Resolutions and infinite-dimensionality

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Abstract. In the present paper we give a partial answer to the question: Let $p: X \to \underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a resolution such that the spaces X_α, $\alpha \in A$, are A-weakly (S-weakly) infinite dimensional. Is it true that X and $\lim \underline{X}$ are A-weakly (S-weakly) infinite-dimensional?

The main result of Section One is the characterisation of a base of X (1.4. Lemma). An important property of a resolution gives Lemma 1.8.

Section Two is devoted to the mappings p_α, $X \to X_\alpha$. The closedness of p_α is proved in Theorem 2.1.

Section Three is the main section. Theorem 3.1. asserts that X and $\lim X$ are A-weakly infinite-dimensional if $p: X \to \underline{X}$ is a resolution and $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is σ-directed inverse system of A-weakly infinite-dimensional spaces X_α. If the mappings satisfy the condition $|\mathrm{Fr}\, f_{\alpha\beta}^{-1}(x_\alpha)| < \aleph_0$ then the converse of Theorem 3.1. holds (Theorem 3.5.). Theorems 3.2. and 3.3. are analogous theorems for the dimension $\dim X$ and $\dim_f X$. If $\underline{X}$ is an iverse system of infinite-dimensional Cantor-manifolds X_α, then X and $\lim \underline{X}$ are infinite-dimensional Cantor-manifolds.

0. Introduction

0.1. We use the notion of inverse systems in the sense of the book [6]. By $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is denoted an inverse system and its limit by $\lim \underline{X}$.

0.2. The notin of a resolution of a space was introduced by S. MARDEŠIĆ. We use the exposition of this notion as in the book [22].

0.3. By $\mathrm{Cl}(A)$ or $\mathrm{Cl}\, A$ we denote the closure of the set A.

0.4. If $f: X \to Y$ is a mapping, the for $A \subseteq X$ we define $f^\#(A) = \{y \in Y: f^{-1}(y) \subseteq A\}$.

0.5. $|A|$ denotes the cardinality of the set A. By $cf(A)$ is denoted the smallest ordinal number which is cofinal in the well-ordered set A.

0.6. The symbol $\mathrm{Fr}\, A$ denotes the boundary of the set i.e. the set $\mathrm{Cl}\, A \cap \mathrm{Cl}(X \setminus A)$.

0.7. We say that $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is σ-directed if for each sequence $\{\alpha_i: i \in N, \alpha_i \in A\}$ there is an $\alpha \in A$ such that $\alpha > \alpha_i$ for each $i \in N$.

0.8. By $w(X)$ is denoted the weight of X. Similarly, by $hl(X)$ is denoted the hereditary Souslin number of X [6: 284].

0.9. We say that a well-ordered inverse system $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is continuous if for every limit ordinal γ, $\gamma \in A$, the space X_γ is homeomorphic with the limit of the inverse system $X' = \{X_\alpha, f_{\alpha\beta}, \alpha \leqq \beta \leqq \gamma\}$.

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1. Basic properties

Let X be a topological space. A resolution of X [22: 74] consists of an inverse system $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ in pro-Top and a morphism $p\colon X\to X$ in pro-Top with the following two properties:

(R1) Let P be an ANR, $\mathscr{V}$ an open covering of P and $h\colon X\to P$ a map. Then there exists a $\alpha\in A$ and a map $f\colon X_\alpha\to P$ such that the maps fp_α and h are $\mathscr{V}$-near (i.e. every $x\in X$ admits a λ such that both $fp_\alpha(x)$ and $h(X)$ belong to $V_\lambda\in\mathscr{V}$).

(R2) Let P be an ANR and $\mathscr{V}$ an open covering of P. Then there is an open covering $\mathscr{V}'$ of P with the following property: If $\alpha\in A$ and $f, f'\colon X_\alpha\to P$ are maps such that the maps fp_α and $f'p_\alpha$ are $\mathscr{V}'$-near, then there exists a $\alpha'>\alpha$ such that maps $ff_{\alpha\alpha'}$ and $f'f_{\alpha\alpha'}$ are $\mathscr{V}$-near.

1.1. Theorem. [22:74, Theorem 1.]. *Let $p\colon X\to\underline{X}$ be a morphism in pro-Cpt. Then p is a resolution of X iff p is an inverse limit of $\underline{X}$.*

1.2. Theorem. [22:87, Corollary 1.]. *Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of normal spaces X_α, X a topological space and $p\colon X\to\underline{X}$ a morphism in pro-Top. Then p is a resolution of X iff it has the following properties:*
(B1) *Let $\alpha\in A$ and let U be an open set in X_α which contains $\mathrm{Cl}\,(p_\alpha(X))$. Then there is $\alpha'\geqq\alpha$ such that $f_{\alpha\alpha'}(X_{\alpha'})\subseteq U$.*
(B2) *For every normal covering $\mathscr{U}$ of X there exists an $\alpha\in A$ and a normal covering $\mathscr{U}_\alpha$ of X_α such that $p_\alpha^{-1}(\mathscr{U}_\alpha)$ refines $\mathscr{U}$.*

Let us recall that an open covering $\mathscr{U}$ is normal iff it admits a metric space M, a map $h\colon X\to M$ and an open covering $\mathscr{V}$ of M such that $h^{-1}(\mathscr{V})$ refines $\mathscr{U}$. A T_1-space X is paracompact iff each open cover of X is normal [6:379]. A locally finite open cover of a normal space is normal [6:379].

We say that $A\subseteq X$ is normally embedded (or $\mathscr{P}$-embedded) in the space X [15: 188] provided every normal covering $\mathscr{U}$ of A admits a normal covering $\mathscr{V}$ of X such that $\mathscr{V}/A$ refines $\mathscr{U}$. A closed subsets of collectionwise normal spaces are normally embedded. In particular, closed subsets of paracompact spaces are normally embedded.

1.3. Definition. An inverse system $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ is called an R-system if the morphism $f=\{f_\alpha\}\colon \lim X\to X$ is a resolution (where $f_\alpha\colon \lim X\to X$, $\alpha\in A$, are the projections). The system X is a weak R-system for a class $\mathscr{K}$ (WR-system for $\mathscr{K}$) if there exist $X\in\mathscr{K}$ and a resolution $p\colon X\to\underline{X}$ such that the induced mapping $p\colon X\to\lim\underline{X}$ is onto.

We start with the following lemma.

1.4. Lemma. *Let $p\colon X\to\underline{X}$ be a resolution. If X is a completely regular space, then the family $\mathscr{B}=\{p_\alpha^{-1}(U_\alpha)\colon U_\alpha$ open in $X, \alpha\in A\}$ is a base for the topology of X.*

Proof. Let U be an open neighborhood of $x\in X$. From the complete regularity of X it follows that there is a function $f\colon X\to I=[0, 1]$ such that $f(x)=0$ and $f(y)=1$ for $y\in X-U$. Let $\mathscr{U}$ be the cover of I which consists of the sets: $U_1=$ $=[0, 1/4)$, $U_2=(1/8, 7/8)$, $U_3=(3/4, 1]$. Property (R1) implies that there is an $\alpha\in A$ and $f_\alpha\colon X_\alpha\to I$ such that f and $f_\alpha p_\alpha$ are $\mathscr{U}$-near. Now, consider the sets $U_\alpha=$

$=f_\alpha^{-1}([0, 1/4))$ and $V=p_\alpha^{-1}(U)$. The points $f(x)$ and $f_\alpha p_\alpha(x)$ are both contained in $U_1\in\mathscr{U}$, but not in U_2-U_1 or in U_3 since $f(x)=0$ and $f, f_\alpha p_\alpha$ are $\mathscr{U}$-near. This means that $x\in V$. On the other hand for every $x'\in V$ we have $f_\alpha p_\alpha(x)\in[0, 1/4]$ i.e. $V\cap(X-U)$ is empty (since $f(X-U)=\{1\}$). Finally, V is an open set about x of the form $p_\alpha^{-1}(U)$, U_α open in X_α, contained in U. The proof is completed.

1.5. Corollary. *Let $p\colon X\to\underline{X}$ be a resolution. If X is completely regular, then for every open set U of X we have $U=\cup\{p_\alpha^{-1}(U_\alpha)\colon \alpha\in A\}$, where U_α is the maximal open subset of X_α with respect to the property $p_\alpha^{-1}(U_\alpha)\subseteq U$.*

1.6. Corollary. *Let $p\colon X\to\underline{X}$ be a resolution for a completely regular space X. Then for every closed $F\subseteq X$ and $x\in X-F$ there is a $\alpha\in A$ such that $p_\alpha(x)\in X_\alpha-\operatorname{Cl}(p_\alpha(F))$.*

1.7. Lemma. *Let p be as in Corollary 1.6. For each closed $F\subseteq X$ we have $F=\cap\{p_\alpha^{-1}\operatorname{Cl}p_\alpha(F)\colon\alpha\in A\}$. Moreover, for each $Y\subseteq X$ the relation $\operatorname{Cl}Y=\cap\{p_\alpha^{-1}\operatorname{Cl}p_\alpha(Y)\colon \alpha\in A\}$ holds.*

Each inverse system has the properties established in 1.4.—1.7. Now we describe some properties of a resolution not possessed by each inverse system.

1.8. Lemma. *Let $p\colon X\to\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be a resolution. If the spaces X, X_α, $\alpha\in A$, are normal, then for every pair of closed disjoint subsets $F_1, F_2\subseteq X$ there is a $a\in A$ such that $Clp_\alpha(F_1)\cap Clp_\alpha(F_2)=\emptyset$.*

Proof. Modify the proof of Lemma 1.4.

1.9. Lemma. *Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of normal spaces X_α. If $\underline{X}$ is a* WR*-system for normal spaces, then* $\lim\underline{X}$ *is normal.*

Proof. From Definition 1.3. it follows that there is a normal space X and a morphism $p\colon X\to\underline{X}$ which is a resolution. Now, let F_1, F_2 be disjoint closed subsets of $\lim\underline{X}$. By virtue of Lemma 1.8. for the sets $p^{-1}(F_1)$ and $p^{-1}(F_2)$ contained in X there is a $\alpha\in A$ such that $\operatorname{Cl}(p_\alpha p^{-1}(F_1))\cap\operatorname{Cl}(p_\alpha p^{-1}(F_2))=\emptyset$, where $p=\lim p_\alpha$, $\alpha\in A$, is the induced mapping $p\colon X\to\lim X$. From the normality of X_α it follows that there exist some open disjoint sets $U_\alpha, V_\alpha\subseteq X_\alpha$ such that $\operatorname{Cl}(p_\alpha p^{-1}(F_1))\subseteq U_\alpha$ and $\operatorname{Cl}(p_\alpha p^{-1}(F_2))\subseteq V_\alpha$. Since $p_\alpha p^{-1}(F_i)=f_\alpha(F_i)$, $i=1, 2$, we have $\operatorname{Cl}f_\alpha(F_1)\subseteq U_\alpha$, $\operatorname{Cl}f_\alpha(F_2)\subseteq V_\alpha$. This means that $f_\alpha^{-1}(U_\alpha)\supseteq F_1$, $f_\alpha^{-1}(V_\alpha)\supseteq F_2$. The normality of $\lim X$ is proved and the proof is completed.

1.10. Remark. It is well-known that the limit of an inverse system of normal spaces need not be normal. The limit is normal if $\underline{X}$ is inverse system of compact (or countably compact, f_{nm} closed and X a sequence) spaces.

1.11. Lemma. *Let $p\colon X\to\underline{X}=\{X_n, f_{nm}, N\}$ be a resolution such that $p\colon X\to\lim\underline{X}$ is onto. If the spaces X_n, $n\in N$, are perfectly normal and X is normal, then X and $\lim\underline{X}$ are perfectly normal.*

Proof. From Lemma 1.9. it follows that $\lim X$ is normal. In order to complete the proof it suffices to prove that each closed subset of X ($\lim\underline{X}$) is a G_δ-set [6:68]. This is an immediate consequence of Lemma 1.7. and the perfect normality of X_n, $n\in N$. The proof is completed.

2. Properties of mappings p_α

The following question is natural: Is the mapping $p_\alpha\colon X\to X_\alpha$ a closed mapping if the mappings $f_{\alpha\beta}\colon X_\beta\to X_\alpha$ are closed?

If $X=\lim \underline{X}$, $p=f$, the answer is negative [28]. That the answer is afirmative in the case of inverse sequence was proved by ZENOR. In contrast to that situation for the morphism f, we now prove

2.1. Theorem. *Let $p\colon X\to\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be a resolution such that X, X_α, $\alpha\in A$, are normal and $p_\alpha\colon X\to X_\alpha$, $\alpha\in A$, are onto mappings. The mappings $p_\alpha\colon X\to X$, $\alpha\in A$, are closed iff the mappings $f_{\alpha\beta}$ are closed.*

PROOF. Let us prove that p_α are closed if $f_{\alpha\beta}$ are closed. Let x_α be a point of X_α and let U be an open set such that $p_\alpha^{-1}(x_\alpha)\subseteq U\subseteq X$. By virtue of Lemma 1.8. it follows that there exists a $\beta\in A$ such that $p_\beta(p_\alpha^{-1}(x_\alpha))\cap \operatorname{Cl} p_\beta(X-U)=\emptyset$. Since A is directed we can assume that $\beta\geqq\alpha$. Then $p_\beta p_\alpha^{-1}(x)=f_{\alpha\beta}^{-1}(x)$. This means that $f_{\alpha\beta}^{-1}(x_\alpha)\cap \operatorname{Cl} p_\beta(X-U)=\emptyset$ i.e. $X_\beta-\operatorname{Cl} p_\beta(X-U)=V_\beta\supseteq f_{\alpha\beta}^{-1}(x_\alpha)$ and $p_\beta^{-1}(V_\beta)\subseteq U$. Since $f_{\alpha\beta}$ is closed, we have an open set U_α about x_α such that $f_{\alpha\beta}^{-1}(x_\alpha)\subseteq f_{\alpha\beta}^{-1}(U_\alpha)\subseteq V_\beta$. Clearly, $p_\alpha^{-1}(U_\alpha)=p_\beta^{-1}f_{\alpha\beta}^{-1}(U_\alpha)\subseteq p_\beta^{-1}(V_\beta)\subseteq U$. The closednes of p_α is proved.

Conversely, let all p_α, $\alpha\in A$, be the closed mappings. From the relation $p_\alpha=f_{\alpha\beta}p_\beta$, $\beta\geqq\alpha$ it follows that $p_\alpha p_\beta^{-1}(F_\beta)=f_{\alpha\beta}(F_\beta)$ for each subset F_β of X_β. If F_β is closed and p_α closed, then $f_{\alpha\beta}(F_\beta)$ is closed. This means that $f_{\alpha\beta}$ is closed. The proof is completed.

2.2. Lemma. *If p is as in Theorem 2.1. and the induced mapping $p\colon X\to\lim X$ is onto, then p is closed.*

PROOF. Let x be a point of $\lim X$ and U an open neighborhood of the se $p^{-1}(x)$. From 1.8. it follows that there is a $\alpha\in A$ and an open $U_\alpha\supseteq p_\alpha p^{-1}(x)$ such that $p_\alpha^{-1}(U_\alpha)\subseteq U$. Let $V=f_\alpha^{-1}(U_\alpha)$. We have $p^{-1}(V)\subseteq U$. The proof is completed.

2.2.1. Corollary. *Under the conditions of Lemma 2.2. the induced mapping $p\colon X\to\lim \underline{X}$ is a homeomorphism.*

PROOF. From Lemma 1.8. it follows that p is 1—1. Apply Lemma 2.2. and Proposition 1.4.18. from [6].

2.2.2. Remark. Let us note that $p\colon X\to\lim X$ is a homeomorphism if the conditions of Theorem 2.6. (Theorem 2.7.) are satisfied.

2.3. Theorem. *Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be a* WR*-system for the class of normal spaces such that the projections $f_\alpha\colon \lim X\to X_\alpha$, $\alpha\in A$, are onto mappings. Then the projections f_α, $\alpha\in A$, are closed iff the mappings $f_{\alpha\beta}\colon X_\beta\to X_\alpha$ are closed.*

PROOF. Let $x_\alpha\in X_\alpha$ and U an open set about $f_\alpha^{-1}(x_\alpha)$. The set $p^{-1}(U)$ is an open neighborhood of $p^{-1}f_\alpha^{-1}(x_\alpha)=p_\alpha^{-1}(x)$. By virtue of 2.1. there is an open $U_\alpha, x_\alpha\in U_\alpha$, such that $p_\alpha^{-1}(U_\alpha)\subseteq p^{-1}(U)$. Clearly, $f_\alpha^{-1}(U_\alpha)\subseteq U$. This means that f_α is closed.

Conversely, for each closed F_β and $\beta\geqq\alpha$ we have the set $F=f_\beta^{-1}(F_\beta)$ and the relation $f_{\alpha\beta}f_\beta(F)=f_\alpha(F)=f_{\alpha\beta}(F_\beta)$. This means that $f_{\alpha\beta}(F_\alpha)$ is closed since f_α and f_β are closed. The proof is completed.

2.4. Theorem. *Let* $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ *be a* WR-*system for the class of normal spaces. If* $f_{\alpha\beta}$ *are closed,* f_α *onto mappings and* X_α *normal, then* $\lim \underline{X}$ *is normal.*

Proof. Let F_1, F_2 be closed disjoint subsets of $\lim X$. If we apply 1.8. and 2.3. on the sets $p^{-1}(F_1)$ anf $p^{-1}(F_2)$ we obtain a $\alpha\in A$ such $f_\alpha(R_1)\cap f_\alpha(F_2)=\emptyset$. By virtue of the normality of X_α and closedness of f_α (Theorem 2.3.) it follows that there exist open disjoint sets U_α, V_α such that $f_\alpha(F_1)\subseteq U_\alpha$ and $f_\alpha(F_1)\subseteq V_\alpha$. The set $f_\alpha^{-1}(U_\alpha)$ and $f_\alpha^{-1}(V_\alpha)$ are open disjoint sets which contain F_1 and F_2. The proof is completed.

2.5. Theorem. *Let* $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ *be a* WR-*system for the class of normal spaces. If* $f_{\alpha\beta}$ *are hereditarily quotient and* p_α *onto, then* $p_\alpha\colon X\to X_\alpha$ *and* $f_\alpha\colon \lim \underline{X}\to X_\alpha$, $\alpha\in A$, *are hereditarily quotient.*

Proof. Let us recall that from the definition of the morphism p it follows that $p_\alpha=f_{\alpha\beta}p_\beta$ and $p_\beta^{-1}f_{\alpha\beta}^{-1}=p_\alpha^{-1}$, $\beta\geqq\alpha$. In order to complete the proof it suffices to prove that for each $x_\alpha\in X_\alpha$ and each open neighborhood U of $p_\alpha^{-1}(x_\alpha)$ $(f_\alpha^{-1}(x_\alpha))$ we have $x_\alpha\in \operatorname{Int} p_\alpha(U)$ $(x\in \operatorname{Int} f_\alpha(U))$. By virtue of 1.8. and directedness of A it follows that there is a $\beta\geqq\alpha$ and an open set $U_\beta\supseteq p_\beta p_\alpha^{-1}(x_\alpha)=f_{\alpha\beta}^{-1}(x_\alpha)$ such that $p_\beta^{-1}(U_\beta)\subseteq U$. Clearly, $U_\beta\subseteq \operatorname{Int} p_\beta(U)$. Since $f_{\alpha\beta}$ is hereditarily quotient, we have $x_\alpha\in \operatorname{Int} f_{\alpha\beta}(U)=\operatorname{Int} p_\alpha(U)$. Thus, $x_\alpha\in \operatorname{Int} p_\alpha(U)$. Similarly we prove that $x_\alpha\in \operatorname{Int} f_\alpha(U)$. The proof is completed.

By the same method of proof on can prove

2.6. Theorem. *Let* $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ *be a resolution with onto mappings* p_α: $X\to X_\alpha$, $\alpha\in A$. *The mappings* p_α *are open iff* $f_{\alpha\beta}$ *are open.*

2.7. Remark. Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system. If the mappings $f_{\alpha\beta}$ are hereditarily quotient, then the projections $f_\alpha\colon \lim X\to X_\alpha$, $\alpha\in A$, need not be hereditarily quotient [6:161]. The projections are hereditarily quotient if $\underline{X}$ is an inverse sequence with hereditarily quotient bonding mappings [6:161].

We say that a mapping $f\colon X\to Y$ is fully closed [8] if for each point $y\in Y$ and each family $\{U_1, \ldots, U_k\}$ of open sets of X such that $f^{-1}(y)\subseteq\{U_i\colon 1\leqq i\leqq k\}$ the set $\{y\}\cup(\cup\{f^{*}(U_i)\colon 1\leqq i\leqq k\})$ is open.

Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$. If $f_{\alpha\beta}$ are fully closed, then the projections $f_\alpha\colon \lim X\to X$, $\alpha\in A$, need not be fully closed. If $f_{\alpha\beta}$ are perfect fully closed, then f_α are perfect fully closed [8]. By virtue of theorems on the non-emptyness of the inverse limit from [17] on can prove that if $\underline{X}=\{X_n, f_{nm}, N\}$ is an inverse sequence of countably compact spaces X_n and fully closed mappings f_{nm}, then $f_n\colon \lim X\to X_n$, $n\in N$, are fully closed.

In contrast to this, we now prove

2.8. Theorem. *Let* $X=\{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system of normal spaces and* $p\colon X\to X$ *a resolution of a normal space* X. *Then the* $p_\alpha\colon X\to X_\alpha$, $\alpha\in A$, *are fully closed iff* $f_{\alpha\beta}$ *are fully closed.*

Proof. *Necessity.* Se [8:113, Lemma 2.].

Sufficiency. Let $\mathcal{U}=\{U_1, \ldots, U_k\}$ be a family of open sets in X such that $p_\alpha^{-1}(x_\alpha)\subseteq\cup\mathcal{U}$. The cover $\mathcal{V}=\{U_1, \ldots, U_k, X-p_\alpha^{-1}(x_\alpha)\}$ is a normal cover of X since X is normal [6:379]. From (B2) it follows that there is a $\beta\in\alpha$ and a normal covering $\mathcal{V}_\beta$ of X_β such that $p_\beta^{-1}(\mathcal{V}_\beta)$ refines $\mathcal{V}$. We can assume that $\beta\geqq\alpha$ since A is directed. Let $U_{i\alpha}$ be the union of all $V_\beta\in\mathcal{V}_\beta$ such that $V_\beta\cap f_{\alpha\beta}^{-1}(x_\alpha)\neq\emptyset$ and $p_\beta^{-1}(V_\beta)\subseteq$

$\subseteq U_i$. The family $\{U_{1\alpha}, \ldots, U_{k\alpha}\}$ covers $f_{\alpha\beta}^{-1}(x_\alpha)$. This means that the family $\{p_\beta^\#(U_1), \ldots, p_\beta^\#(U_k)\}$ covers $f_{\alpha\beta}^{-1}(x_\alpha)$. From the fact that $f_{\alpha\beta}$ is fully closed it follows that $f_{\alpha\beta}$ is closed and that each $p_\beta^\#(U_i)$ is open (apply Theorem 1.4.13. of [6] and Theorem 2.1.). This means that $\{x_\alpha\} \cup f_{\alpha\beta}^\# p_\beta^\#(U_1) \cup \ldots \cup f_{\alpha\beta}^\# p_\beta^\#(U_k)$ is open. Finally, the set $\{x_\alpha\} \cup p_\alpha^\#(U_1) \cup \ldots \cup p_\alpha^\#(U_k)$ is open since $p_\alpha^\#(U) - f_{\alpha\beta}^\# p_\beta^\#(U)$. The proof is completed.

2.9. Lemma. *Let $p: X \to \underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a resolution of a normal space X. If $|f_{\alpha b}^{-1}(x_\alpha)| \leqq k$ for each α, β and x_α, then $|p_\alpha^{-1}(x_\alpha)| \leqq k$ for each α and x_α.*

Proof. Suppose that for some $x_\alpha \in X_\alpha$ we have $|p_\alpha^{-1}(x_\alpha)| \geqq k+1$. Let $p_\alpha^{-1}(x_\alpha) = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots)$. For each pair (x_i, x_j), $i \neq j$, we have (from 1.8.) some $\alpha_{ij} \in A$ such that $p_\beta(x_i) \neq p_\beta(x_j)$, $\beta \geqq \alpha_{ij}$. Let $\alpha \geqq \alpha_{ij}$: $i, j \in \{1, 2, \ldots, k, k+1$. Clearly, $p_\alpha(x_i) \neq p_\alpha(x_j)$ for each pair (i, j). The proof is completed.

We close this Section with the following remark.

2.10. Remark. By a similar method of proof on can prove that if in 2.9. $|\mathrm{Fr}\, f_{\alpha\beta}^{-1}(x_\alpha)| \leqq k$, then $|\mathrm{Fr}\, p_\alpha^{-1}(x_\alpha)| \leqq k$.

If X is σ-directed, then $|\mathrm{Fr}\, f_{\alpha\beta}^{-1}(x_\alpha)| < \aleph_0$ $(|f_{\alpha\beta}^{-1}(x_\alpha)| < \aleph_0)$ implies $|\mathrm{Fr}\, f_\alpha^{-1}(x_\alpha)| < \aleph_0$ $(|f_\alpha^{-1}(x_\alpha)| < \aleph_0)$.

3. Resolutions and weak infinitedimensionality

We say that a space X is A-weakly infinite-dimensional [2] if for every sequence $\{(A_i, B_i): i \in N\}$ of a pairs of disjoint closed subsets of the exist a partition C_i betwen A_i and B_i such that $\cap\{C_i: i \in N\} = \emptyset$. If $C_1 \cap \ldots \cap C_k \neq \emptyset$ for some $k \in N$, then we say that X is S-weakly infinite-dimensional.

A space X is said to be A-strongly (S-strongly) infinite-dimensional if X is not A-weakly (S-weakly) infinite-dimensional.

A space X is called an infinite-dimensional Cantor-manifold [2] if X is compact and if $X - F$ is connected for each closed A-weakly infinite-dimensional subspace $F \subseteq X$.

We start with the following theorem.

3.1. Theorem. *Let $p: X \to \underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a resolution of a normal spaces X_α, $\alpha \in A$, such that $\underline{X}$ is σ-directed, X normal and $p_\alpha: X \to X_\alpha$, $\alpha \in A$, $p: X \to \lim \underline{X}$ are onto mapping. If the spaces X_α, $\alpha \in A$, are A-weakly infinite-dimensional, then X and $\lim \underline{X}$ are A-weakly infinite-dimensional.*

Proof. Let $\{(A_i, B_i): i \in N\}$ be any sequence of a pairs of disjoint closed subsets of X. From Lemma 1.8. it follows that for each pair (A_i, B_i) there is an $\alpha_i \in A$ such that $\mathrm{Cl}\, p_{\alpha_i}(A_i) \cap \mathrm{Cl}\, p_{\alpha_i}(B_i) = \emptyset$. Since X is σ-directed we have an $\alpha \in A$ such that $\alpha \geqq \alpha_i$, $i \in N$. This means that $\mathrm{Cl}\, p_\alpha(A_i) \cap \mathrm{Cl}\, p_\alpha(B_i) = \emptyset$ for each $i \in N$. From A-weak infinite-dimensionality of X_α it follows that there exist the partitions C_i, $i \in N$, between $\mathrm{Cl}\, p_\alpha(A_i)$ and $\mathrm{Cl}\, p_\alpha(B_i)$ such that $\cap\{C_i: i \in N\} = \emptyset$. Since $p_\alpha^{-1}(C_i)$, $i \in N$, are partitions between A_i and B_i, the proof is completed.

If $((A_i, B_i): i \in N)$ is a sequence of a pairs of disjoint closed subsets of $\lim X$, then we repeat the preceding part of the proof for the sets $p^{-1}(A_i)$ and $p^{-1}(B_i)$, $i \in N$.

We say that the inverse system $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ is an S-system if for each pair (F_1, F_2) of disjoint closed subsets of $\lim \underline{X}$ there exists an $\alpha\in A$ such that $\mathrm{Cl} f_\alpha(F_1)\cap \cap \mathrm{Cl} f_\alpha(F_2)=\emptyset$. The system $\underline{X}$ is factorizable (or f-system) [31] if for each real-valued function $f\colon \lim X\to I=[0, 1]$ there is an $\alpha\in A$ and a function $g_\alpha\colon X\to I$ such that $f=g_\alpha f_\alpha$.

3.2. Remark. Each inverse system of compact spaces is a S-system. Each inverse sequence of countably compact spaces an S-system. If $\underline{X}$ is a σ-directed inverse system of compact spaces, then $\underline{X}$ is an f-system [31:28]. Each σ-directed inverse system $\underline{X}$ with Lindelöf is an f-system.

3.3. Lemma. *Let $\underline{X}$ be an f-system. If* $\lim \underline{X}$ *is normal, then $\underline{X}$ is an S-system.*

PROOF. Trivial.

3.4. Lemma. *If $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ is a σ-directed S-system of A-weak infinite-dimensional spaces X_α, then* $\lim X$ *is A-weak infinite-dimensional.*

PROOF is similar to the proof of Theorem 3.1.

3.5. Corollary. *If $\underline{X}$ is a σ-directed inverse system of compact weak infinite-dimensional spaces, then* $\lim \underline{X}$ *is weak infinite-dimensional.*

3.6. Corollary. *Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be a σ-directed inverse system with Lindelöf limit. If X_α, $\alpha\in A$, are A-weak infinite-dimensional spaces, then* $\lim X$ *is A-weak infinite-dimensional.*

If the mappings $f_{\alpha\beta}$ are fully closed, then we have

3.7. Theorem. *Let a morphism $p\colon X\to\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be a resolution in the category pro-Cpt such that the mappings $f_{\alpha\beta}$ are fully closed and* $\dim f_{\alpha\beta}^{-1}(x_\alpha)\leqq k$, *$k\in N$. If the spaces X_α, $\alpha\in A$, are weakly infinite-dimensional, then X and* $\lim X$ *are weakly infinite-dimensional.*

PROOF. First, note that X and $\lim \underline{X}$ are homeomorphic [22:74]. Moreover, from [6:482] it follows that $f_\alpha\colon \lim X\to X_\alpha$, $\alpha\in A$, are weakly infinite-dimensional. Since f_α, $\alpha\in A$, are fully (Theorem 2.8.) it follows from [8: Theorem 9_0] that $\lim X$ is weakly infinite-dimensional.

By the same method of proof we have

3.8. Theorem. *Let $p\colon X\to\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be a resolution such that X, X_α, $\alpha\in A$, are normal and $p_\alpha\colon X\to X_\alpha$, $\alpha\in A$, $p\colon X\to$* $\lim \underline{X}$ *are onto. If A is σ-directed and X_α, $\alpha\in A$, are S-weakly infinite-dimensional, then X and* $\lim \underline{X}$ *are S-weakly infinite-dimensional.*

If $\dim X_\alpha\leqq n$ for each $\alpha\in A$, then we have

3.9. Theorem. *Let $p\colon X\to\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be a resolution of normal spaces X and X_α, $\alpha\in A$, such that $p\colon X\to$* $\lim \underline{X}$ *is onto. If* $\dim X_\alpha\leqq n$, *$\alpha\in A$, then* $\dim X\leqq n$ *and* $\dim(\lim \underline{X})\leqq n$.

PROOF. A straightforward modification of the proof of Theorem 3.1. using Theorem on partitions [6:488].

3.10. Remark. Let us note that an alternate proof of the last Theorem can be found from [18]. In this paper it is proved that Theorem 3.9. holds for $\dim_f X$ i.e. for a covering dimension defined using functionally open covers of a Tychonoff space X [6:472].

Now we prove a partial converse of Theorem 3.1.

3.11. Theorem. *Let $p: X \to \underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a resolution in the category pro-Cpt of a compact spaces and a continuous onto mappings such that A is σ-directed and $|\mathrm{Fr}\, f_{\alpha\beta}^{-1}(x_\alpha)| < \aleph_0$ for each $\alpha, \beta \in A$ and $x_\alpha \in X_\alpha$. A space X and $\lim \underline{X}$ are weakly infinite-dimensional iff the spaces X_α, $\alpha \in A$, are weakly infinite-dimensional.*

Proof. Let us recall that X and $\lim \underline{X}$ are homeomorphic spaces [22:74]. By 2.10. ve have that $|\mathrm{Fr}\, f_\alpha^{-1}(x_\alpha)| < \aleph_0$. Suppose that $\lim \underline{X}$ is weakly infinite-dimensional. In order to prove Theorem it suffices to prove that X_α, $\alpha \in A$, are weakly infinite-dimensional. If we suppose that X_α, for some $\alpha \in A$, is strongly infinite-dimensional, then from Skljarenko's theorem [1: 23] it follows that there exists a point $x_\alpha \in X_\alpha$ such that $|\mathrm{Fr}\, f_\alpha^{-1}(x_\alpha)| \geqq \aleph_0$. This is in a contradiction with $|\mathrm{Fr}\, f_\alpha^{-1}(x_\alpha)| < \aleph_0$. The proof is completed.

3.12. Remark. Theorem 3.11. holds for inverse systems of compact spaces since such systems are a resolutions [22:74].

3.13. Remark. If $f_\alpha: \lim \underline{X} \to X_\alpha$ in Theorem 3.11. is not onto mapping, then we infer that $f_\alpha(\lim \underline{X})$ is weakly infinite-dimensional.

Now we consider the inverse systems of infinite-dimensional Cantor-manifolds.

3.14. Theorem. *Let $p: X \to \underline{X}$ be as in Theorem 3.11. If the spaces X_α, $\alpha \in A$, are infinite-dimensional Cantor-manifolds and if the mappings $f_{\alpha\beta}$ are monotone, then X and $\lim \underline{X}$ are infinite-dimensional Cantor-manifolds.*

Proof. Let F be a weakly infinite-dimensional subspace of $\lim \underline{X} \approx X$. By 3.13. it follows that $f_\alpha(F)$ is weakly infinite-dimensional for each $\alpha \in A$. This means that $Y_\alpha = X_\alpha - f_\alpha(F)$ is connected since X_α is infinite-dimensional Cantor-manifold. Since f_α, $\alpha \in A$, are monotone [6:436] we infer that each $f_\alpha^{-1}(Y_\alpha)$ is connected. Clearly, $\lim X - F = \cup \{f_\alpha^{-1}(Y_\alpha): \alpha \in A\}$. Since the set $\cup \{f_\alpha^{-1}(Y_\alpha): \alpha \in A\}$ is connected [6:435] we infer that $\lim X - F$ is connected. The proof is completed.

3.15. Remark. The assumption that $\underline{X}$ is σ-directed in the above Theorem (and in Theorem 3.11.) can be omitted if $|\mathrm{Fr}\, f_{\alpha\beta}^{-1}(x_\alpha)| \leqq k$ for some fixed integer k and each α, β and x_α.

3.15. Theorem. *Let $p: X \to \underline{X}$ be as in Theorem 3.11. If the spaces X_α, $\alpha \in A$, are infinite-dimensional Cantor-manifolds and if the mappings f_α are open onto mappings, then X and $\lim X$ are infinite-dimensional Cantor-manifolds.*

Proof. Let F be a weakly infinite-dimensional subspace of $\lim X$. For each $\alpha \in A$ a subspace $f_\alpha(F) \subseteqq X_\alpha$ is weakly infinite-dimensional [27: Theorem 1.]. This means that $Y_\alpha = X_\alpha - f_\alpha(F)$ is connected. A subspace $f_{\alpha\beta}^{-1}(f_\alpha(F)) \subseteqq X_\beta$, $\alpha \leqq \beta$, is also weakly inxnite-dimensional since $f_{\alpha\beta}/f_{\alpha\beta}^{-1}(f_\alpha(F))$ is open [27: Theorem 1.]. Thus $Y_{\alpha\beta} = X_\beta - f_{\alpha\beta}^{-1}(f_\alpha(F))$ is connected for each $\beta \geqq \alpha$. The inverse system $Y_\alpha = \{Y_{\alpha\beta}, f_{\beta\gamma}/Y_{\beta\gamma}, \alpha \leqq \beta \leqq \gamma\}$ has open and closed bonding mappings [6: 95]. From the next Lemma it follows that $\lim Y_\alpha = f_\alpha^{-1}(Y_\alpha)$ is connected. From [6: 434] it follows that $\cup\{f_\alpha^{-1}(Y_\alpha): \alpha\in A\}$ is connected. The proof is completed.

3.15. Lemma. *Let* $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system with open and closed mappings* $f_\alpha: \lim \underline{X}\to X$, $\alpha\in A$. *If the spaces* X_α, $\alpha\in A$, *are connected, then* $\lim \underline{X}$ *is connected.*

Proof. Suppose that $\lim \underline{X}$ is not connected. This means that there is a nonempty open and closed subset F of $\lim \underline{X}$ such that $\lim X - F$ is also non-empty. For each $\alpha\in A$ a set $Y_\alpha = f_\alpha(F)$ is open and closed. From the connectedness of X_α it follows that $f_\alpha(F) = X_\alpha$. Since $F = \lim\{f_\alpha(F), f_{\alpha\beta}/f_\beta(F), A\}$ [6:137] we have $F=\lim X$. This is in contradictin with $\lim X - F \neq \emptyset$. The proof is completed.

If $f_{\alpha\beta}$ are open mappings with finite fibers, then we have

3.16. Theorem. *Let* $p: X\to\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ *be a* σ*-directed resolution such that* X, X_α, $\alpha\in A$, *are paracompact and the mappings* $f_{\alpha\beta}$ *open onto mappings with the property* $|\mathrm{Fr}\, f_{\alpha\beta}^{-1}(x_\alpha)| < \aleph_0$. *The spaces* X *and* $\lim \underline{X}$ *are A-weakly infinite-dimensional iff the spaces* X_α, $\alpha\in A$, *are A-weakly infinite-dimensional.*

Proof. If the spaces X_α, $\alpha\in A$, are A-weakly infinite-dimensional, then X and $\lim \underline{X}$ are A-weakly infinite-dimensional (Theorem 3.1.). In order to complete the proof suppose that $\lim X$ is A-weakly infinite-dimensional. From [22: 83] it follows that X is A-weakly infinite-dimensional since $X\approx \lim X$. Moreover, the projections $f_\alpha: \lim X\to X$, $\alpha\in A$, are onto since $1\leqq |\mathrm{Fr}\, f_\alpha^{-1}(x_\alpha)| < \aleph_0$ (Remark 2.10.). This means that $f_{\alpha\beta}$ are open mappings (Theorem 2.6.). By [27: Theorem 1.] we complete the proof.

By the same method of proof, we have

3.17. Theorem. *Let* $p: X\to\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ *be a resolution such that* X, X_α, $\alpha\in A$, *are paracompact spaces and* $f_{\alpha\beta}$ *are open mappings with* $1\leqq |f_{\alpha\beta}^{-1}(x_\alpha)|\leqq k$, $k\in N$. *The spaces* X *and* $\lim \underline{X}$ *are A-weakly infinite-dimensional iff the spaces* X_α, $\alpha\in A$, *are A-weakly infinite-dimensional.*

We close this Section with the following corollary.

3.18. Corollary. *Let* $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system of paracompact spaces* X_α, $\alpha\in A$, *and open perfect mappings* $f_{\alpha\beta}$ *with the property* $1\leqq |f_{\alpha\beta}^{-1}(x_\alpha)|\leqq k$, $k\in N$. *The spaces* $\lim \underline{X}$ *is A-weakly (A-strongly) infinite-dimensional iff the spaces* X_α, $\alpha\in A$, *are A-weakly (A-strongly) infinite-dimensional.*

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