On asymptotic distribution modulo a subdivision

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Dedicated to Prof. Béla Barna on his 80th birthday

Let (R_n) $(n=0, 1, 2, \dots)$ and (x_n) $(n=1, 2, \dots)$ be increasing and unbounded sequences of non-negative real numbers with $R_0=0$. Let (i_n) $(n=1, 2, \dots)$ be the sequence of natural numbers defined by

$$R_{i_n-1} \leqq x_n < R_{i_n}.$$

Define a sequence (Δx_n) by

$$\Delta x_n = \frac{x_n - R_{i_n-1}}{R_{i_n} - R_{i_n-1}}. \tag{1}$$

The sequence (x_n) is said to be uniformly distributed modulo (R_n) if the sequence (Δx_n) is uniformly distributed mod 1. We note that in the case $(R_n)=(n)$ we get the definition of the usual uniform distribution as a special case.

The uniform distribution modulo a subdivision (R_n) of the interval $[0, \infty)$ was introduced by W. J. LE VEQUE [5], who established some conditions for sequences (x_n) and (R_n) such that (x_n) is uniformly distributed mod (R_n).

H. DAVENPORT and W. J. LE VEQUE [2] as well as H. DAVENPORT and P. ERDŐS [1] proved results concerning the sequence $(x_n)=(n\Theta)$, where Θ is a positive irrational number. It follows from their results that the sequence $(n\Theta)$ is uniformly distributed mod (R_n) for almost all Θ if the sequence $(R_{n+1}-R_n)$ is decreasing (see [2]) or if $R_n/R_{n-1}\to 1$ as $n\to\infty$ and the number of terms R_n satisfying the condition $R_n<N$ is less than $c.\ N^{2-\delta}$, where c and δ are some fixed positive numbers (see [1]).

Another type of results was obtained by P. KISS [3]. Let $A(x, N, (\Delta x_n))$ be the number of terms Δx_n defined in (1) for which $n\leqq N$ and $\Delta x_n<x$. If there is an infinite sequence $N_1<N_2<\dots$ of natural numbers and a function $F\colon [0, 1]\to[0, 1]$ such that

$$\lim_{i\to\infty}\frac{1}{N_i} A(x, N_i, (\Delta x_n)) = F(x) \quad (0\leqq x\leqq 1),$$

then (x_n) is said to be almost asymptotically distributed mod (R_n) with respect to the distribution function $F(x)$. If $F(x)=x$, then (x_n) is called almost uniformly distributed mod (R_n) (see also definitions 1.2 and 7.2 in [4]). The following theorem was proved in [3]. Let Θ be a positive irrational number and let (R_n) be a linear recurring

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sequence (satisfying the condition $R_n=c\beta^n+o(\beta^n)$ with some positive real numbers c and β). Then the sequence $(x_n)=(n\Theta)$ is almost uniformly distributed mod (R_n) but it is not uniformly distributed.

The purpose of this note is to give an extension of the result of P. Kiss mentioned above. We shall show that the irrationality condition for Θ is not necessary, the result depends only on growth conditions of the sequences. We investigate a more general case and suppose that the sequences (x_n) and (R_n) grow polynomially and exponentially, respectively. It is possible to suppose slightly more general growth conditions but we do not discuss this in detail.

Theorem 1. *Let (R_n) and (x_n) be sequences of non-negative real numbers satisfying $R_0=0$,*

$$R_n = \alpha e^{an}+\mathcal{O}(e^{an}\cdot \varrho^n) \tag{2}$$

for $n>0$ and

$$x_n = \delta n^d+\mathcal{O}(n^{d-1}), \tag{3}$$

where $a, \alpha, \varrho, \delta$ and d are fixed positive real numbers and $0<\varrho<1$. Further let $F(x)$ be the function defined by

$$F(x) = \frac{(x(e^a-1)+1)^{1/d}-1}{e^{a/d}-1}. \tag{4}$$

Then the sequence (x_n) is almost asymptotically distributed mod (R_n) *with respect to the distribution function $F(x)$. Furthermore there is a subsequence $N_1<N_2<\ldots$ of natural numbers such that*

$$\left|\frac{1}{N_i}A(x, N_i, (\Delta x_n))-F(x)\right| = \mathcal{O}(N_i^{-c}),$$

where c is a positive constant depending only on α, ϱ, δ and d.

Since $F(x)=x$ only if $d=1$, Theorem 1 implies the following corollary, which includes the result of [3] on the sequence $(x_n)=(n\Theta)$.

Corollary. *Let (R_n) and (x_n) be the sequences defined in Theorem 1. Then (x_n) is almost uniformlydi stributed* mod (R_n) *if and only if $d=1$ (i.e. the sequence (x_n) grows as a linear polynomial).*

Furthermore we prove another theorem which shows that under our growth conditions the sequence (x_n) is only "almost" asymptotically distributed mod (R_n) to the distribution function $F(x)$.

Theorem 2. *Let (R_n) and (x_n) be the sequences defined in Theorem 1. Then the limit*

$$\lim_{N\to\infty}\frac{1}{N}A(x, N, (\Delta x_n))$$

does not exist for any x with $0<x\leqq 1$.

Proof of Theorem 1. First we note that

$$(1+\varepsilon)^{\beta} = \mathcal{O}(\beta) \quad (\beta > 0),$$

(5)

$$\frac{1+\varepsilon_1}{1+\varepsilon_2} = 1+\mathcal{O}\big(\max(|\varepsilon_1|, |\varepsilon_2|)\big),$$

if $|\varepsilon|, |\varepsilon_1|, |\varepsilon_2| < \varepsilon_0(\beta)$.

Let N be a positive integer. Define L_N by

(6)
$$L_N = \sum_{0 \leqq x_k < R_N} 1.$$

By (2), (3) and (6) we have

$$\delta L_N^d\big(1+\mathcal{O}(1/L_N)\big) = \alpha e^{aN}\big(1+\mathcal{O}(\varrho^N)\big),$$

which, by (5), yields

$$L_N = \left(\frac{\alpha}{\delta}\right)^{1/d} e^{aN/d}\big[1+\mathcal{O}\big(\max(\varrho^N, 1/L_N)\big)\big] =$$

(7)

$$= \left(\frac{\alpha}{\delta}\right)^{1/d} e^{aN/d} + \mathcal{O}\big(\max(e^{aN/d}\varrho^N, 1)\big).$$

If $\chi_{[0,x)}$ is the characteristic function of the interval $[0, x)$, that is $\chi(y)=1$ if $0 \leqq y < x$ and $\chi(y)=0$ otherwise, and $\Delta R_n = R_n - R_{n-1}$, then

(8)
$$A\big(x, L_N, (\Delta x_n)\big) = \sum_{n=1}^{N} \sum_{R_{n-1} \leqq x_k < R_n} \chi_{[0,x)}\left(\frac{x_k - R_{n-1}}{\Delta R_n}\right).$$

In the second summation, $\chi=1$ if and only if

$$R_{n-1} \leqq x_k < x\Delta R_n + R_{n-1}.$$

Hence by (2), (3) and the definition of ΔR_n,

$$\left[\frac{\alpha e^{a(n-1)}\big(1+\mathcal{O}(\varrho^{n-1})\big)}{\delta\left(1+\mathcal{O}\left(\frac{1}{k}\right)\right)}\right]^{1/d} \leqq k < \left[\frac{\big(x\cdot\alpha e^{a(n-1)}(e^a-1)+\alpha e^{a(n-1)}\big)\big(1+\mathcal{O}(\varrho^n)\big)}{\delta\left(1+\mathcal{O}\left(\frac{1}{k}\right)\right)}\right]^{1/d},$$

and so

$$\sum_{R_{n-1} \leqq x_k < R_n} \chi_{[0,x)}\left(\frac{x_k - R_{n-1}}{\Delta R_n}\right) =$$

$$= \left(\frac{\alpha}{\delta}\right)^{1/d} e^{a(n-1)/d}\big[\big(x(e^a-1)+1\big)^{1/d}-1\big] + \mathcal{O}\big(\max(e^{an/d}\varrho^n, 1)\big).$$

By (8) and the summation formula for geometric series we obtain

(9)
$$A\big(x, L_N, (\Delta(x_n))\big) = \left(\frac{\alpha}{\delta}\right)^{1/d}\big[\big(x(e^a-1)+1\big)^{1/d}-1\big]\frac{e^{aN/d}}{e^{a/d}-1} +$$

$$+\mathcal{O}\big(\max(e^{aN/d}\varrho^N, N)\big).$$

Thus, by (4), (5) and (7), we derive

$$\frac{1}{L_N} A(x, L_N, (\Delta x_n)) = F(x) + \mathcal{O}\left(\max\left(\varrho^N, \frac{N}{e^{aN/d}}\right)\right).$$

However, by (7)

$$N = \mathcal{O}\left(\frac{d}{a}\log L_N\right),$$

and so (note that $0<\varrho<1$)

$$\max\left(\varrho^N, \frac{N}{e^{aN/d}}\right) = \mathcal{O}(L_N^{-c})$$

follows for some $c>0$. This yields the theorem with $N_i=L_i$ $(i=1, 2, \dots)$.

Proof of Theorem 2. It is enough to prove that there is a subsequence $K_1< <K_2<\dots$ of natural numbers such that

$$\lim_{i\to\infty} \frac{1}{K_i} A(x, K_i, (\Delta x_n)) \neq F(x)$$

for any fixed x with $0<x<1$, where $F(x)$ is the function defined in (4).

Let λ be a fixed real number with $0<\lambda<1$, $\lambda>x$ and put $\lambda(n)=R_{n-1}+\lambda\Delta R_n$ for $n>1$. Further let

$$K_i = \sum_{x_0 \leqq x < \lambda(i)} 1.$$

We obtain similarily as in the proof of Theorem 1 that

$$K_N = \left(\frac{\alpha}{\delta}\right)^{1/d} e^{a(N-1)/d}(1+\lambda e^a-\lambda)^{1/d} + \mathcal{O}\big(\max(e^{aN/d}\varrho^N, 1)\big). \tag{10}$$

Using the definitions of $F(x)$, L_N and the condition $\lambda>x$, we obtain by (7), (9) and (10)

$$A(x, K_N, (\Delta x_n)) = A(x, L_N, (\Delta x)) =$$

$$= \left(\frac{\alpha}{\delta}\right)^{1/d} e^{aN/d} F(x) + \mathcal{O}\big(\max(e^{aN/d}\varrho^N, N)\big).$$

From this, by (10), it follows that

$$\lim_{N\to\infty} \frac{1}{K_N} A(x, K_N, (\Delta x_n)) = \frac{e^{a/d}F(x)}{(1+\lambda e^a-\lambda)^{1/d}},$$

which equals to $F(x)$ only if $\lambda=1$. Thus our assertion is proved.

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