Boundedness, convergence and global stability of solutions of a nonlinear differential equation of the second order

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Introduction. In this paper we shall discuss the asymptotic behaviour of solutions of the second order differential equation

$$x''+f(t,x,x')x'+g_1(t,x)g_2(x')+h(t,x,x')+e(t,x,x') = 0, \tag{1}$$

which is equivalent to the system

$$x' = y, \quad y' = -f(t,x,y)y-g_1(t,x)g_2(y)-h(t,x,y)-e(t,x,y). \tag{2}$$

In the paper [1] the author discussed the uniform boundedness of solutions and their derivatives and further the global asymptotic stability of the trivial solution of the second order differential equation

$$x''+f(t,x,x')x'+\psi_1(x)\psi_2(x')+h(t,x,x')+e(t,x,x') = 0.$$

I. W. BAKER in [4] investigated the continuation and boundedness of solutions and their derivatives of the differential equation

$$u''+\Phi(t,u,u')u'+p(t)f(u)g(u') = h(t,u,u').$$

Further in [5], M. YAMAMOTO and S. SAKATA discussed the boundedness and the attractivity properties of solutions and their derivatives of the differential equations

$$\bigl(a(t)x'\bigr)'+b(t)f_1(x)g_1(x')x'+c(t)f_2(x)g_2(x') = e(t,x,x')$$

and

$$\bigl(a(t)x'\bigr)'+h(t,x,x')+c(t)f(x)g(x') = e(t,x,x').$$

In the first part of this paper there are introduced some sufficient conditions for a solution $x(t)$ of the equation (1) to be defined and bounded together with its first derivative. These results generalize some results of [4] and [5]. In the second part there are given some results concerning the uniform boundedness of solutions and the global asymptotic stability of the trivial solution of the system (2). Further in this part there are introduced some sufficient conditions for the convergence of all solutions $(x(t), y(t))$ of the system (2) to the origin as $t\to\infty$. Our results of this part generalize some results of [1] and [5].

Definitions and Propositions. In this paper we shall use the following definitions and propositions of [1]—[3]. Let $\varphi(t)=\varphi(t;\, t_0,\, x_0)$ denote a solution of the system

$$x' = f(t, x),\ x\in R_n,\ t\in I = \langle 0, \infty),\ f(t, x)\in C(I\times R_n) \tag{3}$$

through x_0 at $t=t_0$.

Definition 1. The solutions of (3) are uniformly bounded if for any $(t_0, \alpha)\in I\times R_1$ there exists $\beta=\beta(\alpha)>0$ such that $|x_0|\leqq\alpha$ implies $|\varphi(t;\, t_0,\, x_0)|\leqq\beta(\alpha)$ for every $t\geqq t_0$.

Proposition 1 (T. Yoshizava [1]). *Let there exist continuous functions $V(t, x)$ and $W_i(x)$, $i=1, 2$ in $I\times R_n$ such that the following conditions hold:*

1. $0 < W_1(x) \leqq V(t, x) \leqq W_2(x),\ W_1(x) \to \infty,\ |x| \to \infty.$

2. $V'(t, x) = \dfrac{\partial V}{\partial t} + \dfrac{\partial V}{\partial x} f(t, x) \leqq 0.$

Then the solutions of (3) *are uniformly bounded for* $t\geqq 0$.

Definition 2 (see [1]). A point $x\in R_n$ is called an ω-limit point of $\varphi(t)$ of the system (3) if there exists a sequence $\{t_n\}_{n=1}^{\infty}\subset I$ such that $\varphi(t_n)\to x$ as $t_n\to\infty$ and $n\to\infty$.

Definition 3 (see [1]). The set of all ω-limit points of a solution $\varphi(t)$ of the system (3) is called the ω-limit set of a solution $\varphi(t)$ of the system (3). This set will be denoted by $\Omega(\varphi)$.

Definition 4 (see [2]). A function $V(t, x)$ is called uniformly small if there exists a continuous, positively definite function $W(x)$ such that

$$W(t, x) \leqq W(x)$$

in $I\times R_n$.

Proposition 2 (see [2]). *If there exists a positively definite uniformly small function $V(t, x)$, which has a negatively definite derivative with respect to t, then the trivial solution of the system* (3) *is uniformly asymptotically stable.*

Definition 5 (see [3]). The trivial solution of system (3) is called globally asymptotically stable if it is asymptotically stable and for all solutions $\varphi(t;\, t_0, x_0)$ of the system (3) $\varphi(t;\, t_0, x_0)\to 0$ as $t\to\infty$.

Proposition3 (see [1]). *Let the assumptions of Proposition 1 be fulfilled. Moreover, suppose that the functions on the right sides of the system* (3) *are bounded in $I\times P$, where $P\subset R_n$ is any compact set, and there exists a continuous function $W_3(x)$ such that*

$$V'(t, x) \leqq -W_3(x) \leqq 0$$

in $I\times R_n$. Then $\varphi(t)\to\{x\colon |x|\leqq\beta,\ W_3(x)=0\}$, $t\to\infty$, where the constant β is as in Definition 1.

Theorems. *Let* $f(t, x, y)$, $h(t, x, y)$, $e(t, x, y)\in C(D_0)$, $\dfrac{\partial g_1(t, x)}{\partial t}$, $g_1(t, x)\in$ $\in C(D_1)$, $g_2(y)\in C^1(R_1)$, *where* $R_1=(-\infty, \infty)$, $D_0=I\times R_2$, $R_2=R_1\times R_1$ *and* $D_1=I\times R_1$.

Let us introduce the following notation:

$$G_1(t, x) = \int_0^x g_1(t, s)\, ds, \quad G_2(y) = \int_0^y \frac{s}{g_2(s)}\, ds$$

and

$$\frac{\partial G_1(t, x)}{\partial t} = \int_0^x \frac{\partial g_1(t, s)}{\partial t}\, ds.$$

In what follows suppose, in addition to the assumptions given above, that
(I) there exist functions $p_i(x)\in C(R_1)$ such that

$$xp_i(x) > 0, \quad x \neq 0, \quad i = 1, 2$$

and

$$|p_1(x)| \leqq |g_1(t, x)| \leqq |p_2(x)| \quad \text{in} \quad I\times R_1,$$

$$P_i(x) = \int_0^x p_i(s)\, ds, \quad i = 1, 2 \quad \text{and} \quad P_1(x) \to \infty \quad \text{as} \quad |x| \to \infty.$$

(II) $xg_1(t, x) > 0, \quad x \neq 0, \ t\in I.$

(III) $G_2(y) \to \infty$ as $|y| \to \infty$.

(IV) $g_2(y) > 0, \ y\in R_1$.

(V) $g_2'(y) \operatorname{sgn} y \geqq 0, \ y\in R_1$.

(VI) there exists a nonnegative function $f_1(t)\in C(I)$ such that

$$f(t, x, y) \geqq -f_1(t) \quad \text{in} \quad D_0 \quad \text{and} \quad \int_0^\infty f_1(t)\, dt < \infty.$$

(VII) there exists a nonnegative function $e_1(t)\in C(I)$ such that

$$|e(t, x, y)| \leqq \frac{1}{2}\, |y|\, e_1(t) \quad \text{in} \quad D_0$$

and

$$\int_0^\infty e_1(t)\, dt = M < \infty.$$

(VIII) $yh(t, x, y) \geqq 0$ in D_0.

(IX) $\dfrac{\partial G_1(t, x)}{\partial t} \leqq 0$ in D_1.

(X) there exists a positive number M_1 such that

$$\frac{y^2}{g_2(y)} \leqq M_1 G_2(y) \quad \text{for all} \quad |y| \geqq 1.$$

We have

Theorem 1. *Suppose that the assumptions* (II)—(IV) *and* (VI)—(X) *hold. Then all solutions* $(x(t), y(t))$ *of* (2) *are defined for every* $t \geqq 0$.

Proof. Let a solution $(x(t), y(t))$ of (2) be defined on $\langle 0, T)$ and $|x(t)|+|y(t)| \to \infty$ as $t \to T_-$. Define the function

$$V(t, x, y) = \big(G_1(t, x)+G_2(y)+K\big) \exp\big(-E(t)\big),$$

where $E(t)=\int_0^t e_1(s)ds$ and K is any positive constant. Differentiate $V(t) = V\big(t, x(t), y(t)\big)$ with respect to t for any solution of (2). Then we have

$$V'(t) = \Big(\frac{\partial G_1(t, x)}{\partial t} - \frac{y^2}{g_2(y)} f(t, x, y) - \frac{y}{g_2(y)} h(t, x, y) - \tag{4}$$

$$- \frac{y}{g_2(y)} e(t, x, y) - e_1(t)\big(G_1(t, x)+G_2(y)+K\big)\Big) \exp\big(-E(t)\big)$$

for every $t \in \langle 0, T)$.

By (X) there exist positive constants M_1 and M_2 such that

$$\frac{y^2}{g_2(y)} \leqq M_1 G_2(y) + M_2$$

for every $y \in R_1$. Hence by (II), (IV) and (VI)—(IX) with respect to (4) we have

$$V'(t) \leqq M_2 \Big(f_1(t) + \frac{1}{2} e_1(t)\Big) + M_1 \Big(f_1(t) + \frac{1}{2} e_1(t)\Big) V(t)$$

in $\langle 0, T)$.

Integrating the above inequality from 0 to $t \in (0, T)$ and using Bellman's lemma we obtain

$$V(t) \leqq C_0 \exp\Big(M_1 \int_0^t \Big(f_1(s) + \frac{1}{2} e_1(s)\Big) ds\Big) = C_1 < \infty \tag{5}$$

for $t \in \langle 0, T)$, where

$$C_0 = V(0) + \sup_{t \in \langle 0, T)} M_2 \int_0^t \Big(f_1(s) + \frac{1}{2} e_1(s)\Big) ds.$$

From (II) and (IV) there follows

$$G_2(y) \leqq \big(\exp E(t)\big) \cdot V(t)$$

for $t \in \langle 0, T)$, hence we have in $\langle 0, T)$

$$G_2(y) \leqq C_1 \exp E(t) < \infty.$$

The condition (III) gives that $y(t)$ and $x(t)$ are bounded on $\langle 0, T)$, which is a contradiction. The theorem is proved.

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled and let moreover* (I) *hold. Then all solutions* $(x(t), y(t))$ *of* (2) *are bounded for* $t \geqq 0$.

Proof. By (VI) and (VII) from (5) we get

$$V(t) \leqq \left(V(0)+M_2 \int_0^\infty \left(f_1(s)+\frac{1}{2}e_1(s)\right)ds\right)\cdot \exp\left(M_1 \int_0^\infty \left(f_1(s)+\frac{1}{2}e_1(s)\right)ds\right) =$$

$$= K_0 < \infty.$$

From (II), (IV) and (VII) it follows

$$G_2(y) \leqq V(t) \exp M$$

for $t \in I$, hence for every $t \geqq 0$

$$G_2(y) \leqq K_0 \exp M < \infty.$$

Furthermore (III) gives that $y(t)$ is bounded for $t \geqq 0$. Besides, (II), (IV) and (VII) yield

$$G_1(t, x) \leqq V(t) \exp M$$

for $t \in I$. Hence

$$G_1(t, x) \leqq K_0 \exp M < \infty \tag{6}$$

for every $t \geqq 0$.

The last inequality implies that $x(t)$ is bounded for $t \geqq 0$, for otherwise it would be possible to find a sequence $\{t_n\}_{n=1}^\infty$ such that $x(t_n) \to \infty$ as $t_n \to \infty$. Then for positive x and all t_n sufficiently large we would have by (I) that

$$G_1(t_n, x(t_n)) = \int_0^{x(t_n)} g_1(t_n, s)\, ds \geqq \int_0^{x(t_n)} p_1(s)\, ds.$$

This contradicts (6), since the right side tends to ∞ as $t_n \to \infty$. For negative x we could get a contradiction similarly as before. This completes the proof.

Remark. Theorem 1 and Theorem 2 extend Theorems 2.1, 3.1, 3.4 and 3.5 of [4] to the more general equation (1).

If in (VI) $f_1(t)=0$ for $t \geqq 0$, then the following theorems generalize Theorems 2.1, 2.2 and Corollaries 2.2, 2.3 of [1] and their proofs are similar to those in [1].

Theorem 3. *Let the assumptions* (I)—(X) *hold. Then the solutions of* (2) *are uniformly bounded for* $t \geqq 0$.

Proof. We use the same function $V(t, x, y)$ as in the proof of Theorem 1. The assumptions (I), (II), (IV) and (VII) imply

$$0 < \left(P_1(x)+G_2(y)+K\right) \exp(-M) = W_1(x, y) \leqq V(t, x, y) \leqq$$

$$\leqq P_2(x)+G_2(y)+K = W_2(x, y),$$

where by (I) and (III) $W_1(x, y) \to \infty$ as $|x| \to \infty$, $|y| \to \infty$. Using the fact that for $y \in R_1$

$$G_2(y) = \frac{y^2}{2g_2(y)} + \frac{1}{2} \int_0^y \frac{s^2 g_2'(s)}{g_2^2(s)} ds, \tag{7}$$

we get from (4) by (II) and (IV)—(IX)

$$V'(t) \leqq -\frac{1}{2} \left(\int_0^y \frac{s^2 g_2'(s)}{g_2^2(s)} ds \right) \cdot e_1(t) \cdot \exp(-E(t)) \leqq 0 \quad \text{in} \quad D_0.$$

By Proposition 1 the theorem is proved.

Theorem 4. *Let the assumptions* (I)—(X) *be fulfilled. Moreover, suppose that*

(XI) *for any positive constant c there exists a positive constant K_c such that*

$$|p_2(x)| \leqq K_c, \ |g_1(t, x_1) - g_1(t, x_2)| \leqq K_c |x_1 - x_2|,$$

where $|x_1| \leqq c$, $|x_2| \leqq c$ and $|f(t, x, y)| \leqq K_c$ for $|x| \leqq c$, $|y| \leqq c$ and $t \in I$.

(XII) *there exists a positive function $f_2(x, y) \in C(R_2)$ such that*

$$f(t, x, y) \geqq f_2(x, y) \quad in \quad D_0.$$

(XIII) *for any positive constant c and $x(t), y(t) \in C(I)$*

$$\int_t^{t+1} h(s, x(s), y(s))\, ds \to 0 \quad \text{as} \quad t \to \infty,$$

where $|x(t)| \leqq c$ and $|y(t)| \leqq c$.

Then for all solutions $(x(t), y(t))$ of (2) *$x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$.*

Proof. Using the same function $V(t, x, y)$ as in the proof of Theorem 3, by (II), (IV), (V), (VII)—(IX), (XII) and (7), we get

$$V'(t, x, y) \leqq -\frac{y^2}{g_2(y)} f_2(x, y) \cdot \exp(-M) = -W_3(x, y) \leqq 0$$

in D_0. By Theorem 3 all solutions $(x(t), y(t))$ of (2) are uniformly bounded for $t \geqq 0$. Hence there exists a positive constant A such that $|x(t)| \leqq A$, $|y(t)| \leqq A$ for $t \in I$. Therefore the set

$$B = \{(x(t), y(t)) \colon |x(t)| \leqq A, \ |y(t)| \leqq A, \quad t \in I\} \subset R_2$$

is compact. By (I), (VII), (XI) and (XIII) the right sides of (2) are bounded in $I \times B$. Hence by Proposition 3 we have

$$(x(t), y(t)) \to L \quad \text{as} \quad t \to \infty,$$

i.e. the ω-limit set $\Omega(x, y)$ is a subset of

$$L = \left\{(x, y) \colon |x| \leqq A, \ |y| \leqq A; \ \frac{y^2 f_2(x, y)}{g_2(y)} = 0\right\}.$$

From (IV) and (XII) it follows

$$L = \{(x, y)\colon |x| \leqq A;\ y = 0\},$$

e.i.

(8) $$y(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Further we prove that $\Omega(x, y) = \{(0, 0)\}$. Let $(a, 0) \in \Omega(x, y)$. Then by Definition 2 there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to \infty$, $n \to \infty$ and

(9) $$x(t_n) \to a,\ y(t_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Now, it suffices to show that $a=0$. Let $a \neq 0$. Integrating the second equation of (2) from t_n to t_n+1, we get

$$y(t_n+1)-y(t_n) = -\int_{t_n}^{t_n+1} f(s, x(s), y(s))\,ds - g_1(t_n, x(t_n))g_2(y(t_n)) -$$

$$-\int_{t_n}^{t_n+1} \big(g_1(s, x(s)) - g_1(t_n, x(t_n))\big) g_2(y(s))\,ds -$$

$$-\int_{t_n}^{t_n+1} \big(g_2(y(s)) - g_2(y(t_n))\big) g_1(t_n, x(t_n))\,ds -$$

$$-\int_{t_n}^{t_n+1} h(s, x(s), y(s))\,ds - \int_{t_n}^{t_n+1} e(s, x(s), y(s))\,ds,$$

i.e.

$$g_1(t_n, x(t_n))g_2(y(t_n)) = y(t_n) - y(t_n+1) -$$

$$-\int_{t_n}^{t_n+1} f(s, x(s), y(s))\,ds - \int_{t_n}^{t_n+1} \big(g_1(s, x(s))) - g_1(t_n, x(t_n)\big) g_2(y(s))\,ds -$$

$$-\int_{t_n}^{t_n+1} \big(g_2(y(s)) - g_2(y(t_n)\big) g_1(t_n, x(t_n))\,ds - \int_{t_n}^{t_n+1} h(s, x(s), y(s))\,ds -$$

$$-\int_{t_n}^{t_n} e(s, x(s), y(s))\,ds.$$

Further we have

(10) $$\big|g_1(t_n, x(t_n))\big| g_2(y(t_n)) \leqq |y(t_n)| + |y(t_n+1)| +$$

$$+\int_{t_n}^{t_n+1} \big|f(s, x(s), y(s))\big|\,|y(s)|\,ds + \int_{t_n}^{t_n+1} \big|g_1(s, x(s)) - g_1(t_n, x(t_n))\big| g_2(y(s))\,ds +$$

$$+\int_{t_n}^{t_n+1} \big|g_2(y(s)) - g_2(y(t_n))\big| \cdot \big|g_1(t_n, x(t_n))\big|\,ds +$$

$$+\Big|\int_{t_n}^{t_n+1} h(s, x(s), y(s))\,ds\Big| + \int_{t_n}^{t_n+1} \big|e(s, x(s), y(s))\big|\,ds.$$

By the well-known Lagrange's theorem there exist $\xi\in(t_n, s)$ and a positive constant c_1 such that for $g_2(y)\in C^1(R_1)$

$$\left|g_2(y(x))-g_2(y(t_n))\right| = |y(s)-y(t_n)|\left|g_2'(y(\xi))\right| \leqq \left(|y(s)|+|y(t_n)|\right)c_1.$$

Integrating the first equation of (2) from t_n to t we get

$$x(t)-x(t_n) = \int_{t_n}^{t} y(s)\,ds \quad \text{and for} \quad t_n \leqq t \leqq t_n+1$$

$$|x(t)-x(t_n)| \leqq \sup_{t_n\leqq t<\infty} |y(t)|.$$

Let $K_s>0$ be the maximum of all constants in this proof, then by the estimates given above the inequality (10) yields

$$\left|g_1(t_n, x(t_n))\right|\cdot g_2(y(t_n)) \leqq (3K_s^2+K_s+2)\cdot \sup_{t_n\leqq t<\infty} |y(t)|+$$

$$+\left|\int_{t_n}^{t_n+1} h(s, x(s), y(s))\,ds\right|+\frac{1}{2}\int_{t_n}^{t_n+1} |y(s)|\,e_1(s)\,ds.$$

By (I) and (IV), the last inequality gives

$$\left|p_1(x(t_n))\right|\cdot g_2(y(t_n)) \leqq (3K_s^2+K_s+2)\cdot \sup_{t_n\leqq t<\infty} |y(t)|+$$

$$+\left|\int_{t_n}^{t_n+1} h(s, x(s), y(s))\,ds\right|+\frac{1}{2}\sup_{t_n\leqq t<\infty} |y(t)|\cdot\int_{t}^{t_n+1} e_1(s)\,ds.$$

In view of (I), (VII), (XIII) and (8) we have

$$p_1(x(t_n)) \to 0 \quad \text{as} \quad n\to\infty.$$

Since the function $p_1(x)$ is continuous in R_1, by (9) $p_1(a)=0$ for $a\neq 0$. This contradicts (I), hence $x(t)\to 0$ as $t\to\infty$. The theorem is proved.

Theorem 4 yields the following.

Corollary. *The trivial solution of* (2) *is globally asymptotically stable.*

Proof. Since $h(t, x, y)\in C(D_0)$ and (VIII) holds, we have $h(t, x, 0)=0$ in D_1. Analogously by (II) $g_1(t, 0)=0$ for every $t\geqq 0$. So by (VII) the system (2) has the trivial solution. Let

$$V(t, x, y) = \left(G_1(t, x)+G_2(y)\right)\exp\left(-E(t)\right)$$

in D_0. Then by (I)

$$V_1(x, y) = \left(P_1(x)+G_2(y)\right)\exp(-M) \leqq V(t, x, y) \leqq P_2(x)+G_2(y) = V_2(x, y)$$

in D_0, where $V_i(x, y)>0$ in $R_2\setminus(0, 0)$ and $V_i(0, 0)=0$, $i=1, 2$. Differentiate

$V(t, x(t), y(t))$ with respect to t. By (2), (II), (IV), (V), (VII)—(IX) and (XII) we get

$$V'(t, x, y) \leqq -\frac{y^2}{g_2(y)} f_2(x, y) \exp(-M) = -W_3(x, y) \leqq 0$$

in D_0. By Proposition 2 the trivial solution of (2) is uniformly asymptotically stable. Therefore by Theorem 4 and Definition 5 the proof is finished.

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