Radicals of Γ-near-rings

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Abstract. Γ-near-rings were defined by Satyanarayana, and the left and right operator near-rings, L and R, respectively, of a Γ-near-ring M were proposed by the present author. In this paper, we discuss the prime radical $\mathscr{P}(M)$ of M and show that, if M has a strong left and a right unity, then $\mathscr{P}(L)^{+}=\mathscr{P}(M)$, where $\mathscr{P}(L)$ is the prime radical of the near ring L. $\mathscr{P}(M)$ is a Hoehnke radical in the variety of Γ-near-rings. We also define the Levitzki and nil radicals, $\mathscr{L}(M)$ and $\mathscr{N}(M)$, respectively. Both are Kurosh—Amitsur radicals. If M has a strong left unity, then $\mathscr{L}(L)^{+}=\mathscr{L}(M)$, where $\mathscr{L}(L)$ is the Levitzki radical of the near-ring L. A similar result holds for the nil radical. s-prime ideals of M are defined, and $\mathscr{N}(M)$ is the intersection of the s-prime ideals of M.

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1. Preliminaries

A Γ-near-ring is a triple $(M, +, \Gamma)$ where

(i) $(M, +)$ is a (not necessarily abelian) group;

(ii) Γ is a nonempty set of binary operators on M such that for each $\gamma\in\Gamma$, $(M, +, \gamma)$ is a right near-ring;

(iii) $x\gamma(y\mu z)=(x\gamma y)\mu z$ for all $x, y, z\in M, \gamma, \mu\in\Gamma$.

If, in addition, the following condition is satisfied:

(iv) If $\gamma, \mu\in\Gamma$ and $x\gamma y=x\mu y$ for all $x, y\in M$, then $\gamma=\mu$ then M is called a strong Γ-near ring. We remark that every Γ-near ring M is a strong Γ'-near ring for some choice of Γ'. For $\gamma, \mu\in\Gamma$, we define $\gamma\sim\mu$ if $x\gamma y=x\mu y$ for all $x, y\in M$. Clearly, this is an equivalence relation on Γ. Let $[\gamma]$ denote the equivalence class containing γ, and let $\Gamma'=\{[\gamma]:\gamma\in\Gamma\}$. Clearly, M is a strong Γ'-ring with the operation $x[\gamma]y=x\gamma y(x, y\in M, \gamma\in\Gamma)$.

Let M and M' be Γ-near rings (for the same Γ) and let $f: M\to M'$ be a group homomorphism. Then, if $f(x\gamma y)=f(x)\gamma f(y)$ for all $x, y\in M$ and $\gamma\in\Gamma$, f is called a Γ-near-ring homomorphism. A subset I of M is called an ideal of M (denoted by $I\lhd M$) if it is the kernel of some Γ-near-ring homomorphism. It is easily verified that I is an ideal of M if and only if:

(i) $(I, +)$ is a normal divisor of $(M, +)$;

(ii) For all $a\in I$, $x, y\in M$ and $\gamma\in\Gamma$, $a\gamma x\in I$ and $x\gamma(a+y)-x\gamma y\in I$.

If $I\lhd M$, the factor group M/I is a Γ-near-ring with the normal addition operation, and $(x+I)\gamma(y+I)=x\gamma y+I(x, y\in M, \gamma\in\Gamma)$. If $x\in M$, the ideal generated by x is the smallest ideal of M containing x (i.e. the intersection of all ideals of M containing x), and will be denoted $\langle x\rangle$. Similar notation will be used in the variety of near-rings. In both varieties, the notation $I \lhd M$ will mean "I is an ideal of M". If $U, V \subseteq M$ and $\Phi \subseteq \Gamma$, we denote $U\Phi V = \{u\gamma v: u \in U, \gamma \in \Phi, v \in V\}$.

Γ-near rings constitute a generalization of Γ-rings in the sense of Barnes [1], and also of near-rings, in the sense that every near ring $(N, +, \cdot)$ is a Γ-near-ring with $\Gamma = \{\cdot\}$. They also form a variety of Ω-groups, and the ideals and homomorphisms as defined above coincide with the corresponding definitions given in [9]. Subdirect sums of Γ-near-rings are defined in the natural way. The usual isomorphism theorems for rings are valid for Γ-near-rings ([9] Theorems 3A—3D).

Kurosh—Amitsur radical classes for Ω-groups have been defined by RJABUHIN. A class $\mathcal{R}$ of Γ-near rings is a radical class in this context if:

R1: $\mathcal{R}$ is closed under homomorphisms.

R2: If M is a Γ-near ring, and $\mathcal{A}$ is a class of $\mathcal{R}$-ideals of M which is linearly ordered by inclusion, then

$$\bigcup_{A \in \mathcal{A}} A \in \mathcal{R}.$$

R3: If M is a Γ-near ring and $A \lhd M$, then $A \in \mathcal{R}$, $M/A \in \mathcal{R}$ implies $M \in \mathcal{R}$.

If, in addition, the following axiom is satisfied:

R4: $M \in \mathcal{R}$, $A \lhd M$ implies $A \in \mathcal{R}$

then $\mathcal{R}$ is called a hereditary radical class.

2. The operator near-rings

Let M be a Γ-near-ring. In [3], the left and right operator near-rings of M were defined. Let $\mathcal{L}$ be the set of all mappings of M into itself which act on the left. Then $\mathcal{L}$ is a right near-ring with the operations pointwise addition and composition of mappings. Let $x \in M$, $\gamma \in \Gamma$. Define $[x, \gamma]: M \to M$ by $[x, \gamma]y = x\gamma y$ for all $y \in M$. The sub-near-ring L of $\mathcal{L}$ generated by the set $\{[x, \gamma]: x \in M, \gamma \in \Gamma\}$ is called the left operator near-ring of M.

If $I \subseteq L$, then $I^+ = \{x \in M: [x, \gamma] \in I$ for all $\gamma \in \Gamma\}$.

If $J \subseteq M$, $J^{+\prime} = \{l \in L: lx \in J$ for all $x \in M\}$. It is shown in [3], Proposition 5, that $I \lhd L$ implies $I^+ \lhd M$ and that $J \lhd M$ implies $J^{+\prime} \lhd L$. Furthermore, it is easily seen that these mappings preserve intersections of sets.

If $l \in L$, $x \in M$ and $\gamma \in \Gamma$, then it can be shown that $l[x, \gamma] = [lx, \gamma]$. This identity, which is a consequence of the right distributivity of M, will be of use later.

A right operator near-ring R of M is defined analogously to the definition of L. Let $\mathcal{R}$ be the left-near-ring of all mappings of M into itself which act on the right. If $\gamma \in \Gamma$, $y \in M$, we define $[\gamma, y]: M \to M$ by $x[\gamma, y] = x\gamma y$ for all $x \in M$. R is the sub-near-ring of $\mathcal{R}$ generated by the set $\{[\gamma, y]: \gamma \in \Gamma, y \in M\}$.

Example 2.1. (SATYANARAYANA [10])

Let $(G, +)$ be a (not necessarily abelian) group and let X be a nonempty set. Let M be the set of all mappings of X into M, and Γ the set of all mappings of M into X, mappings being taken to act on the left. It is easily seen that M is a Γ-near-ring with the operations pointwise addition and composition of mappings. Such a Γ-near-ring will be referred to as a Γ-near-ring of mappings.

Let M be a Γ-near-ring, and let M' be a Γ'-near-ring. We say that M is embeddable in M', if there exist a group monomorphism $f: M \to M'$ and an injective mapping $\varphi: \Gamma \to \Gamma'$ satisfying, for all $x, y \in M$, $\gamma \in \Gamma$:

$$f(x\gamma y) = f(x)\varphi(\gamma)f(y).$$

Proposition 2.2. *Every strong Γ-near-ring M is embeddable in a gamma-near-ring of mappings.*

PROOF. Let $R_0 = \{[\gamma, x]: \gamma \in \Gamma, x \in M\}$ and let $R_1 = R_0 \cup \{\infty\}$, where ∞ denotes any element which is distinct from the elements of R_0. Let M' and Γ' denote, respectively the sets of all mappings of R_1 into M and of M into R_1. We show that the Γ-near-ring M is embeddable in the Γ'-near-ring M'.
Suppose that $x \in M$. We define $\hat{x}: R_1 \to M$ by

$$\hat{x}(r) = xr \quad (r \in R_0)$$

$$\hat{x}(\infty) = x.$$

Note that, if $x, y, z \in M$ and $\gamma \in \Gamma$, then

$$\begin{aligned} \widehat{x+y}([\gamma, z]) &= (x+y)\gamma z \\ &= x\gamma z + y\gamma z \\ &= (\hat{x}+\hat{y})([\gamma, z]) \\ \widehat{x+y}(\infty) &= x+y \\ &= (\hat{x}+\hat{y})(\infty). \end{aligned}$$

Hence $\widehat{x+y} = \hat{x}+\hat{y}$, i.e. the mapping $x \to \hat{x}$ is a group homomorphism. Moreover, suppose that $\hat{x} = \hat{y}$. Then $\hat{x}(\infty) = \hat{y}(\infty)$ i.e. $x = y$. Thus $x \to \hat{x}$ defines a group monomorphism.

Now suppose that $\gamma \in \Gamma$. Define $\hat{\gamma}: M \to R_1$ by $\hat{\gamma}(x) = [\gamma, x]$ for all $x \in M$. Then if $\gamma, \mu \in \Gamma$ and $\hat{\gamma} = \hat{\mu}$, it follows that $[\gamma, y] = [\mu, y]$ for all $y \in M$, whence $x\gamma y = x\mu y$ for all $x, y \in M$. Since M is a strong Γ-near-ring, $\gamma = \mu$. Hence the mapping $\gamma \to \hat{\gamma}$ is injective.

Finally, suppose that $x, y, z \in M$ and $\gamma, \mu \in \Gamma$. Then

$$\begin{aligned} \hat{x}\hat{\gamma}\hat{y}([\mu, z]) &= \hat{x}\hat{\gamma}(y\mu z) \\ &= \hat{x}([\gamma, y\mu z]) \\ &= x\gamma y\mu z \\ &= \widehat{x\gamma y}([\mu, z]). \\ \hat{x}\hat{\gamma}\hat{y}(\infty) &= \hat{x}\hat{\gamma}(y) \\ &= \hat{x}([\gamma, y]) \\ &= x\gamma y \\ &= \widehat{x\gamma y}(\infty). \end{aligned}$$

Hence $\widehat{x\gamma y} = \hat{x}\,\hat{\gamma}\,\hat{y}$, and *the proof is complete.*

3. Prime ideals and the prime radical

In this section, let M be a Γ-near-ring, and let L be its left operator near-ring. If $P \lhd M$, and $A, B \lhd M$, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, then P is called a prime ideal of M. The following characterization of prime ideals of M is proved in exactly the same way as its counterpart for near-rings ([11], Lemma 2).

Proposition 3.1. *Let $P \lhd M$. Then the following are equivalent:*
(a) *P is a prime ideal of M.*
(b) *If $x, y \in M$, $x, y \notin P$, then there exist $x' \in \langle x\rangle$, $y' \in \langle y\rangle$ and $\gamma \in \Gamma$ such that $x'\gamma y' \notin P$.*

If $\sum_i [d_i, \delta_i] \in L (d_i \in M, \gamma_i \in \Gamma)$ has the property that $\sum_i d_i \delta_i x = x$ for all $x \in M$, then $\sum_i [d_i \delta_i]$ is called a left unity for M. A strong left unity for M is an element $[d, \delta]$ of L such that $d\delta x = x$ for all $x \in M$. Right unities and strong right unities are defined analogously. It may be verified that if l is a left unity for M, then l is the unity of the near-ring L.

Proposition 3.2. *Suppose M has a right unity $\sum_i [\delta_i, d_i]$. Then, if P is a prime ideal of M, $P^{+'}$ is a prime ideal of L.*

Proof. Suppose that $l, l' \in L - P^{+'}$. Then there exist $x, y \in M$ such that lx, $ly \notin P$. Since P is a prime ideal of M, there exist $u \in \langle lx\rangle$, $v \in \langle ly\rangle$ and $\gamma \in \Gamma$ such that $u\gamma v \notin P$. We claim that $[u\gamma v, \delta_j] \notin P^{+'}$ for at least some j. For if $[u\gamma v, \delta_j] \in P^{+'}$ for all j, it follows that $\sum_i u\gamma v\delta_i d_i \in P$, i.e. $u\gamma v \in P$, contradicting our choice of u, v and γ. Let j be such that $[u\gamma v, \delta_j] \notin P^{+'}$, i.e. $[u, \gamma][v, \delta_j] \notin P^{+'}$. Now let $I \lhd L$ be such that $l \in I$. Then, if $\mu \in \Gamma$, then $[lx, \mu] = l[x, \mu] \in I$. Hence, $lx \in I^+$. It follows that $\langle lx\rangle \subseteq I^+$, and hence that $u \in I^+$. Consequently, $[u, \gamma] \in I$. This implies that $[u, \gamma] \in \langle l\rangle$. Similarly, $[v, \delta_j] \in \langle l'\rangle$. This completes the proof that $P^{+'}$ is a prime ideal of L.

Proposition 3.3. *Let M be a Γ-near-ring with a strong left unity $[d, \delta]$. If Q is a prime ideal of L, then Q^+ is a prime ideal of M.*

Proof. Suppose $x, y \notin Q^+$. Then there exist $\gamma, \mu \in \Gamma$ such that $[x, \gamma]$, $[y, \mu] \notin Q$. Since Q is a prime ideal of L, there exist $l_1 \in \langle [x, y]\rangle$ and $l_2 \in \langle [y, \mu]\rangle$ such that $l_1 l_2 \notin Q$. Now $l_1 l_2 = l_1[d, \delta] l_2 [d, \delta] = [(l_1 d)\delta(l_2 d), \delta]$. It follows that $(l_1 d)\delta(l_2 d) \notin Q^+$. Now let $z \in M$. Then $x\gamma z \in \langle x\rangle$, whence $[x, \gamma] \in \langle x\rangle^{+'}$. Hence $\langle [x, \gamma]\rangle \subseteq \langle x\rangle^{+'}$. It follows that $l_1 \in \langle x\rangle^{+'}$, whence $l_1 d \in \langle x\rangle$. Similarly, $l_2 d \in \langle y\rangle$. This completes the proof that Q^+ is a prime ideal of M.

Proposition 3.4. *Let M be a Γ-near-ring with left operator near-ring L. If M has a strong left unity $[d, \delta]$ and a right unity $\sum_i [\varepsilon_i, e_i]$, then*

$$\mathscr{P}(L)^+ = \mathscr{P}(M).$$

Proof. Let P be a prime ideal of L. Then, by Proposition 3.2, P^+ is a prime ideal of M. Moreover, $(P^+)^{+'} = P$ by [3], Proposition 5. Suppose Q is a prime ideal of M. Then by Proposition 3.3, $Q^{+'}$ is a prime ideal of L, and $(Q^{+'})^+ = Q$, by [3], Proposition 5. Thus the mapping $P \to P^+$ defines a one-to-one correspondence between the sets of prime ideal of L and M. Hence

$$\begin{aligned}\mathscr{P}(L)^+ &= (\cap P)^+ \quad \text{(as } P \text{ runs through the prime ideals of } L)\\ &= \cap P^+\\ &= \mathscr{P}(M).\end{aligned}$$

Remarks

1. The above result was shown by Coppage and Luh ([8], Theorem 4.1) to hold for arbitrary Γ-rings. It is an open question whether the assumption of the existence of unities can be dropped for Γ-near-rings.
2. The above definition for primeness in Γ-near-rings coincides with that given for Ω-groups by Buys and Gerber [6]. It is well known that the prime radical is not a Kurosh—Amitsur radical in the variety of near-rings. Since Γ-near-rings are a generalization of near-rings, the same will be true in this variety.
3. A Hoehnke radical in a variety G of Ω-groups is a mapping P which assigns to each $G\in\mathscr{G}$ an ideal $P(G)$ of G such that:
   (i) If $f\colon G\to G'$ is a surjective Ω-homomorphism, then $f(P(G))\subseteq P(G')$;
   (ii) $p(G/P(G))=\langle 0\rangle$ for all $G\in\mathscr{G}$.

In view of the definition in [5] and [6], Corollary 2.5 and Theorems 3.3 and 3.5, the prime radical is a Hoehnke radical in the variety of Γ-near-rings.

4. The Levitzki radical

The Levitzki radical for near-rings has been defined by Bhandari and Sexana [2]. If N is a near-ring, and $A\lhd N$, then A is locally nilpotent, if for every finite subset F of A, $F^n=\{0\}$ for some positive integer n. The Levitzki radical of N, $\mathscr{L}(N)$, is the sum of the locally nilpotent ideals of N. Buys and Gerber [7] extended the notion of local nilpotence to arbitrary varieties of Ω-groups, in a way that generalizes the definition of Bhardari and Sexana.

Following Coppage and Luh [8], we define an ideal A of a Γ-near-ring M to be locally nilpotent if for all finite subsets F and Φ of A and Γ, respectively, $(F\Phi)^nF=F\Phi F\dots\Phi F=\{0\}$, for some positive integer n. The Levitzki radical, $\mathscr{L}(M)$, is the sum of the locally nilpotent ideals of M.

Another notion of local nilpotence for Γ-near-rings is available using the definition of Buys and Gerber. In terms of this definition, $A\lhd M$ is locally nilpotent if for any finite subset F of A, there exist $\gamma_1, \gamma_2, \dots, \gamma_n\in\Gamma$ such that $F\gamma_1F\gamma_2\dots\gamma_nF=\{0\}$. It is clear that local nilpotence in the sense of Coppage and Luh implies local nilpotence in this sense. The coverse is not, however true. Let $M=Z_6=\Gamma$. Clearly, M is a Γ-near-ring with the addition and multiplication operations on Z_6. The set $A=\{\bar{0}, \bar{3}\}$ is an ideal of M. Moreover, A is locally nilpotent in the sense of Buys and Gerber, for if $F\subseteq A$, then $F\{\bar{2}\}F=\{0\}$. It is not, however, locally nilpotent in the sense of Coppage and Luh, since if $F=\Phi=\{\bar{3}\}$, then $(F\Phi)^nF\neq\{\bar{0}\}$ for any positive integer n. Throughout this section, the term locally nilpotent will mean locally nilpotent in the sense of Coppage and Luh. The next two results can be proved by easy modifications of the corresponding results in [7].

Proposition 4.1. *Let A be an ideal of the Γ-near-ring M. Then M is locally nilpotent if and only if the Γ-near-rings A and M/A are locally nilpotent.*

Proposition 4.2. *Let M be an arbitrary Γ-near-ring. Then $\mathscr{L}(M)=\cap\{P\colon P$ is a prime ideal of M and $\mathscr{L}(M/P)=\langle 0\rangle\}$.*

Remarks

1. The Levitzki radical is a hereditary Kurosh—Amitsur radical in the variety of Γ-near-rings. The fact that the Levitzki radical class $\mathscr{L}$ satisfies axioms R1, R2 and R4 is easily verified, while the fact that it satisfies R3 is a consequence of Proposition 4.1.
2. It follows immediately from Proposition 4.2 that for an arbitrary Γ-near-ring M, $\mathscr{P}(M)\subseteqq\mathscr{L}(M)$.

Proposition 4.3. *Let M be a Γ-near-ring with left operator near-ring L. Then:*
(a) $\mathscr{L}(L)^+\subseteqq\mathscr{L}(M)$;
(b) *If M has a strong left unity, then* $\mathscr{L}(L)^+=\mathscr{L}(M)$.

PROOF.

(a) Since $\mathscr{L}$ is a Kurosh—Amitsur radical in the variety of near-rings ([7], Theorem 3.8), $\mathscr{L}(L)$ is a locally nilpotent ideal of L. Let F, Φ be finite subsets of $\mathscr{L}(L)^+$ and Γ, respectively. Then, if $G=\{[x,\gamma]\colon x\in F, \gamma\in\Phi\}$, G is a finite subset of $\mathscr{L}(L)$. Hence, $G^n=\{0\}$ for some positive integer n. It follows that $(F\Phi)^n F=\{0\}$, i.e. $\mathscr{L}(L)^+$ is a locally nilpotent ideal of M. Since $\mathscr{L}$ is a Kurosh—Amitsur radical in the variety of Γ-near-rings, $\mathscr{L}(L)^+\subseteqq\mathscr{L}(M)$.

(b) Clearly, we need only prove $\mathscr{L}(M)\subseteqq\mathscr{L}(L)^+$. Let $l\in\mathscr{L}(M)^{+\prime}$. Then $l=l[d,\delta]=[ld,\delta]$. Hence, every element of $\mathscr{L}(M)^{+\prime}$ can be written in the form $[x,\delta]$ for some $x\in\mathscr{L}(M)$. Let $F=\{[x_1,\delta],\ldots,[x_n,\delta]\}$ be a finite subset of $\mathscr{L}(M)^{+\prime}$, where $x_i\in\mathscr{L}(M)$ $(1\leqq i\leqq n)$. Since $\mathscr{L}$ is a Kurosh—Amitsur radical in the variety of Γ-near-rings $\mathscr{L}(M)$ is locally nilpotent. Let $G=\{x_1,\ldots,x_n\}$ and $\Phi=\{\delta\}$. Then there exists a positive integer n such that $(F\Phi)^n F=\{0\}$. It follows that $F^{n+1}=\{0\}$, and hence that $\mathscr{L}(M)^{+\prime}$ is a locally nilpotent ideal of L. Hence, $\mathscr{L}(M)^{+\prime}\subseteqq\mathscr{L}(L)$. Thus $\mathscr{L}(M)\subseteqq(\mathscr{L}(M)^{+\prime})^+\subseteqq\mathscr{L}(L)^+$, and the proof is complete.

Remark. Proposition 4.3 (b) was shown by COPPAGE and LUH ([8], Theorem 7.2) to hold for arbitrary varieties of Γ-rings (i.e. without making any assumptions about unities). Whether this result holds for arbitrary varieties of Γ-near-rings is an open question.

5. The nil radical

The nil radical of a Γ-ring was defined by COPPAGE and LUH [8]. Let M be a Γ-near-ring, and let $\gamma\in\Gamma$. Then $x\in M$ is said to be γ-nilpotent if there exists a positive integer n such that $(x\gamma)^n x=x\gamma x\ldots\gamma x=0$. If $A\lhd M$, then A is called a γ-nil ideal of M if A consists entirely of γ-nilpotent elements. The γ-nil radical of M, $\mathscr{N}_\gamma(M)$, is the sum of the γ-nil ideals of M.

Proposition 5.1. *The class $\mathscr{N}_\gamma$ of γ-nil Γ-near-rings is a hereditary radical class.*

The proof of this proposition is straightforward, and will be omitted.

A subset S of a Γ-near-ring M is called a γ-system if, for all $x, y\in S$, $x\gamma y\in S$. An ideal P of M is called $s-\gamma$-prime if $M-P$ contains a γ-system S such that $I\lhd M$, $I\subsetneqq P$ implies that $S\cap I\neq\emptyset$. M is called an $s-\gamma$-prime Γ-near-ring if $\langle 0\rangle$ is an $s-\gamma$-prime ideal of M.

Lemma 5.2. *Let M be a Γ-near-ring, and let $P \lhd M$. Then P is an s—γ-prime ideal of M if and only if M/P is an $s-\gamma$-prime Γ-ring.*

PROOF. Let P be an $s-\gamma$-prime ideal of M. Suppose that S is a γ-system in $M-P$ such that $I \cap S \neq \emptyset$ for every ideal I of M which is not contained in P. Then let $S_0=\{x+P\colon x \in S\}$. Clearly S_0 is a γ-system in M/P and $P \notin S_0$. Suppose that I is a nonzero ideal of M/P. Then by [9], Theorem 3B (applied to the natural homomorphism of M onto M/P), $I=J/P$ for some ideal J of M which properly contains P. Hence $J \cap S \neq \emptyset$, from which $I \cap S_0 \neq \emptyset$. Hence M/P is an $s-\gamma$-prime Γ-near-ring.

Conversely, suppose that M/P is an $s-\gamma$-prime Γ-near-ring. Then $M/P-\{P\}$ contains a γ-system S such that $S \cap I \neq \emptyset$ for every nonzero ideal I of M/P. Let $S_1=\{x \in M\colon x+P \in S\}$. Suppose that $J \lhd M$ and $J \subsetneqq P$. Then $J+P \lhd M$ ([9], Theorem 3D) and $(J+P)/P$ is a nonzero ideal of M/P. Hence $(J+P)/P \cap S \neq \emptyset$, i.e. there exist $j \in J$ and $p \in P$ such that $j+p+P \in S$, whence $j+P \in S$. Hence $j \in J \cap S_1$. Clearly, S_1 is a γ-system in M which is contained in $M-P$. Thus, P is an $s-\gamma$-prime ideal of M, as required.

Lemma 5.3. *Let M be an $s-\gamma$-prime Γ-near-ring. Then $\mathcal{N}_\gamma(M)=\langle 0\rangle$.*

PROOF. Let S be a γ-system of M which does not contain 0, such that $S \cap I \neq \emptyset$ for all $0 \neq I \lhd M$. Then if $0 \neq I \lhd M$, let $x \in S \cap I$. It follows that $(x\gamma)^n x \in S$ for all positive integers n. Since $0 \notin S$, x is not γ-nilpotent, and hence I is not a γ-nil ideal of M. Since I is arbitrary, it follows that $\mathcal{N}_\gamma(M)=\langle 0\rangle$.

Proposition 5.4. *Let M be an arbitrary Γ-near-ring. Then*

$$N_\gamma(M) = \cap P$$

where the intersection runs over the $s-\gamma$-prime ideals of M.

PROOF. Let P be an $s-\gamma$-prime ideal of M. Then M/P is an $s-\gamma$-prime Γ-near-ring by Lemma 5.2, whence $\mathcal{N}_\gamma(M/P)=\langle 0\rangle$ by Lemma 5.3. It follows from the fact that $\mathcal{N}_\gamma$ is a radical class that $\mathcal{N}_\gamma(M) \subseteqq P$. Hence, $\mathcal{N}_\gamma(M) \subseteqq \cap P$.

Now suppose that $x \in M-\mathcal{N}_\gamma(M)$. Then $\langle x\rangle$ is not a γ-nil ideal of M. Hence, there exists $y \in \langle x\rangle$ such that $(y\gamma)^n y \neq 0$ for all positive integers n. Let $S=\{y, y\gamma y, \dots, (y\gamma)^n y, \dots\}$. It is easily seen that S is a γ-system of M which does not meet $\mathcal{N}_\gamma(M)$. The family I of all ideals of M which contain $\mathcal{N}_\gamma(M)$, but do not meet S is nonempty, since $\mathcal{N}_\gamma(M) \in I$. By Zorn's Lemma, I has a maximal element P. Now let $S_0= =\{z+P\colon z \in S\}$. Clearly, S_0 is a γ-system of M/P, which does not contain 0. Let $0 \neq U \lhd M/P$. Then $U=V/P$ for some ideal V of M which properly contains P. Since $V \cap S \neq \emptyset$, $V/P \cap S_0 \neq \emptyset$. Hence, M/P is an $s-\gamma$-prime Γ-near-ring. By Lemma 5.2, P is an $s-\gamma$-prime ideal of M. But $x \notin P$, for this would imply that $\langle x\rangle \subseteqq P$, whence $y \in P$, which contradicts the definition of P. Hence, $\cap P \subseteqq \mathcal{N}_\gamma(M)$, and the proof is complete.

Corollary 5.5. *Let M be an arbitrary Γ-near-ring. Then $\mathcal{N}_\gamma(M)=\langle 0\rangle$ if and only if M is isomorphic to a subdirect sum of $s-\gamma$-prime Γ-near-rings.*

An ideal A of a Γ-near-ring M is called nil if A is γ-nil for each $\gamma \in \Gamma$. The nil radical of M, $\mathcal{N}(M)$, is the sum of the nil ideals of M. It is easily shown that $\mathcal{N}$ is a hereditary radical class in the variety of Γ-near-rings. The next result is proved exactly as for Γ-rings ([4], Proposition 2.1).

Proposition 5.6. *Let M be an arbitrary Γ-near-ring. Then*

$$\mathcal{N}(M) = \bigcap_{\gamma \in \Gamma} \mathcal{N}_\gamma(M).$$

Remark. The nil radical $\mathcal{N}(A)$ of a near-ring A has been defined by VAN DER WALT [11]. If we define an ideal A of a Γ-near-ring to be s-prime if A is $s-\gamma$-prime for some $\gamma \in \Gamma$, then the next result generalises [11], Theorem 10.

Proposition 5.7. *Let M be an arbitrary Γ-near-ring. Then $\mathcal{N}(M) = \langle 0 \rangle$ if and only if M is isomorphic to a subdirect sum of s-prime Γ-near-rings.*

Proposition 5.8. *Let M be a Γ-near-ring with left operator near-ring L. Then*
(a) $\mathcal{N}(L)^+ \subseteqq \mathcal{N}(M)$.
(b) *If M has a strong left unity $[d, \delta]$ then*

$$\mathcal{N}(L)^+ = \mathcal{N}(M).$$

PROOF.
(a) Let $x \in \mathcal{N}(L)^+$ and $\gamma \in \Gamma$. Then $[x, \gamma] \in \mathcal{N}(L)$, from which $[x, \gamma]^n = 0$ for some positive integer n. It follows that $(x\gamma)^n x = 0$, and hence that $\mathcal{N}(L)^+$ is a nil ideal in M. Since $\mathcal{N}$ is a radical class in the variety of Γ-near-rings, $\mathcal{N}(L)^+ \subseteqq \mathcal{N}(M)$, as required.
(b) Let $l \in \mathcal{N}(M)^{+\prime}$. As in the proof of Proposition 4.3 (b), $l = [x, \delta]$ for some $x \in \mathcal{N}(M)$. Let n be a positive integer such that $(x\delta)^n x = 0$. Clearly, $[x, \delta]^{n+1} = 0$, so $\mathcal{N}(M)^{+\prime}$ is a nil ideal of L.

By [11], Theorem 9, $\mathcal{N}(M)^{+\prime} \subseteqq \mathcal{N}(L)$. Hence, $\mathcal{N}(M) \subseteqq (\mathcal{N}(M)^{+\prime})^+ \subseteqq \mathcal{N}(L)^+$, and the proof is complete.

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