Characterization of pairs of additive functions with some regularity property

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Dedicated to Prof. Béla Gyires on his 80th birthday

1. For a group G let $\mathcal{A}_G^*$ denote the set of completely additive functions f: $\mathbf{N}\to G$, i.e. those functions for which $f(mn)=f(m)+f(n)$, $\forall m, n\in\mathbf{N}$.

Let $\mathcal{K}$ be the set of all infinite monotonically increasing sequences of positive integers. For typical elements $N=\{n_1<n_2<\dots\}$, $M=\{m_1<m_2<\dots\}$ of $\mathcal{K}$ let $N\oplus M=\{n_1+m_1<n_2+m_2<\dots\}$, $N*M=\{n_1m_1<n_2m_2\dots\}$. Let furthermore for $k\in Z$, $k\oplus N$ denote the sequence of the positive elements of $k+n_j$ $(j=1, 2, \dots)$, and let $kN=\{kn_1<kn_2<\dots\}$ for $k\in\mathbf{N}$.

In what follows under a topological group we always mean a T_0-group, that guarantees that the metric can be chosen to be translation invariant (see [4]).

Let G_1, G_2 be metrically compact Abelian groups, $\varphi\in\mathcal{A}_{G_1}^*$, $\psi\in\mathcal{A}_{G_2}^*$. Let $\mathcal{K}_\varphi$ (resp. $\mathcal{K}_\psi$) be the set of those $N\in\mathcal{K}$ for which the limit $a_\varphi(N):=\lim_j \varphi(n_j)$ (resp. $a_\psi(N):=\lim_j \psi(n_j)$) exists.

We should like to determine all those pairs of functions (φ, ψ) for which $N\in\mathcal{K}_\varphi$ implies that $-1\oplus N\in\mathcal{K}_\psi$. The case $G_1=G_2$, $\varphi=\psi$ under the similar condition $N\in\mathcal{K}_\varphi\Rightarrow 1\oplus N\in\mathcal{K}_\varphi$ has been considered in our earlier papers [1], [2]. It was proved

Lemma 1. *If G is a metrically compact Abelian group, $\varphi\in\mathcal{A}_G^*$, with the property $N\in\mathcal{K}_\varphi\Rightarrow 1\oplus N\in\mathcal{K}_\varphi$, then there exists a continuous homomorphism F: $\mathbf{R}_x\to G$, $\mathbf{R}_x$ denotes the multiplicative group of the positive reals, such that φ is a restriction of F on the set $\mathbf{N}$. Conversely, if F: $R_x\to G$ is a continuous homomorphism, then $\varphi(n):=$ $:=F(n)$ $(\forall n\in\mathbf{N})$ is such a function, for which $Z\in\mathcal{A}_G^*$ and $N\in\mathcal{K}_\varphi$ implies $1\oplus N\in\mathcal{K}_\varphi$.*

In the proof we used the next theorem due to Wirsing [3] which will be formulated now as

Lemma 2. *Let T denote the one-dimensional torus, $\varphi\in\mathcal{A}_T^*$, $\Delta\varphi(n):=\varphi(n+1)-$ $-\varphi(n)\to 0$ $(n\to\infty)$. Then there exists a $\tau\in\mathbf{R}$ such that*

$$\varphi(n)\equiv\tau\log n \pmod 1.$$

2. The following assertions can be proved easily, arguing as it was done in [1], [2].

Lemma 3. *Let K_φ, resp. K_ψ denote the set of all accumulation points of $\{\varphi(n)|n\in\mathbf{N}\}$, resp. $\{\psi(n)|n\in\mathbf{N}\}$. Then $K_\varphi\subseteq G_1$, $K_\psi\subseteq G_2$, and $\varphi(n)\in K_\varphi$, $\psi(n)\in K_\psi$ $(\forall\in\mathbf{N})$.*

Lemma 4. *Let $\varphi\in\mathcal{A}^*_{G_1}$, $\psi\in\mathcal{A}^*_{G_2}$ and assume that $N\in\mathcal{K}_\varphi$ implies that $k\oplus N\in\mathcal{K}_\psi$, with a given $k\in\mathbf{Z}$. Then there exists a function $H\colon K_\varphi\to K_\psi$ with the following property:*

If $N\in\mathcal{K}_\varphi$ and $a_\varphi(N)=g$, then $a_\psi(k\oplus N)=H(g)$. Furthermore $H[K_\varphi]=K_\psi$, H is a continuous function.

Lemma 5. *Assume that the conditions of Lemma 4 hold with $k=-1$. Then they hold with $k=1$, as well.*

Proof. From Lemma 4 we have that for $N\in\mathcal{K}_\varphi$, $a_\varphi(N)=g$, $a_\psi(-1\oplus N)=L(g)$, where $L\colon K_\varphi\to K_\psi$ is a continuous function. By using the additivity of φ and ψ, and the identity $n^2-1=(n-1)(n+1)$, we deduce that $\varphi(n_\nu)\to g$, $\varphi(n_\nu^2)\to 2g$, $\psi(n_\nu-1)\to L(g)$, $\psi(n_\nu^2-1)\to L(2g)$, whence $\psi(n_\nu+1)=\psi(n_\nu^2-1)-\psi(n_\nu-1)$, $\psi(n_\nu+1)\to L(2g)-L(g)=:S(g)$. □

Now and in the sequel we assume that $\varphi(n_\nu)\to g\Rightarrow\varphi(n_\nu-1)\to L(g)$. Then, by Lemma 5, $\varphi(n_\nu+1)\to S(g)$, $S(g)=L(2g)-L(g)$. From Lemma 4 we have that L, S are continuous, $L[K_\varphi]=S[K_\varphi]=K_\psi$.

Let the translation invariant metrics defined in G_1, G_2 be ϱ, σ, respectively. For $A,B\subseteqq G_1$, $C,D\subseteqq G_2$, let

$$\varrho(A,B)=\inf_{x\in A,\,y\in B}\varrho(x,y);\ \varrho(C,D)=\inf_{u\in C,\,v\in D}\sigma(u,v).$$

Let furthermore

$$F_h:=\{g\,|\,g\in K_\varphi,\ L(g)=h\},$$

$$E_h:=\{g\,|\,g\in K_\varphi,\ S(g)=h\}.$$

Lemma 6. *The sets F_h, E_h are closed for each $h\in K_\psi$.*

Proof. It is clear, since L,S are continuous.

Lemma 7. *The condition $\psi(n_\nu)\to h$ holds for some $h\in K_\psi$ if and only if $\varrho(\varphi(n_\nu-1),E_h)\to 0$ $(\nu\to\infty)$.*

Proof. Assume that $\psi(n_\nu)\to h$ and $\varrho(\varphi(n_\nu-1),E_h)\not\to 0$. Then

$$\varrho(\varphi(n_{\nu'}-1),E_h)>\delta(>0)$$

for a suitable infinite subsequence $n_{\nu'}$ of n_ν. Then for a suitable rarefied subsequence $n_{\nu''}$ of $n_{\nu'}$, $\varphi(n_{\nu''}-1)\to\tau$ $(\in K_\varphi)$. Then $\varrho(\tau,E_h)>\delta/2$, consequently $\tau\notin E_h$, $\psi(n_{\nu''})\to S(\tau)\neq h$, which is a contradiction. Assume now that $\varrho(\varphi(n_\nu-1),E_h)\to 0$ and $\psi(n_\nu)\not\to h$. Then there exists such a rarefied subsequence $n_{\nu'}$ for which $\psi(n_{\nu'})\to\tau(\neq h)$. After a further rarefaction we have $\varphi(n_{\nu''}-1)\to\eta$, $\psi(n_{\nu''})\to\tau$. From the continuity of $\varrho(\cdot,E_n)$ we have $\varrho(\eta,E_h)=0$, which implies that $\eta\in E_h$, since E_h is closed. □

Lemma 8. *The condition $\psi(n_\nu)\to h$ holds for some $h\in K_\psi$ if and only if $\varrho(\varphi(n_\nu+1),E_h)\to 0$ $(\nu\to\infty)$.*

Proof. The same as the proof of Lemma 7.

Lemma 9. *If $N\in K_\varphi$, $a_\varphi(N)=0$, then $a_\psi(N)=0$.*

PROOF. Assume that $\varphi(n_\nu)\to 0$. Then $\varphi(n_\nu^3)=3\varphi(n_\nu)\to 0$, $\psi(n_\nu+1)\to S(0)= =L(2\cdot 0)-L(0)=0$, $\psi(n_\nu^3+1)\to 0$. Consequently

$$\psi\left(\frac{n_\nu^3+1}{n_\nu+1}\right)=\psi(n_n^2-n_\nu+1)=\psi\big(n_\nu(n_\nu-1)+1\big)\to 0.$$

From Lemma 7 we have

$$\varrho\big(\varphi(n_\nu(n_\nu-1)),E_0\big)\to 0,$$

and by $\varphi(n_\nu)\to 0$, $\varrho\big(\varphi(n_\nu-1),E_0\big)\to 0$. Applying Lemma 7 again, we get $\varphi(n_\nu)\to 0$. □

Lemma 10. *There exists a continuous homomorphism $H\colon K_\varphi\to K_\psi$, such that $N\in\mathcal{K}_\varphi$ implies that $a_\psi(N)=H(a_\varphi(N))$.*

PROOF. Let $g\in K_\varphi$. From Lemma 3 it follows that there exists an $M\in K_\varphi$, such that $a_\varphi(M)=-g$.

After a suitable rarefaction, if it is needed, we can assume that $a_\psi(M)$ exists as well. Let us assume that M is so chosen. Let now N be an arbitrary sequence $N\in\mathcal{K}_\varphi$, such that $a_\varphi(N)=g$. Then $\varphi(n_\nu m_\nu)=\varphi(n_\nu)+\varphi(m_\nu)\to g-g=0$, and so by Lemma 9, $\psi(n_\nu m_\nu)\to 0$. Since $a_\psi(M)$ exists, therefore $N\in\mathcal{K}_\psi$ and $a_\psi(N)=-a_\psi(M)$. This means that $N\in\mathcal{K}_\varphi\Rightarrow N\in\mathcal{K}_\psi$ and that the value $a_\psi(N)$ depends only on g.

If $a_\varphi(N)=g_1$, $a_\varphi(M)=g_2$, then $\varphi(n_\nu m_\nu)\to g_1+g_2$, and by the additivity of φ and ψ we deduce immediately that $H(g_1+g_2)=H(g_1)+H(g_2)$. Hence (or from Lemma 9) we have $H(0)=0$.

Finally, the continuity of H is straightforward, so we omit the proof. □

Lemma 11. *Let $U=\{g|g\in K_\varphi, H(g)=0\}$. Then U is a compact subgroup of K_φ.*

PROOF. Clear.

3. Let $M\in\mathcal{K}_\varphi$ be a fixed sequence such that $a_\varphi(M)=0$. Let

$$r_1(m,k):=\sup_{n\geqq k}\sigma\big(\psi(nm-1),\psi(n-1)\big), \tag{3.1}$$

$$r_2(m,k):=\sup_{n\geqq k}\sigma\big(\psi(nm+1),\psi(n+1)\big). \tag{3.2}$$

Lemma 12. *For an arbitrary sequence $k_\nu\nearrow\infty$, we have, $r_1(m_\nu,k_\nu)\to 0$, $r_2(m_\nu,k_\nu)\to 0$ $(\nu\to\infty)$.*

PROOF. Assume that $r_1(m_\nu,k_\nu)\not\to 0$. Let $\{\nu'\}$ be a suitable rarefaction of $\{\nu\}$, such that $\psi(n_{\nu'}m_{\nu'}-1)-\psi(n_{\nu'}-1)\to\tau\neq 0$. After a further rarefaction $\{\nu''\}$ of $\{\nu'\}$ we can assume that $\lim\varphi(n_{\nu''})=\eta$ exists. But then $\varphi(n_{\nu''}m_{\nu''})\to\eta$, consequently $\psi(n_{\nu''}-1)\to L(\eta)$, $\psi(n_{\nu''}m_{\nu''}-1)\to L(\eta)$, that contradicts to $\tau\neq 0$. So the first assertion is true. The proof of the second assertion is the saeme, therefore we omit is.

Lemma 13. *We have, $\psi(n+1)-\psi(n)\to 0$ as $n\to\infty$.*

PROOF. In $\sigma\big(\psi(nm-1),\psi(n-1)\big)$ write first $n=k+1$, then we get $\sigma\big(\psi(km+ +n-1),\psi(km)\big)$, substituting now $k=(m-1)s$, and applying the translation invariant property of σ, this is the same as $\sigma\big(\psi(ms+1),\psi(s)\big)$. So we have $r_3(m_\nu,k_\nu)\to$

$\to 0$, where

$$r_3(m, k) := \sup_{n \geq k} \sigma(\psi(mn+1), \psi(n)). \tag{3.3}$$

From (3.2), (3.3) it follows immediately, $\psi(n+1) - \psi(n) \to 0$. □

The condition $\psi(n+1)-\psi(n)\to 0$ is equivalent with $H(\varphi(n+1)-\varphi(n))\to 0$ $(n\to\infty)$.

Let us consider now the factor-group $K_\lambda := K_\varphi | U$, with the natural metric r defined as follows. If $U_g = g_i + U (\in K_\lambda)$, then $r(U_{g_1}, U_{g_2}) = \varrho(g_1 - g_2, U)$. It is clear that K_λ is compact. Let $\lambda \in \mathscr{A}^*_{K_\lambda}$ defined by $\lambda(n) := \varphi(n) + U$. This induces a mapping $B: K_\lambda \to K_\psi$, which is a topological isomorphism. Indeed, let $B(\varphi(n)+U) = \psi(n) = H(\varphi(n))$, and in general $B(U_g) = H(g)$. Since $U_{g_1+g_2} = U_{g_1} + U_{g_2}$, and $H(g_1+g_2) = H(g_1) + H(g_2)$, therefore

$$B(U_{g_1} + U_{g_2}) = B(U_{g_1}) + B(U_{g_2}).$$

It is clear that $U_{g_\nu} \to U_h$ if and only if $H(g_\nu) \to H(h)$, which guarantees that B and the inverse mapping B^{-1} are continuous. From now on we may assume that $\psi = \lambda$.

Let us assume now that for $\varphi = \varphi_1$, $\varphi_2 \in \mathscr{A}^*_{G_1}$ the conditions $N \in \mathscr{K}_{\varphi_i} \Rightarrow -1 \oplus N \in \mathscr{K}_\lambda$ hold for $i = 1, 2$, and that $K_{\varphi_1} = K_{\varphi_2}$. Then the function $H: K_{\varphi_i} \to K_\lambda$ is well defined, $H(\varphi_i(n)) = \lambda(n)$, consequently $H(\varphi_1(n) - \varphi_2(n)) = 0$ $\forall n \in \mathbf{N}$, i.e. $\varphi_1(n) - \varphi_2(n) \in U$, $\varphi_1(n) = \varphi_2(n) + u(n)$, $u \in \mathscr{A}^*_U$.

Let φ, λ be such a pair for which $N \in \mathscr{K}_\varphi \Rightarrow -1 \oplus N \in \mathscr{K}_\lambda$. Then the same condition holds for each pair (φ_1, λ) as well, where $\varphi_1(n) = \varphi(n) + u(n)$, $u \in \mathscr{A}^*_U$, U is defined in Lemma 11.

Since $K_\varphi = K$ is defined as the set of all limit points of sequences $\{\varphi(n_\nu) | n_1 < n_2 < \dots\}$, therefore its cardinality is not greater than the continuum. Since G_1 is a T_0 group, therefore G_1 and so K is connected.

Since $\lambda \in \mathscr{A}^*_{K/U}$ satisfies the condition $\lambda(n+1) - \lambda(n) \to 0$, therefore there exists a continuous homomorphism $\Lambda: \mathbf{R}_x \to K/U$ such that $\Lambda(n) := \lambda(n)$ $n \in \mathbf{N}$.

Conversely, let $U \subseteqq K \subseteqq G_1$ be metrically compact Abelian group, G_1 is a T_0-group. Let Λ: $\mathbf{R}_x \to K/U$ be a continuous homomorphism. Let $\lambda(n) := \Lambda(n)$, $n \in \mathbf{N}$. Then $\lambda \in \mathscr{A}^*_{K/U}$, and the condition $\Delta\lambda(n) = \lambda(n+1) - \lambda(n) \to 0$ holds. Let $\mathscr{P} = \{p\}$ be the set of primes. We define the value $\varphi_1(p)$ by choosing an arbitrary element $h \in U_g$ where U_g is the coset that corresponds to $\lambda(p)$. Then we shall define $\varphi_1(n)$ for composite integers to be a completely additive function in K. Let $H: K \to K/U$ be the natural homomorphism, $H(g) = U_g$. Then we have $H(\varphi_1(n)) = \lambda(n)$, and the condition $N \in \mathscr{K}_{\varphi_1} \Rightarrow -1 \oplus N \in \mathscr{K}_\lambda$ obviously holds.

Collecting our results we get the following

Theorem. *Let G_1, G_2 be metrically compact Abelian groups, G_1 be a T_0 group. Let $\varphi \in \mathscr{A}^*_{G_1}$, $\psi \in \mathscr{A}^*_{G_2}$ be such functions for which $N \in \mathscr{K}_\varphi$ implies that $-1 \oplus N \in \mathscr{K}_\psi$. Then $\psi(n+1) - \psi(n) \to 0$ $(n \to \infty)$. Let U be the same as in Lemma 11. Let $\lambda(n) := \varphi(n) + U$. Then $\lambda \in \mathscr{A}^*_{K_\varphi/U}$, furthermore K_φ/U and K_ψ are topologically isomorphic, $\lambda(n)$ corresponds to $\psi(n)$ $\forall n \in \mathbf{N}$, There exists a continuous homomorphism Λ: $\mathbf{R}_x \to K_\varphi/U$ such that $\lambda(n) = \Lambda(n)$ $(\forall n \in \mathbf{N})$. Furthermore for the natural homomorphism $H: K_\varphi \to K_\varphi/U$ we have $H(\varphi(n+1) - \varphi(n)) = \lambda(n+1) - \lambda(n) \to 0$.*

Conversely, let $U \subseteqq K(\subseteqq G_1)$ *be compact subgroups of* G_1. *Let* $\Lambda: \mathbf{R}_x \to K/U$ *be a continuous homomorphism;* $\lambda(n) := \Lambda(n)$. *Let* $\varphi(p)$ *be any element of the coset* $g+U=\lambda(p)$, *for each prime p. Extend the domain of* φ *to the set n such that* $\varphi \in \mathscr{A}_K^*$. *Then* $H(\varphi(n+1)-\varphi(n))=\lambda(n+1)-\lambda(n)\to 0$.

Consequently, $N \in \mathscr{K}_\psi \Rightarrow -1 \oplus N \in \mathscr{K}_\lambda$.

The special case $G_1 = T$ is formulated as

Corollary. *Let T denote the one-dimensional torus, and* G_2 *be an arbitrary metrically compact Abelian group. Let* $\varphi \in \mathscr{A}_T^*$, $\psi \in \mathscr{A}_{G_2}^*$.

Assume that $N \in \mathscr{K}_\varphi \Rightarrow -1 + N \in \mathscr{K}_\psi$. *Assume furthermore that* $\psi(n)$ *is not identically zero.*

Then there exist $\tau \in \mathbf{R}$, $M \in \mathbf{N}$, $u \in \mathscr{A}_{Z_M}^*$ *such that* $\varphi(n) = \frac{\tau}{M}\log n + u(n) \pmod 1$.

Let $\lambda(n) = M\varphi(n)$.

Then the correspondence $\lambda(n) \leftrightarrow \psi(n)$ $(\forall n \in \mathbf{N})$ *generate a topological isomorphism between* K_λ *and* K_ψ. *The converse assertion is true as well.*

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