number of elements from $\bar{P}$ will occur. We choose $\gamma\in I$, such that $P_\gamma\in\Delta_1$ and in P_γ the above-mentioned elements from $\bar{P}$ are contained.

Then

$$a, -a\in H(P_\gamma, \{s_i-g_i, s_i-g_i', t_i-s_i/i=1, \dots, n\})$$

which contradicts the condition that (A) is satisfied for each order P_γ.

We suppose that for $\bar{P}$ condition (B) is not satisfied. Then for some elements a, b for which there exist elements $(l_i, l_i')\in\bar{R}$ and $v_i\in R$, $i=1, \dots, n$, such that

$$v_i-l_i,\ v_i-l_i'\in H(\bar{P}, \{w_k-l_k, w_k-l_k'/w_k=f(l_k, l_k'),\ k=1, \dots, i\})$$

and for which the condition is satisfied

$$a, b\in H(-\bar{P}, \{l_i-w_i, l_i'-w_i, w_i-v_i/i=1, \dots, n\})$$

we have

$$f(0, f(a, b))\neq 0.$$

Continuing this reasoning, we come to the conclusion that there is an order $P_\gamma\in\Delta_1$ for which the condition (B) is not satisfied, a contradiction.

Thence, $\bar{P}\in\Delta$, which means that in Δ there is a maximal element Q.

We shall show that Q is a lattice order in the ring R.

I. Because with respect to the order Q condition (A) is in force for the given function f, it follows that for any elements $(a_0, b_0)\in\bar{R}$ and $s_0=f(a_0, b_0)$ the semiring

$$Q'=H(Q, s_0-a_0, s_0-b_0)$$

is a conic one.

We shall show that for the order Q' conditions (A) and (B) are satisfied.

We suppose that condition (A) is not satisfied, i.e. there exists a set of elements $(g_i, g_i')\in\bar{R}$ and a set of such elements t_i in R, $i=1, \dots, n$, that

$$t_i-g_i,\ t_i-g_i'\in H(Q', \{s_k-g_k, s_k-g_k'/s_k=f(g_k, g_k'),\ k=1, \dots, i\})$$

at each $i=1, \dots, n$, for which the semiring

$$H(Q', \{s_i-g_i, s_i-g_i', t_i-s_i/i=1, \dots, n\})$$

is not a conic one.

From

$$H(Q', \{s_i-g_i, s_i-g_i', t_i-s_i/i=1, \dots, n\})\subseteqq$$
$$\subseteqq H(Q, s_0-a_0, s_0-b_0, \{s_i-g_i, s_i-g_i', t_i-s_i/i=1, \dots, n\})$$

it follows that the hypothesis is not true.

Thus, for the order Q' condition (A) is satisfied.

We suppose that condition (B) is not satisfied, i.e. for some $(a, b)\in\overline{R}$ for which some elements exist $(l_i, l_i')\in\overline{R}$ and $v_i\in R$, $i=1, \ldots, n$, subjected to the condition

$$v_i-l_i,\ v_i-l_i'\in H(Q', \{w_k-l_k, w_k-l_k'/w_k = f(l_k, l_k'),\ k=1, \ldots, i\})$$

for each $i=1, \ldots, n$, despite that

$$a, b\in H(-Q', \{l_k-w_k, l_k'-w_k, w_k-v_k/k=1, \ldots, n\})$$

then

$$f(0, f(a, b))\neq 0.$$

Because

$$H(-Q', \{l_i-w_i, l_i'-w_i, w_i-v_i/i=1, \ldots, n\})\subseteqq$$
$$\subseteqq H(-Q, a_0-s_0, b_0-s_0, \{l_i-w_i, l_i'-w_i, w_i-v_i/i=1, \ldots, n\})$$

it follows that condition (B) is not satisfied for the order Q. The contradiction obtained shows that Q' satisfies condition (B).

Hence, $Q'\in\Delta$. But Q is a maximal element in Δ, hence

$$Q'=Q$$

i.e. for each pair of elements $a_0, b_0\in R$, $a_0\neq b_0$ there exists $s_0\in R$ such that $s_0-a_0\in Q$, $s_0-b_0\in Q$. This shows that Q is a directed order in R.

II. Let $(a_0, b_0)\in\overline{R}$ and let $t_0\geqq a_0$, $t_0\geqq b_0$, $t_0\in R$, for the order Q (such a t_0 exists according to what proved above).

Because the order Q satisfies condition (A),

$$H(Q, s_0-a_0, s_0-b_0, t_0-s_0)$$

is a conic semiring, where $s_0=f(a_0, b_0)$. Hence,

$$Q''=H(Q, t_0-s_0)$$

is a partial order (c.s.), because $s_0-a_0\in Q$, $s_0-b_0\in Q$ according to has been already proved under I.

We shall show that the order Q'' satisfies conditions (A) and (B).

We admit that Q'' does not satisfy condition (A), i.e., there exist a set of elements $(g_i, g_i')\in\overline{R}$ and $t_i\in R$, $i=1, \ldots, n$, such that

$$t_i-g_i,\ t_i-g_i'\in H(Q'', \{s_k-g_k, s_k-g_k'/s_k=f(g_k, g_k'),\ k=1, \ldots, i\})$$

for each $i=1, \ldots, n$, but the semiring

$$H(Q'', \{s_i-g_i, s_i-g_i', t_i-s_i/i=1, \ldots, n\})$$

is not a conic one.

Because Q is a directed order, $s_i\geqq g_i$, $s_i\geqq g_i'$ for Q and

$$H(\{s_i-g_i, s_i-g_i'/i=1, \ldots, n\})\subset Q.$$

Hence,

$$H(Q'', \{s_k-g_k, s_k-g_k'/s_k=f(g_k, g_k'),\ k=1, \ldots, i\})\subset H(Q, t_0-s_0)$$

for each $i=1, \ldots, n$.

Now for the above-selected elements $g_i, g_i', t_i, i=1, \dots, n,$ the condition

$$t_i-g_i,\ t_i-g_i'\in H(Q, t_0-s_0), \quad i = 1, \dots, n,$$

is satisfied, or

$$(1) \qquad t_i-g_i = q_i+c_i$$

$$t_i-g_i' = q_{i1}+c_{i1}$$

where

$$q_i, q_{i1}\in Q;\ c_i, c_{i1}\in H(Q, t_0-s_0).$$

Thus

$$-c_{i1}, -c_i\in H(-Q, -(t_0-s_0))$$

and if

$$f(-c_i, -c_{i1}) = v_i;\ v_i > -c_i, v_i > -c_{i1},$$

then

$$f(0, f(-c_i, -c_{i1})) = 0,$$

because for Q condition (B) is satisfied. Hence, $v_i\in -Q$. From (1) we obtain

$$t_i-c_i = g_i+q_i$$

$$t_i-c_{i1} = g_i'+q_{i1}$$

which can be written in another way

$$v_i+t_i+(-v_i-c_i) = g_i+q_i$$

$$v_i+t_i+(-v_i-c_{i1}) = g_i'+q_{i1}$$

or

$$(v_i+t_i)-g_i = q_i+(v_i+c_i)$$

$$(v_i+t_i)-g_i' = q_{i1}+(v_i+c_{i1}).$$

From the inclusions

$$H(Q'', t_i-s_i) \subseteqq H(Q, t_0-s_0, (v_i+t_i)-s_i-v_i) \subseteqq$$

$$\subseteqq H(Q, t_0-s_0, (v_i+t_i)-s_i)$$

for each $i=1, \dots, n$ (we use that $v_i\in Q$), it follows that if (g_i, g_i') are any elements from $\bar{R}$ and the elements v_i+t_i from R, $i=1, \dots, n$, are such that

$$(t_i+v_i)-g_i,\ (v_i+t_i)-g_i'\in H(Q, t_0-s_0, \{s_k-g_k, s_k-g_k'/s_k = f(g_k, g_k'),\ k = 1, \dots, i\})$$

for each $i=1, \dots, n$, then

$$H(Q, s_0-a_0, s_0-b_0, t_0-s_0, \{s_i-g_i, s_i-g_i', (v_i+t_i)-s_i/i = 1, \dots, n\})$$

is not a conic semiring which contradicts condition (A) for Q.

Hence, for the order Q'' condition (A) is satisfied.

We suppose that if $(a, b)\in\bar{R}$ are such elements for which there exist some elements $(g_i, g_i')\in\bar{R}$ and $t_i\in R$, $i=1, \dots, n$, satisfying the condition

$$t_i-g_i,\ t_i-g_i'\in H(Q'', \{s_k-g_k, s_k-g_k'/s_k = f(g_k, g_k'),\ k = 1, \dots, i\})$$

for each $i=1, \ldots, n$, and if

$$a, b \in H(-Q'', \{g_i - s_i, g_i' - s_i, s_i - t_i / i = 1, \ldots, n\})$$

then

$$f(0, f(a, b)) \neq 0.$$

From the above-said it follows that this should be true also for the order Q, and this contradicts condition (B) for the order Q.

Hence, for Q'' condition (B) is valid. Thus $Q'' \in \Delta$. But Q is a maximal element in Δ. Hence,

$$Q'' = Q.$$

This implies that for any two elements $a_0, b_0 \in R$, $a_0 \neq b_0$ there exists $s_0 \in R$, with $s_0 \geqq a_0$, $s_0 \geqq b_0$, and if the element t_0 from R satisfies $t_0 \geqq a_0$, $t_0 \geqq b_0$, then $t_0 \geqq s_0$ for the order Q in R. This shows that Q is a lattice order in the ring R.

This proves the theorem.

References

[1] L. Fuchs, Partially ordered algebraic systems. 1963.
[2] S. A. Todorinov—G. Teneva, Lattice ordered rings. *Publ. Math. Debrecen,* **26** (1979), 51—53.

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