On certain properties of centralizers hereditary to the factor group

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The structure of minimal non-p-nilpotent groups was decribed by ITO [1] in the following way:

Theorem 1. *Let G be a finite group, p prime. Let us suppose that every proper subgroup of G is p-nilpotent but G is not. Then*

1. *Every proper subgroup of G is nilpotent*
2. $|G|=p^a q^b$, *where q is a prime different from p*
3. *If* $P\in \mathrm{Syl}_p(G)$ *then* $P \lhd G$ *and if* $p>2$ *then* $\exp(P)=p$, *if* $p=2$ *then* $\exp(P)\leqq 4$
4. *If* $Q\in \mathrm{Syl}_q(G)$ *then Q is cyclic.*

In the following we shall call such a group a (p, q)-group. The fact that G contains a (p, q)-subgroup we shall denote by $G\geqq(p, q)$ (otherwise we shall write $G\not\geqq(p, q)$).

The aim of this paper is to prove the following results:

Theorem A. Let G be a finite group, $p\neq q$ primes. If $G\not\geqq(p, q)$ then this property is inherited to every homomorphic image of G.

Theorem B. Let G be a finite group $r, p\neq q$ primes. If for every r-element $x\in G^{\#}$, $C_G(x)\not\geqq(p, q)$ then this property is inherited to every homomorphic image of G.

Theorem C. Let G be a finite group, let $\pi(G)=\{p, q\}$ be the set of prime divisors of G. If for every r-element $x\in G^{\#}$, $r\in\pi(G)$, $C_G(x)$ is supersolvable then this property is inherited to every homomorphic image of G.

The proof of the following lemmas can be easily obtained from Theorem 1:

Lemma 1. *Let G be a finite group, then G is p-nilpotent if and only if $G\not\geqq(p, q)$ for every prime q different from p.*

Lemma 2. *Let G be a finite solvable group, $\pi\leqq\pi(G)$. Then G has a normal π-complement if and only if $G\not\geqq(p, q)$ for every $p\in\pi$, $q\in\pi'$.*

PROOF OF THEOREM A. Let G be a counterexample of minimal order. So $G\not\geqq(p, q)$ but $G/H\geqq(p, q)$ for a suitable $H\lhd G$.

1. By the inductive hypothesis we may suppose that H is a minimal normal subgroup of G and G/H is a (p, q)-group.
2. Applying the Frattini-argument to $R\in\mathrm{Syl}_r(H)$ we have $G/H\cong N_G(R)/H\cap N_G(R)$ so by the minimality of G, $N_G(R)=G$. As H is a minimal normal subgroup we have that H is an elementary abelian r-group.
3. We can distinguish two cases: a) $r\notin\{p, q\}$, b) $r=p$ or $r=q$. In the case a)

$(|H|, |G: H|)=1$ so by the theorem of Zassenhaus there exists a $K \leqq G$ with $HK=G$ and $H \cap K=1$. But then $G \geqq K \cong G/H \geqq (p, q)$, which is a contradiction. In the case b) $\pi(G)=\{p, q\}$ so by Lemma 1 $Q \in \mathrm{Syl}_q(G)$ is normal in G so G/H is nilpotent, which is also a contradiction.

PROOF OF THEOREM B: Let G be a couterexample of minimal order, $H \lhd G$ such that $C_{\bar{G}}(\bar{x}) \geqq (p, q)$ for some r-element $\bar{x} \in \bar{G}^{\#}$, where $\bar{G}=G/H$.

1. By induction we may suppose that $H \leqq \Phi(G)$, H is a minimal normal subgroup of G and $\bar{x} \in Z(G/H)$.
2. We may also suppose that $|\bar{x}|=r$ and we may choose an r-element x in the inverse image of $\bar{x}$.
3. As in step 2. in the proof of Theorem A we may suppose that H is elementary abelian t-group for some prime t.
4. We may suppose that $(|x|, |H|) \neq 1$:
Otherwise $\langle x \rangle \in \mathrm{Syl}_r(H\langle x \rangle)$. By the Frattini argument $G=HN_G(\langle x \rangle)=N_G(\langle x \rangle)$ as $H \leqq \Phi(G)$. Let $n \in N_G(\langle x \rangle)$ then $n \in C_G(x)$. So $G=C_G(x)$ and by applying Theorem A to $G/H=C_{\bar{G}}(\bar{x})$ we have that $C_G(x) \geqq (p, q)$, contradiction.
5. Let $B=H\langle x \rangle$. Then B is an elementary abelian r-group: If $B' \neq 1$ then as $|B/B'| \geqq r^2$ and $1 \neq B' \leqq H$ we have that $B' < H$, which contradicts to the minimality of H. So $B'=1$. As $\mho_1(B) \leqq H$, by the minimality of H we have that $\mho_1(B)=1$ or $\mho_1(B)=H$. In the second case $|B: \Phi(B)|=r$ that is B is cyclic so $|B| \leqq r^2$. If $|B|=r^2$ then $|x|=r^2$ and $H=\langle x^r \rangle$. As $[x, G] \leqq H=\langle x^r \rangle$ $1=[x, y]^r=$ $=[x^r, y]$ for every $y \in G$ so $x^r \in Z(G)$. Applying Theorem A we have that $C_G(x^r) \geqq$ $\geqq (p, q)$, contradiction.
6. We can distinguish three cases: a) $r \notin \{p, q\}$, b) $r=p$, c) $r=q$. In the case a) by induction we may suppose that $G/H=\langle \bar{x}, \bar{U} \rangle=\langle \bar{x} \rangle \times \bar{U}$, where $\bar{U}$ is a (p, q)-group. In particular G is solvable. Let U be the inverse image of $\bar{U}$ in G, and let $T \in \mathrm{Hall}_{r'}(U)$ so $T \in \mathrm{Hall}_{r'}(G)$ as well. By the Frattini argument $G=UN_G(T)$. Obviously $N_G(T) > T$. On the other hand $G=BT$ so $N_G(T)=(B \cap N_G(T))$. T and $B \cap N_G(T) \neq 1$. But $C_G(B \cap N_G(T)) \geqq BT=G$ and again by Theorem A we cannot have $G/H \geqq (p, q)$.
In the case b) B is elementary abelian p-group. Let $B \leqq P \in \mathrm{Syl}_p(G)$ $Q \in \mathrm{Syl}_q(G)$. As in case a) we may suppose that $G/H=\langle \bar{x}, \bar{U} \rangle$, where $\bar{U}$ is a (p, q)-group. Then $\bar{P} \in \mathrm{Syl}_p(\bar{G})$ is normal so $P \lhd G$ and $H \cap Z(P) \neq 1$ which by the minimality of H yields $H \leqq Z(P)$. By the theorem of Maschke $B=H \times T$ where T is a Q-invariant complement to H. Then $C_G(T) \geqq BQ$. As $\pi(G)=\{p, q\}$ and $C_G(T) \ngeqq (p, q)$, by Lemma 1 we have that $Q \lhd C_G(T)$ and so $H \leqq Z(G)$, which by Theorem A yields contradiction.
In the case c) B is an elementary abelian q-subgroup. Let $P \in \mathrm{Syl}_p(G)$, $B \leqq Q \in$ $\in \mathrm{Syl}_q(G)$ and we may suppose that $\bar{G}=\langle \bar{x}, \bar{U} \rangle$, where $\bar{U}$ is a (p, q)-group. Then $G/H \rhd \bar{P} \in \mathrm{Syl}_p(\bar{G})$ so $G \rhd HP$ and as $H \leqq \Phi(G)$ the Frattini argument yields $P \lhd G$. But $1 \neq Z(Q) \cap H$ and $C_G(Z(Q) \cap H) \geqq QP=G$ so by Theorem A we have a contradiction.

Corollary 1. *If G is a finite group, r prime and for every r-element $x \in G^{\#}$ $C_G(x)$ has a Sylow-tower corresponding to a fixed ordering of primes then this property is inherited to every homomorphic image of G.*

PROOF. Let $\pi(G)=\{p_1 \succ p_2 \succ \ldots \succ p_n\}$ be the ordered set of prime divisors of G. Then the existence of a Sylow-tower in G corresponding to this ordering is equivalent to the fact that $G \not\equiv (p_k, p_m)$ for every $k, m \leqq n$ where $k > m$.

Corollary 2. *Let G be a finite solvable group, π a set of primes. Let us suppose that for every π-element $x \in G^\#$ $C_G(x)$ has a normal π-complement N and $C_G(x)/N$ is nilpotent. Then every irreducible π-character of G is monomial.*

PROOF. Let G be a counterexample of minimal order and $\chi \in \mathrm{Irr}(G)$ is a π-character which is not monomial. We may assume that $\chi(1)>1$ and χ is primitive. By Theorem B and Lemma 2 the hypotheses are inherited to every homomorphic image of G. So we may assume that $\mathrm{Ker}\,\chi=1$. As $F(G)\neq 1$ so $1\neq Z(F(G))\leqq Z(G)$ so G has a normal π-complement, namely $O_{\pi'}(G)$, provied $F(G)$ is not a π'-group. But the solvability of G gives $C_G(F(G))\leqq F(G)$. So $F(G)\leqq O_{\pi'}(G)$ leads to $F(G)\leqq Z(G)$, — a contradiction — as the next argument shows. As the constituents of $\chi|O_{\pi'}(G)$ are linear, $O_{\pi'}(G)'\leqq \mathrm{Ker}\,\chi=1$ so $O_{\pi'}(G)\leqq Z(G)$. But then $G/O_{\pi'}(G)$ is nilpotent so G is nilpotent, contradiction.

PROOF OF THEOREM C. Let G be a counterexample of minimal order, $H \lhd G$ such that $C_{\bar G}(\bar x)$ is not supersolvable, where $\bar G = G/H$.

1. As in steps 1—5 in the proof of Theorem B we may assume that H is a minimal normal subgroup of G, $H\leqq \Phi(G)$, $\bar x \in Z(G/H)$, $|\bar x|=p$, x is a p-element in the inverse image of $\bar x$, H and B are elementary abelian p-groups.
2. We can distinguish two cases: a) $p<q$, b) $p>q$.
   In the case a) let $B\leqq P\in \mathrm{Syl}_p(G)$, $Q\in \mathrm{Syl}_q(G)$. From Corollary 1 we have that there exists a Sylow-tower in $C_{\bar G}(\bar x)$ so that $C_{\bar G}(\bar x) \rhd \bar Q \in \mathrm{Syl}_q(\bar G)$. By the Frattini argument we have that $Q \lhd G$ so $C_G(Z(P)\cap H)\geqq PQ=G$ and G is supersolvable, contradiction. In the case b) similarly we have that $P\lhd G$ so $H\leqq Z(P)$. By the theorem of Maschke $B=H\times T$, where T is a Q-invariant complement to H in B. We may suppose that $B\not\leqq Z(P)$, otherwise $C_G(T)\geqq PQ=G$ and G would be supersolvable, which is not the case. In the following we shall construct two supersolvable subgroups of G of index p and q respectively. By a result of ASAAD [2] if a group possesses two supersolvable subgroups of index of the maximal and minimal prime divisor of its order respectively then the group is supersolvable. So in our case this would yield a contradiction.
   Let us consider the subgroup $1\neq [T, P]\leqq H$. As $C_G(Q)\geqq T$ we have that $[T, P]=[T, G]$ and it is easy to see that this subgroup is normal in G so $[T, G]=H$. Hence $H\leqq P'$. There exists a subgroup L of index q in G. We shall show that L is supersolvable. If $|Q|=q$ then it is trivially true. So we may assume that $P\neq L$. As $H\leqq P'\leqq \Phi(P)\leqq \Phi(L)$ and by induction L/H is supersolvable so L is also supersolvable. Now let us consider the supersolvable subgroup $C_G(x)$. We may suppose that $T=\langle x\rangle$ so $Q\leqq C_G(x)$. Using the fact that $H\leqq Z(P)$ we have that every minimal normal subgroup $H_1\leqq H$ of $C_G(x)$ is normal in G. So $|H|=p$ and $|B|=p^2$, which yields $|P\colon C_P(x)|=p=|G\colon C_G(x)|$. So by the theorem of Asaad we are done.

Corollary 3. *Let G be a finite group, $|\pi(G)|=2$, r prime. Let us suppose that for every r-element $x\in G^*$ $C_G(x)$ is supersolvable then G is an M-group.*

PROOF. Let G be a counterexample of minimal order, $\chi\in \mathrm{Irr}(G)$ is not monomial. Then we may assume that $\chi(1)>1$ and χ is primitive. As by Theorem C our hypothesis is inherited to $G/\mathrm{Ker}\,\chi$, we may suppose that $\mathrm{Ker}\,\chi=1$. As G is solvable $1\neq Z(F(G))\leqq Z(G)$ so G is supersolvable, contradiction.

References

[1] N. Itô, Note on (LM)-groups of finite order. *Kodai Math. Seminar Report* 1951, 1—6.
[2] M. Asaad, Investigations in the theory of finite groups. *Ph. D. Thesis*, 1976. Budapest.

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