The period length of Voronoi's algorithm for certain cubic orders

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1. Introduction. Let D be a positive integer which is not a perfect square. Also, let $\Phi=(P+\sqrt{D})/Q$, where $P, Q\in\mathbf{Z}$ (the rational integers) and $Q|D-P^2$. It is well known (see, for example PERRON [6] or CHRYSTAL [1]), that the Regular Continued Fraction expansion of $\Phi=\Phi_0$ is given by

$$\Phi = \langle q_0, q_1, q_2, \ldots, q_{n-1}, \Phi_n\rangle,$$

where $\Phi_m=(P_m+\sqrt{D})/Q_m=1/(\Phi_{m-1}-q_{m-1})$, $q_m=[\Phi_m]$ $(m=0, 1, 2, 3, \ldots)$. We also know that this continued fraction must ultimately become periodic; that is, for some $k\in\mathbf{Z}^+$ we will get

$$\Phi_{m+k} = \Phi_m \tag{1.1}$$

for all $m\geqq 0$. The least positive value of k for which (1.1) occurs is called the *period length* of the continued fraction expansion of Φ.

If we put $A_{-2}=0$, $A_{-1}=1$, $B_{-2}=1$, $B_{-1}=0$ and define

$$A_m = q_mA_{m-1}+A_{m-2},\ B_m = q_mB_{m-1}+B_{m-2}\quad (m = 0, 1, 2, \ldots)$$

then

$$A_m/B_m = \langle q_0, q_1, q_2, \ldots, q_m\rangle.$$

Also, (see, for example, WILLIAMS and WUNDERLICH [12]), if we define

$$\theta_n = A_{n-2}-\bar{\Phi}B_{n-2},$$

we get

$$N(\theta_n) = (-1)^{n-1}Q_{n-1}/Q_0. \tag{1.2}$$

Here, as usual, we use $N(\alpha)$ to denote the norm of α; in this case $N(\alpha)=\alpha\bar{\alpha}$, where $\bar{\alpha}$ is the conjugate of α. Note that

$$\theta_n = (Q_{n-1}\prod_{i=1}^{n-1}\Phi_i)/Q_0 = \prod_{i=1}^{n-1}\theta_g^{(i)},$$

where

$$\theta_g^{(i)} = (P_i+\sqrt{D})/Q_{i-1} = -(\bar{\Phi}_i)^{-1}.$$

When $P=0$, $Q=1$, we have

$$A_{n-2}^2-DB_{n-2}^2 = (-1)^{n-1}Q_{n-1} \tag{1.3}$$

from (1.2). We also have

Theorem 1.1. *If* $gcd(x, y)=1$, $x, y>0$, *and*

$$x^2-Dy^2 = N, \tag{1.4}$$

where $|N|<\sqrt{D}$, *then* $x=A_m$ *and* $y=B_m$ *for some* $m\geqq 0$. □

Because of (1.3) we see that the continued fraction expansion of $\sqrt{D}$ provides us with the vaules of $|N|<\sqrt{D}$ for which (1.4) has solutions. In view of this it is of some interest to be able to determine the continued fraction expansion of $\sqrt{D}$. For most values of $\sqrt{D}$, however, this is rather difficult to do because k tends to be large; but for certain, simple values of D we have small values for k and the continued fractions expansion of D is easy to find. In our presentation of some of these below we use $\delta=\sqrt{D}$.

(i) For $D=M^2+1$, we have $k=1$, $q_0=M$, $P_1=M$, $Q_1=1$, $q_1=2M$, $\theta_2=M+\delta$.

(ii) For $D=M^2-1$, we have $k=2$, $q_0=M-1$, $P_1=M-1$, $Q_1=2M-2$, $q_1=1$, $P_2=M-1$, $Q_2=1$, $q_2=2M-2$, $\theta_2=M-1+\delta$, $\theta_3=M+\delta$.

(iii) For $D=M^2+2$, we have $k=2$, $q_0=M$, $P_1=M$, $Q_1=2$, $q_1=M$, $P_2=M$, $Q_2=1$, $q_2=2M$, $\theta_2=M+\delta$, $\theta_3=M^2+1+M\delta$.

(iv) For $D=M^2-2$ $(M>2)$, we have $k=4$, $q_0=M-1$, $P_1=M-1$, $Q_1=2M-3$, $q_1=1$, $P_2=M-2$, $Q_2=2$, $q_2=M-2$, $P_3=M-2$, $Q_3=2M-3$, $q_3=1$, $P_4=M-1$, $Q_4=1$, $q_4=2M-2$, $\theta_2=M-1+\delta$, $\theta_3=M+\delta$, $\theta_4=M^2-M-1+(M-1)\delta$, $\theta_5=M^2-1+M\delta$.

(v) For $D=M^2+M$, we have $k=2$, $q_0=M$, $P_1=M$, $Q_1=M$, $q_1=2$, $P_2=M$, $Q_2=1$, $q_2=2M$, $\theta_2=M+\delta$, $\theta_3=2M+1+2\delta$.

(vi) For $D=M^2-M$, we have $k=2$, $q_0=M-1$, $P_1=M-1$, $Q_1=M-1$, $q_1=2$, $P_2=M-1$, $Q_2=1$, $q_2=2(M-1)$, $\theta_2=M-1+\delta$, $\theta_3=2M-1+2\delta$,

Let $\Delta_{\mathcal{O}}$ denote the discriminant of an order $\mathcal{O}$ of $\mathcal{K}=\mathcal{Q}(\delta)$, the quadratic field formed by adjoining δ to the rationals $\mathcal{Q}$. Now $\mathcal{O}$ has a unitary basis $\{1, \Phi\}$ where $\Phi\in\mathcal{O}$ and $\Delta_{\mathcal{O}}=(\Phi-\bar{\Phi})^2=I^2\Delta$, where $I\in\mathbf{Z}$ and Δ is the discriminant of the maximal order $\mathcal{O}_{\mathcal{K}}\supseteqq\mathcal{O}$. Also, if $D=E^2D'$, where D' is square-free, we have $\Delta=4D'$ when $D'\equiv 2, 3 \pmod 4$ and $\Delta=D'$ when $D\equiv 1 \pmod 4$. The following result is a generalization of Theorem 1.1.

Theorem 2.2. *Let* $\mathcal{O}$ *be any order of* $\mathcal{K}=\mathcal{Q}(\delta)$ *with basis* $\{1, \Phi\}$. *If* $\alpha=x+y\Phi$ $(x, y\in\mathbf{Z})$ *with* $\alpha>1$ *and* $gcd(x, y)=1$ *and*

$$|N(\alpha)| < \sqrt{\Delta_{\mathcal{O}}}/2, \tag{1.5}$$

then

$$\alpha = A_m-\bar{\Phi}B_m$$

for some $m\geqq 1$. *Here the values of* A_m *and* B_m *are computed from the regular continued fraction expansion of* Φ.

PROOF. Similar to the proof of Theorem 1.3 of WILLIAMS and DUECK [11]. □

Of particular interest is the maximal order $\mathcal{O}_{\mathcal{K}}$. Note that for (ii), (iii), (iv), (v), (vi) above we already have the continued fraction expansion for the maximal order when D is square-free. However, for case (i) when $2|M$ and D is square-free, we have

$D\equiv 1 \pmod 4$ and $\mathcal{O}_{\mathcal{K}}$ has basis $\{1, \Phi\}$ with $\Phi=(1+\sqrt{D})/2$. In this case we find that the continued fraction expansion of $(1+\sqrt{D})/2$ has $k=3$, $q_0=M/2$, $P_1=M-1$, $Q_1=M$, $q_1=1$, $P_2=1$, $Q_2=M$, $q_2=1$, $P_3=M-1$, $Q_3=2$, $q_3=M-1$, $\theta_2=(M-1+\sqrt{D})/2$, $\theta_3=(M+1+\sqrt{D})/2$, $\theta_4=M+\sqrt{D}$.

Another problem of some interest is the determination of those square-free values of D such that the regular continued fraction for the maximal order of $\mathcal{Q}(\delta)$ has $k=1$. For this case it is a simple matter to show that $k=1$ only for $D=M^2+1$ $(2\nmid M)$ and for $D=M^2+4$ $2\nmid M)$.

All of these results are very easy to obtain in the case of a quadratic field. If, however, we wish to examine the analogous problems in $\mathcal{Q}(\delta)$, where $\delta^3=D$, matters become rather more difficult. The purpose of this paper is to provide results analogous to (i)—(vi) in $\mathcal{Q}(\sqrt[3]{D})$. We shall also partially solve the problem of when the period length of Voronoi's continued fraction for the maximal order of $\mathfrak{R}(\sqrt[3]{D})$ is 1. In order to do this we must first describe Voronoi's extension of the regular continued fraction algorithm to cubic fields with negative discriminant.

2. Voronoi's algorithm. In this section we will give a brief description of Voronoi's algorithm. For more details the reader is referred to Delone and Faddeev [2], Williams, Cormack, Seah [10] or Williams and Dueck [11]. Here and in the sequel we will use the symbol $\mathcal{K}$ to denote a cubic field of negative discriminant. If $\alpha\in\mathcal{K}$, denote its conjugates by α' and α'' and we denote the norm of α by $N(\alpha)=\alpha\alpha'\alpha''$. We also define the point $A\in\mathcal{R}^3$ corresponding to α by

$$A = (\alpha, \eta_\alpha, \zeta_\alpha),$$

where $\eta_\alpha=(\alpha'-\alpha'')/2i$, $\zeta_\alpha=(\alpha'+\alpha'')/2$, $i^2+1=0$.

If $\lambda, \mu, \nu\in\mathcal{K}$ and λ, μ, ν are rationally independent, we define the lattice $\mathcal{L}(\subseteq\mathcal{R}^3)$ of $\mathcal{K}$ with basis $\{\lambda, \mu, \nu\}$ by

$$\mathcal{L} = \{a\Lambda+bM+cN | a, b, c\in\mathbf{Z}\}.$$

If $a\in\mathcal{K}$ and $A\in\mathcal{L}$ for the sake of brevity we will often use the notation $\alpha\in\mathcal{L}$ to denote that it is really the corresponding point A that is in $\mathcal{L}$. We also use $\alpha\mathcal{L}$ to denote the lattice with basis $\{\alpha\lambda, \alpha\mu, \alpha\nu\}$. If A (or α) is any point of $\mathcal{L}$, we define the *normed body* $\mathcal{N}(A)$ of A to be

$$\mathcal{N}(A) = \mathcal{N}(\alpha) = \{(x, y, z) | x, y, z\in\mathcal{R};\ |x| < \alpha;\ y^2+z^2 \leqq |\alpha'|^2\}$$

Here if $|\alpha'|=|\beta'|$, we must have $\alpha=\pm\beta$. ([2], p. 274). We say that $\Phi(\neq 0)$ is a (relative) *minimum* of $\mathcal{L}$ if $\mathcal{N}(\Phi)\cap\mathcal{L}=\{(0, 0, 0)\}$. If ψ and Φ are minima of $\mathcal{L}$ and $\psi>\Phi$, we say that ψ and Φ are *adjacent* minima of L when there does not exist a non-zero $\theta\in\mathcal{L}$ such that $\Phi<\theta<\psi$ and $|\theta'|<|\Phi'|$. If

$$(2.1)\qquad \theta_1, \theta_2, \theta_3, \ldots, \theta_n, \ldots$$

is a sequence of minima in $\mathcal{L}$ such that $\theta_{i+1}>\theta_i$ and θ_{i+1}, θ_i are adjacent $(i=1, 2, 3, \ldots)$, we say that (2.1) is a *chain* of minima of $\mathcal{L}$. By Minkowski's theorem (see [2]) we can prove that there always exist such chains in $\mathcal{L}$. Further, if θ is any minimum of $\mathcal{L}$ and $\theta>\theta_1$, then $\theta=\theta_m$ for some m $(\in\mathbf{Z})>1$.

If $1\in\mathscr{L}$ (i.e. $(1, 0, 1)\in\mathscr{L}$) and 1 is also a minimum of $\mathscr{L}$, we say that $\mathscr{L}$ is a *reduced* lattice of $\mathscr{K}$. Voronoi's continued fraction algorithm determines a chain of minima in such a lattice. If $\mathscr{L}$ is any reduced lattice of $\mathscr{K}$, let θ_g be the minimum adjacent to 1 in $\mathscr{L}$. Since $(1/\theta_g)\mathscr{L}$ is also a reduced lattice of $\mathscr{L}$. (See, for example, [11] p. 686), we can easily show how a chain like (2.1) can be determined. We let $\mathscr{L}_1=\mathscr{L}$ and $\theta_g^{(1)}$ be the minimum adjacent to 1 in $\mathscr{L}_1$. Define $\mathscr{L}_{n+1}=(1/\theta_g^{(n)})\mathscr{L}_n$, where $\theta_g^{(n)}$ is the minimum adjacent to 1 in $\mathscr{L}_n$. Clearly

$$\theta_n = \prod_{i=1}^{n-1} \theta_g^{(i)}. \tag{2.2}$$

Thus, to determine a chain (2.1), we must show how to solve the problem of finding the minimum adjacent to 1 in a reduced lattice $\mathscr{L}$ of $\mathscr{K}$. We simply mention here that one solution of this problem has been given in [10]; many other solutions have also been given (see [2]).

Let $\alpha\in\mathscr{L}$ and define its *puncture* $P(\alpha)$ to be the point $(\xi_\alpha, \eta_\alpha)\in\mathscr{R}^2$, where $\xi_\alpha=(2\alpha-\alpha'-\alpha'')/2$. Note that $\alpha=\xi_\alpha+\zeta_\alpha$ and that ξ, η, ζ, considered as functions of α, are additive. Thus, the set of all punctures $\{P(\alpha)|\alpha\in\mathscr{L}\}$ forms a lattice. Also, $P(\alpha)=P(\beta)$ if and only if $\alpha-\beta\in\mathbf{Z}$. In [10] it is shown that there exist $\Phi, \psi\in\mathscr{L}$ such that $P(\Phi)$, $P(\psi)$ form a basis of this lattice of punctures and

$$\xi_\Phi > \xi_\psi > 0, \quad \eta_\Phi\eta_\psi < 0, \quad |\eta_\psi| > 1/2, \quad |\eta_\Phi| < 1/2. \tag{2.3}$$

We also have

Theorem 2.1. *If $\theta_g(>1)$ is the minimum adjacent to* 1 *in $\mathscr{L}$, then $P(\theta_g)$ must be one of $P(\theta)$, $P(\psi)$, $P(\theta-\psi)$, $P(\theta+\psi)$ or $P(2\theta+\psi)$.* □

Thus, in order to find θ_g, we must first find θ and ψ. This is usually done by using a form of the regular continued fraction algorithm in the lattice of punctures. Since $P(\theta_g)$ is limited to one of 5 possibilities and $|\theta_g'|<1$, we only have a few choices for θ_g. It must be that one of all of the elements of $\mathscr{L}$ which have puncture $P(\Phi)$, $P(\psi)$, $P(\Phi-\psi)$, $P(\Phi+\psi)$ or $P(2\Phi+\psi)$ such that $|\theta_g'|<1$ and $\theta_g(>1)$ is least. Since

$$|\alpha''|^2 = |\alpha'|^2 = \alpha'\alpha'' = \zeta_\alpha^2+\eta_\alpha^2, \tag{2.4}$$

we must have $|\zeta_\alpha|<1$ when $|\alpha'|<1$. Now $\theta_g=a+\gamma$ for some $\gamma\in\{\Phi, \psi, \Phi-\psi, \Phi+\psi, 2\Phi+\psi\}$ and $a\in\mathbf{Z}$. Further $|\zeta_{\theta_g}|=|a+\zeta_\gamma|<1$; hence, $a=[-\zeta_\gamma]$ or $[-\zeta_\gamma]+1$. Thus, there are only 10 possibilities for θ_g.

Let $\mathcal{O}$ be any order of $\mathscr{K}$ and let $\mathcal{O}_{\mathscr{K}}$ (the ring of algebraic integers of $\mathscr{K}$) be the maximal order of $\mathscr{K}$. If $\{1, \mu, \nu\}$ is a unitary basis of $\mathcal{O}$ and $\Delta_{\mathcal{O}}$ is the discriminant of $\mathcal{O}$, then $\Delta_{\mathcal{O}}=I^2\Delta$, where $I\in\mathbf{Z}$ and Δ is the discriminant of $\mathscr{K}$ (or $\mathcal{O}_{\mathscr{K}}$). Also, if $\mathscr{L}$ is the lattice with basis $\{1, \mu, \nu\}$ then $\mathscr{L}$ must be a reduced lattice. For, if it were not, there would exist $\alpha\in\mathscr{L}$ such that $0<\alpha<1$ and $|\alpha'|<1$. Hence $|N(\alpha)=|\alpha||\alpha'|^2<1$. Since $\alpha\in\mathcal{O}\subseteqq\mathcal{O}_{\mathscr{K}}$, we have $N(\alpha)\in\mathbf{Z}$, a contradiction. Let η be any unit (>1) and let $\varepsilon_0(>1)$ be the fundamental unit of $\mathcal{O}$. By using similar reasoning we see that η must also be a minimum of $\mathscr{L}$; hence, if $\theta_1=1$, we have $\theta_{p+1}=\varepsilon_0$ for some $p>0$. Since $\mathscr{L}_1=\theta_{p+1}\mathscr{L}_{p+1}=\varepsilon_0\mathscr{L}_{p+1}=\mathscr{L}_{p+1}$, this algorithm, like the continued fraction algorithm for an order in $\mathscr{Q}(\sqrt{D})$, becomes periodic with period p.

We can also prove a theorem analogous to Theorem 1.2.

Theorem 2.2. *Let $\mathcal{O}$ be an order of $\mathcal{K}$ with unitary basis $\{1, \mu, \nu\}$ and let $\mathcal{L}$ be the corresponding lattice with basis $\{1, \mu, \nu\}$. Also, let $\alpha \in \mathcal{O}$, $\alpha > 1$ and $\alpha = x + y\mu + z\nu$ with $x, y, z \in \mathbf{Z}$ and $gcd(x, y, z) = 1$. If*

$$(2.5) \qquad |N(\alpha)| < \sqrt[4]{|\Delta_{\mathcal{O}}|/27},$$

then α must be an element of the chain (2.1) of $\mathcal{L}$ with $\theta_1 = 1$.

PROOF. (c.f. [11], Theorem 4.5 and [9] Theorem 9.1). Since $\alpha > 1$ and the sequence $\theta_1, \theta_2, \theta_3, \dots, \theta_n, \dots$ is not bounded above, we must have some k such that

$$\theta_k \leqq \alpha < \theta_{k+1}.$$

Since $|\alpha'|^2 = \alpha'\alpha'' = \alpha'\alpha'' + \alpha'\alpha + \alpha''\alpha - \alpha(\alpha'' + \alpha' + \alpha) + \alpha^2$, we have $N(\alpha)/\alpha \in \mathcal{O}$ and $\varrho = N(\alpha)\theta_k/\alpha \in \mathcal{O}$.

If $\alpha = \theta_k$, the theorem is true; suppose $\alpha \neq \theta_k$. Since α is not a minimum, we must have $|\alpha'| > |\theta_k'|$; hence,

$$0 < \varrho < |N(\alpha)| \quad \text{and} \quad |\varrho'| < |N(\alpha)|.$$

Thus, if

$$d(\varrho) = (\varrho - \varrho')^2(\varrho' - \varrho'')^2(\varrho'' - \varrho)^2,$$

then (see, for example, [9] p. 649)

$$(2.6) \qquad |d(\varrho)| < 27N(\alpha)^6.$$

Also,

$$d(\varrho) = N(\alpha)^6 \left(\frac{\theta_k}{\alpha} - \frac{\theta_k'}{\alpha'}\right)^2 \left(\frac{\theta_k'}{\alpha'} - \frac{\theta_k''}{\alpha''}\right)^2 \left(\frac{\theta_k''}{\alpha''} - \frac{\theta_k}{\alpha}\right)^2$$

$$= N(\alpha)^6 \begin{vmatrix} \alpha^2 & \alpha\theta_k & \theta_k^2 \\ \alpha'^2 & \alpha'\theta_k' & \theta_k'^2 \\ \alpha''^2 & \alpha''\theta_k'' & \theta_k''^2 \end{vmatrix}.$$

Since α^2, $\alpha\theta_k$, $\theta_k^2 \in \mathcal{O}$, we have

$$(2.7) \qquad d(\varrho) = N(\alpha)^2 J^2 \Delta_{\mathcal{O}},$$

where $J \in \mathbf{Z}$. Also, if $J = 0$, then $d(\varrho) = 0$ and $\varrho \in \mathbf{Z}$. If $\theta_k = a + b\mu + c\nu$, then $gcd(a, b, c) = 1$. Also, $\varrho x = N(\alpha)a$, $\varrho y = N(\alpha)b$, $\varrho z = N(\alpha)c$, together with $gcd(x, y, z) = 1$, imply that $\varrho = |N(\alpha)|$, which is not so. Thus, the theorem follows on comparing (2.6) and (2.7). □

Thus, just as in the quadratic case, it is of some interest to be able to present the complete Voronoi continued fraction for an order of $\mathcal{K}$. Unfortunately, Voronoi's algorithm is much more intricate than the regular continued fraction algorithm; nevertheless, we will be able to obtain some results for certain orders in special subic fields.

3. Some simple lemmas. In this section we will present some simple results which have been found useful in the development of Voronoi's continued fraction for orders in $\mathcal{K}$. We assume here and in the sequel that $\mathcal{L}$ is a reduced lattice of $\mathcal{K}$.

Lemma 3.1. *If* $\omega, \chi \in \mathscr{L}$ *and* $|\omega'|<1$, $|\chi'|<1$, $\xi_\omega > \xi_\chi + 2$, *then* $\omega > \chi$.

PROOF. Since $|\omega'|<1$ and $|\chi'|<1$, by (2.4) we have $|\zeta_\omega|<1, |\zeta_\chi|<1$. Since $\omega = \xi_\omega + \xi_\omega$ and $\chi = \zeta_\chi + \xi_\chi$, the result follows immediately. □

Lemma 3.2. *If* $\chi \in \mathscr{L}$ *and* $|\xi_\chi| < 1/2$, *then* $|\chi'|>1$.

PROOF. Suppose $|\chi'| \leqq 1$. Since 1 is a minimum of $\mathscr{L}$, we must have $|\chi|>1$. Indeed, since $|\xi_\chi| = 1 > 1/2$ when $|\chi|=1$, we must have $|\chi|>1$; thus, if $\chi>0$ and $\omega = \chi - 1$, then $\omega > 0$. Since $\zeta_\chi < 1$ and $\xi_\chi < 1/2$, we also have $\omega < 1/2$; hence, $|\omega'|>1$. Since

$$|\omega'|^2 = |\chi'|^2 - 2\zeta_\chi + 1,$$

we get $\zeta_\chi < 1/2$ and $\chi = \zeta_\chi + \xi_\chi < 1$, a contradiction. Similarly, if $\chi<0$, we can put $\omega = \chi + 1$ and arrive at another contradiction. □

Lemma 3.3. *If* $\chi \in \mathscr{L}$, $\xi_\chi > 0$, $|\chi'|<1$ *and* $\chi < 2$, *then* $|\chi' - 1| > 1$.

PROOF. Since $|\chi'|<1$, we have $\zeta_\chi > -1$ and $\chi = \zeta_\chi + \xi_\chi > -1$. Also, since $|\chi'|<1$, we must have $|\chi| \geqq 1$; hence $\chi > 1$ and $0 < \chi - 1 < 1$. Since $\mathscr{L}$ is a reduced lattice, we get $|\chi' - 1| > 1$. □

If $\omega \in \mathscr{L}$ and $\omega \notin \mathscr{Q}$, let ω^* be defined as that element of $\mathscr{L}$ such that $P(\omega^*) = P(\omega)$, $|\omega^{*\prime}|<1$ and $|\omega^*|$ is minimal. Clearly, by (2.4) ω^* cannot exist if $|\eta_\omega|>1$. On the other hand, if $|\eta_\omega| < \sqrt{3}/2$, then $\eta_\omega^2 + (\zeta_\omega - a)^2 < 1$ for some $a \in \mathbf{Z}$; that is, ω^* exists in this case. We also note that if ω^* exists and $\omega^* > 0$, then $|\omega^{*\prime} - 1| > 1$. For if $|\omega^{*\prime} - 1| \leqq 1$, then $\omega^* > 1$ and $|\omega^* - 1| < |\omega^*|$. Since $P(\omega^*) = P(\omega^* - 1)$, this contradicts the definition of ω^*. Further, if $\omega > 0$, $|\omega'|<1$ and $|\omega' - 1| > 1$, then $\omega^* = \omega$. We will also require the following Lemma from [10].

Lemma 3.4. (See [10], Lemma 4.3). *Let* $\chi, \omega \in \mathscr{L}$ *and suppose that* $\xi_\chi > 0$, $|\eta_\chi| < \sqrt{3}/2$ *and* $|\omega'|<1$. *If* $\xi_\omega > \xi_\chi$ *and* $\eta_\chi \eta_\omega > 0$, *then* $\omega > \chi^*$. □

If θ is the minimum adjacent to 1 in $\mathscr{L}$, it is sometimes fairly easy to find ω, the minimum of $\mathscr{L}$ adjacent to θ. We give these conditions in

Theorem 3.1. ([11], Theorem 7.5). *When* $\zeta_\theta < -1/2$ *the puncture* $P(\omega)$ *of* ω *must be one of the punctures given in the second column of Table 3.1.*

Table 3.1

$P(\theta)$	Possible choices for $P(\omega)$
$P(\Phi)$	$P(\Phi)$, $P(\psi)$, $P(\Phi+\psi)$
$P(\psi)$	$P(\Phi)$, $P(\psi)$, $P(\Phi-\psi)$
$P(\Phi-\psi)$	$P(\psi)$, $P(\Phi-\psi)$
$P(\Phi+\psi)$	$P(\Phi)$, $P(\Phi+\psi)$
$P(2\Phi+\psi)$	$P(\Phi)$

In this paper we shall consider the fields $\mathcal{Q}(\delta)$, where $\delta^3=D\in\mathbf{Z}$, $\delta\notin\mathcal{Q}$, and $D=M^3\pm1$, $M^3\pm3$, $M^3\pm M$, $M^3\pm3M$ to be fields analogous to the quadratic fields discussed in Section 1. While it is true that we did not discuss the case of $M^2\pm2M$ in that section, we simply point out that $M^2\pm2M=(M\pm1)^2-1$, a case that was discussed. These fields $\mathcal{Q}(\delta)$, like those of §1 have small fundamental units (see NAGELL [5], STENDER [8], RUDMAN [7]); hence, the values for p should not be very large.

In order to develop our continued fraction expansions we will also require some inequalities and we must assume some lower bounds on M. We summarize these in Table 3.2 below.

Table 3.2

$D(=\delta^3)$	Inequalities	Lower Bound on M
M^3+1	$M<\delta<M+1/(3M^2)$	$M\geqq4$
M^3-1	$M-2/(3M^2)<\delta<M$	$M\geqq4$
M^3+3	$M<\delta<M+1/\big(3(M+1)\big)$	$M\geqq4$
M^3-3	$M-1/(3M)<\delta<M$	$M\geqq4$
M^3+M	$M<\delta<M+1/(3M)$	$M\geqq3$
M^3-M	$M-1/(2M)<\delta<M$	$M\geqq5$
M^3+3M	$M+1/M-1/M^3<\delta<M+1/M$	$M\geqq3$
M^3-3M	$M-2/M<M-1/M-2/M^3<\delta<M-1/M<M$	$M\geqq4$

We also point out that if

$$\omega = (m_1+m_2\delta+m_3\delta^2)/r,$$

where $m_1, m_2, m_3, r\in\mathbf{Z}$ and $\delta^3\in\mathbf{Z}$, then

$$\xi_\omega = \frac{3(\delta m_2+\delta^2 m_3)}{r}, \quad \eta_\omega = \frac{\sqrt{3}(\delta m_2-\delta^2 m_3)}{r},$$

$$\zeta_\omega = \frac{2m_1-m_2\delta-m_3\delta^2}{2r}.$$

Further, if $\mathcal{L}$ has a basis of the form

$$\{1, (m_1+m_2\delta+m_3\delta^2)/r,\ (n_1+n_2\delta+n_3\delta^2)/r\},$$

then $\{1, \mu, \nu\}$ is a basis of $\mathcal{L}$ if and only if $\mu, \nu\in\mathcal{L}$ and

$$\begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix} = \pm\begin{vmatrix} \bar{m}_2 & \bar{m}_3 \\ \bar{n}_2 & \bar{n}_3 \end{vmatrix},$$

where $\mu=(\bar{m}_1+\bar{m}_2\delta+\bar{m}_3\delta^2)/r$, $\nu=(\bar{n}_1+\bar{n}_2\delta+\bar{n}_3\delta^2)/r$, and $\bar{m}_1, \bar{m}_2, \bar{m}_3, \bar{n}_1, \bar{n}_2, \bar{n}_3\in\mathbf{Z}$.

A special case. The method we use to derive our results is most conveniently demonstrated by the complete development of the Voronoi continued fraction for a simple, special case. We will use $D=M^3+3M$ here. We will deal first with the order $\mathcal{O}$ with basis $\{1, \delta, \delta^2\}$, where $\delta^3=D$. Note that $\Delta_{\mathcal{O}}=27D^2$. If D is square-free, then in most of the cases mentioned above we have $D\not\equiv\pm1 \pmod 9$ and $\mathcal{O}=\mathcal{O}_{\mathcal{K}}$, the maximal order; however, when $D=M^3+1$ $(3|M)$ and $D=M^3+M$ $(M\equiv\pm2 \pmod 9))$, we do not have $\mathcal{O}=\mathcal{O}_{\mathcal{K}}$. In these cases $\Delta=3D^2$.

Theorem 4.1. *If* $D=M^3+3M$ *and* $\mathcal{K}=\mathcal{Q}(\delta)$, *where* $\delta^3=D$, *then the Voronoi continued fraction expansion for the order* $\mathcal{O}$ *of* $\mathcal{K}$ *with basis* $\{1, \delta, \delta^2\}$ *has period length* $p=4$.

Proof. Since D is square-free, M must be even and $3\nmid M$; hence, we may assume that $M\geqq 10$. If $\mu=(M^2+1)\delta+M\delta^2$ and $\nu=M^2+1+M\delta+\delta^2$, then $\{1, \mu, \nu\}$ is a basis of $\mathcal{O}(=\mathcal{O}_{\mathcal{K}})$. Now

$$\eta_\mu = \sqrt{3}\delta(M^2-1-M\delta)/2 > 0$$

and

$$\eta_\mu < (\sqrt{3}\delta)/2M^2 < 1/2.$$

Also,

$$\xi_\mu = 3\delta(M^2+1+M\delta)/2 > \xi_\nu = 3\delta(M+\delta)/2 > 0$$

and

$$\eta_\nu = \sqrt{3}\delta(M-\delta)/2 < 0.$$

Since

$$-2\eta_\nu = \sqrt{3}\delta(M-\delta) > \sqrt{3}(1/M-1/M^3)(M+1/M-1/M^3) > 1,$$

we see that we may put $\Phi=\mu$ and $\psi=\nu$. Thus, if θ is the minimum adjacent to 1 in $\mathcal{L}_1$, the lattice with basis $\{1, \delta, \delta^2\}$, then $P(\theta)$ must be one of $P(\Phi)$, $P(\psi)$, $P(\Phi-\psi)$, $P(\Phi-\psi)$ or $P(2\Phi+\psi)$. Now $|\psi'|^2=\psi'\psi''=1-(M-\delta)^2<1$ and

$$\xi_\Phi-2\xi_\psi = 3\delta\big((M-1)^2+(M-2)\delta\big)/2 > 2;$$

hence,

$$\xi_{2\Phi+\psi} > \xi_{\Phi+\psi} > \xi_\Phi > \xi_{\Phi-\psi} > \xi_\psi+2.$$

By Lemma 3.1, we can only have $P(\theta)=P(\psi)$. Since $|\psi'|<1$ and $|\psi'-1|= =3M(\delta-M)>3M(1/M-1/M^3)>1$, we must have $\theta=\theta_g^{(1)}=\psi^*=\psi$. It follows that $\mathcal{L}_2$ has basis $\{1, 1/\psi, \Phi/\psi\}$.

Now

$$(3M^2+1)/\psi = 1-(M-\delta)^2,$$

$$(3M^2+1)\Phi/\psi = M^5+2M^3-3M+(-2M^4-3M^2+1)\delta+(M^3+3M)\delta^2.$$

Thus, if we put

$$(3M^2+1)\mu = M^3-M+(M^2+1)\delta+M\delta^2 > 0,$$

$$(3M^2+1)\nu = M^3+M^2-M-1+(M-1)^2\delta+(M+1)\delta^2 > 0,$$

we find that $\{1, \mu, \nu\}$ is a basis of $\mathcal{L}_2$, $0<\eta_\mu<1/2$, $\xi_\mu>\xi_\nu>0$, $\eta_\nu<-\sqrt{3}/2$. Thus, we can put $\Phi=\mu$ and $\psi=\nu$ in $\mathcal{L}_2$. We have

$$(3M^2+1)\psi'\psi'' = (M+1)(M-1)^2+(M+1)^2\delta-2(M-1)\delta^2 < 3M^2+1$$

and

$$(3M^2+1)(\psi'-1)(\psi''-1) = 3M^2+1+(-(M^2-1)M+3+(3M^2-2M+3)\delta+4\delta^2) > 3M^2+1;$$

hence $|\psi'|<1$ and $|\psi'-1|>1$. Also

(4.1) $$\xi_\Phi > 3M(2M^2+1)/(3M^2+1) > 2$$

and

(4.2) $$0 < \xi_{\Phi-\psi} = 3\delta(2M-\delta)/(2(3M^2+1)) < 1/2.$$

By (4.1), Lemma 3.1, and the fact that $|\psi'|<1$, we see that if θ is the minimum adjacent to 1 in $\mathscr{L}_2$, then $P(\theta)\neq P(\Phi+\psi)$, $P(2\Phi+\psi)$. Also, by (4.2) and Lemma 3.2, we have $P(\theta)\neq P(\Phi-\psi)$. Now

$$(3M^2+1)\Phi'\Phi'' = -2M^2+M\delta+\delta^2 < 3M^2+1,$$

$$(3M^2+1)(\Phi'-1)(\Phi''-1) = 3M^2+1+(-2M^3-2M^2+2M+(M^2+M+1)\delta+ +(M+1)\delta^2) > 3M^2+1,$$

and

$$(3M^2+1)(\Phi-\psi) = -M^2+1+2M\delta+\delta^2 > 0;$$

hence, $|\Phi'|<1$, $|\Phi'-1|>1$ and $\Phi>\psi$. It follows that $\theta=\theta_g^{(2)}=\psi^*=\psi$. Also, $\{1, 1/\psi, \Phi/\psi\}$ is a basis of $\mathscr{L}_3=(1/\theta_g^{(2)})\mathscr{L}_2$.

Here

$$(3M^3-3M^2+9M-1)/\psi = (M+1)(M-1)^2+(M+1)^2\delta-2(M-1)\delta^2$$

$$(3M^3-3M^2+9M-1)\Phi/\psi = M^3-2M^2+5M+(M-1)^2\delta+(M+1)\delta^2.$$

Thus, if

$$(3M^3-3M^2+9M-1)\mu = M^3-2M^2+5M+(M-1)^2\delta+(M+1)\delta^2 > 0$$

$$(3M^3-3M^2+9M-1)\nu = 2M^3-3M^2+4M+1+2(M^2+1)\delta-(M-3)\delta^2 > 0,$$

then $\{1, \mu, \nu\}$ is a basis of $\mathscr{L}_3$ and $\xi_\mu>\xi_\nu>0$. Further, $1/2<\eta_\nu<\sqrt{3}/2$, $-1/4<\eta_\mu<0$, and we can put $\Phi=\mu$ and $\psi=\nu$ in $\mathscr{L}_3$. We also have

$$(3M^3-3M^2+9M-1)\zeta_\psi = 2M^3-3M^2+4M+1-(M^2+1)\delta-(M-3)\delta^2/2 > 0 < (1/2)(3M^3-9M^2+8M-5);$$

hence, $\zeta_\psi<1/2$ and $|\psi'|^2=\zeta_\psi^2+\eta_\psi^2<1$. Now it is easy to verify that $\xi_\psi<1$ and from this it follows that $\psi-1=\xi_\psi+\zeta_\psi-1<1$. Thus, $|\psi'-1|>1$. Also,

$$(3M^3-3M^2+9M-1)\Phi'\Phi'' = -M^2+3M+2M\delta-\delta^2 < 3M^3-3M^2+9M-1$$

and

$$(3M^3-3M^2+9M-1)\Phi < (3M^3-3M^2+9M-1)+2;$$

thus, $|\Phi'|<1$ and $\Phi<2$; by Lemma 3.3 we get $|\Phi'-1|>1$. If $\omega=\Phi+\psi$ or $2\Phi+\psi$, then $\eta_\theta\eta_\omega>0$, $|\eta_\psi|<\sqrt{3}/2$ and $\xi_\omega>\xi_\psi$. By Lemma 3.4, we see that if θ is the minimum adjacent to 1 in $\mathscr{L}$, then $P(\theta)\neq P(\Phi+\psi)$, $P(2\Phi+\psi)$. Also, since $\Phi^*=\Phi$, $\psi^*=\psi$ and $\psi-\Phi>0$, we cannot have $P(\theta)=P(\psi)$. If $P(\Phi)=P(\Phi-\psi)$, then $\theta=a+\Phi-\psi$, where $a\in\mathbf{Z}$ and $a>0$. Since $1+\Phi-\psi<1$, we cannot have $a=1$; but, if $a\geqq 2$, then

$$a+\Phi-\psi \geqq \Phi-\psi+2 > \Phi-\zeta_\psi-\xi_\psi+2 > \Phi+1/2 > \Phi;$$

thus $\theta=\theta_g^{(3)}=\Phi^*=\Phi$. Also, $\mathscr{L}_4=(1/\theta_g^{(3)})\mathscr{L}_3$ has basis $\{1, 1/\Phi, \psi/\Phi\}$.

We have

$$3M/\Phi = -M^2+3M+2M\delta-\delta^2,$$

$$3M\psi/\Phi = -2M^2+3M+M\delta+\delta^2;$$

hence, $\{1, \mu, \nu\}$ is a basis of $\mathscr{L}_4$, where $3M\mu=M^2+M\delta+\delta>0$ and $3M\nu=-M^2+$ $+2M\delta-\delta^2$; thus, we can put $\Phi=\mu$ and $\chi=\nu$ in $\mathscr{L}_4$. Since $\eta_\psi>1$, $\eta_{\Phi+\psi}>1$, $\eta_{2\Phi+\psi}>1$, $|\eta_{\Phi-\psi}|>1$, we see that if θ is the minimum adjacent to 1 in $\mathscr{L}_4$, then $P(\theta)=P(\Phi)$. Now

$$\Phi'\Phi'' = (\delta-M)/(3M) < 1$$

and

$$6M(-\Phi'-\Phi''+\Phi'\Phi'') = -2M^2-2M+M\delta+\delta^2 > 0.$$

It follows that $|\Phi'|<1$, $|\Phi'-1|>1$ and $\Phi^*=\Phi$. Thus, $\theta=\theta_g^{(4)}=\Phi^*=\Phi$ and $\mathscr{L}_5=(1/\theta_g^{(4)})\mathscr{L}_4$ has basis $\{1, 1/\Phi, \psi/\Phi\}$. Since $1/\Phi=4-M$ and $\psi/\Phi=-1+M\delta+\delta^2$, we have $\mathscr{L}_5=\mathscr{L}_1$; hence, $p=4$. □

5. Summary of results. As we have seen from Section 4 the process of obtaining the Voronoi continued fraction is both lengthy and tedious. Since the example given in that section is illustrative of the kind of techniques needed to develop these continued fractions, we will present only the final results in this section. The interested reader can easily verify that these results will hold for the values of D indicated. We first assume that $\mathcal{O}$ has basis $\{1, \delta, \delta^2\}$, where $\delta^3=D$.

(i) $D = M^3+1,\ p = 1,$

$\theta_g^{(1)} = M^2+M\delta+\delta^2,$

$\theta_2 = M^2+M\delta+\delta^2,\ N(\theta_2) = 1.$

(ii) $D = M^3-1,\ p = 2,$

$\theta_g^{(1)} = M^2-1+M\delta+\delta^2,\ (3M^2-3M)\theta_g^{(2)} = (M-1)^2+(M-1)\delta+\delta^2,$

$\theta_2 = M^2-1+M\delta+\delta^2,\ N(\theta_2) = 3M^2-3M,\ \theta_3 = M^2+M\delta+\delta^2,$

$N(\theta_3) = 1.$

(iii) $D = M^3+3,\ p = 3,$

$\theta_g^{(1)} = M^2+M\delta+\delta^2,\ \theta_g^{(2)} = (M^2+M\delta+\delta^2)/3 = \theta_g^{(3)},$

$\theta_2 = M^2+M\delta+\delta^2,\ N(\theta_2) = 9,\ \theta_3 = M^4+2M+(M^3+1)\delta+M^2\delta^2,$

$N(\theta_3) = 3,\ \theta_4 = M^6+3M^3+1+(M^5+2M^2)\delta+(M^4+M)\delta^2,$

$N(\theta_4) = 1.$

(iv) $D = M^3-3,\ p = 6,$

$\theta_g^{(1)} = M^2-1+M\delta+\delta^2,\ (3M^2-9M+8)\theta_g^{(2)} = (M-3)^2+(M-3)\delta+\delta^2,$

$3\theta_g^{(3)} = M^2-3+M\delta+\delta^2,\ (3M^2-3M-2)\theta_g^{(4)} = (M-1)^2+(M-1)\delta+\delta^2,$

$3\theta_g^{(5)} = M^2-3+M\delta+\delta^2,\ (3M^2-3M-2)\theta_g^{(6)} = (M-1)^2+(M-1)\delta+\delta^2,$

$\theta_2 = M^2-1+M\delta+\delta^2,\ N(\theta_2) = 3M^2-9M+8,$

$\theta_3 = M^2+M\delta+\delta^2,\ N(\theta_2) = 9,$

$\theta_4 = M^4-M^2-2M+(M^3-M-1)\delta+(M^2-1)\delta^2,$

$N(\theta_4) = 9M^2-9M-6,$

$\theta_5 = M^4-2M+(M^3-1)\delta+M^2\delta^2,\ N(\theta_5) = 3,$

$\theta_6 = M^6-M^4-3M^3+2M+1+(M^5-M^3-2M^2+1)\delta+$
$+(M^4-M^2-M)\delta^2,$

$N(\theta_6) = 3M^2-3M-2,$

$\theta_7 = M^6-3M^3+1+(M^5-2M^2)\delta+(M^4-M)\delta^2,\ N(\theta_7) = 1.$

For this particular case much of the work can be simplified by using Theorem 3.1. Indeed, the nearest integer variant of Voronoi's algorithm (see [11]) has period length 3 here.

(v) $D = M^3+M,\ p = 3,$

$\theta_g^{(1)} = M^2+M\delta+\delta^2,\ \theta_g^{(2)} = (M^2+M\delta+\delta^2)/M = \theta_g^{(3)},$

$\theta_2 = M^2+M\delta+\delta^2,\ N(\theta_2) = M^2,\ \theta_3 = 3M^3+2M+(3M^2+1)\delta+3M\delta^2,$

$N(\theta_3) = M,\ \theta_4 = 9M^4+9M^2+1+(9M^3+6M)\delta+(9M^2+3)\delta^2,$

$N(\theta_4) = 1.$

(vi) $D = M^3-M,\ p = 5.$

$\theta_g^{(1)} = M^2+M\delta+\delta^2,\ (M+1)\theta_g^{(2)} = M^2-1+(M+1)\delta+\delta^2,$

$(3M^2-2M-1)\theta_g^{(3)} = M^2-M+(M+1)\delta+\delta^2,\ M\theta_g^{(4)} = M^2-M+M\delta+\delta^2,$

$(3M^2-4M+1)\theta_g^{(5)} = (M-1)^2+(M-1)\delta+\delta^2,$

$\theta_2 = M^2-1+(M+1)\delta+\delta^2,\ N(\theta_2) = M^2-1,$

$\theta_3 = 3M^3-2M^2-2M+1+(3M^2-2M-1)\delta+(3M-2)\delta^2,$

$N(\theta_3) = (M-1)^2(3M+1),\ \theta_4 = 3M^3-2M+(3M^2-1)\delta+3M\delta^2,$

$N(\theta_4) = M,$

$\theta_5 = 9M^4-3M^3-9M^2+2M+1+(9M^3-3M^2-6M+1)\delta+$
$+(9M^2-3M-3)\delta^2,$

$N(\theta_5) = 3M^2-4M+1,$

$\theta_6 = 9M^4-9M^2+1+(9M^3-6M)\delta+(9M^2-3)\delta^2,\ N(\theta_6) = 1.$

(vii) $D = M^3+M3,\ p = 4,$

$\theta_g^{(1)} = M^2+1+M\delta+\delta^2,\ (3M^2+1)\theta_g^{(2)} = M^3+M^2-M-1+(M-1)^2\delta+$
$+(M+1)\delta^2,$

$(3M^3-3M^2+9M-1)\theta_g^{(3)} = M^3-2M^2+5M+(M-1)^2\delta+(M+1)\delta^2,$

$3M\theta_g^{(4)} = M^2+M\delta+\delta^2,$

$\theta_2 = M^2+1+M\delta+\delta^2,\ N(\theta_2) = 3M^2+1,$

$\theta_3 = M^3+2M-1+(M^2+1)\delta+M\delta^2,\ N(\theta_3) = 3M^3-3M^2+9M-1,$

$\theta_4 = M^3+2M+(M^2+1)\delta+M\delta^2,\ N(\theta_4) = 3M,$

$\theta_5 = M^4+3M^2+1+(M^3+2M)\delta+(M^2+1)\delta^2,\ N(\theta_5) = 1.$

(viii) $D = M^3-3M,\ p = 5,$

$\theta_g^{(1)} = M^2-2+M\delta+\delta^2,\ (3M^2-8)\theta_g^{(2)} = M^3-2M^2-2M+4+$
$+(M^2+M-4)\delta+(M-2)\delta^2,\ (3M^3-9M^2+6M+1)\theta_g^{(3)} = M^3-4M^2+$
$+4M+(M^2-M-1)\delta+(M-1)\delta,$

$3M\theta_g^{(4)} = M^2-3M+M\delta+\delta^2,\ (3M^2-6M+1)\theta_g^{(5)} = (M-1)^2+$
$+(M-1)\delta+\delta^2,$

$\theta_2 = M^2-2+M\delta+\delta^2,\ N(\theta_2) = 3M^2-8,$

$\theta_3 = M^3-M^2-2M+1+(M^2-M-1)\delta+(M-1)\delta^2,\ N(\theta_3) = 3M^3-$
$-9M^2+6M+1,$

$\theta_4 = M^3-2M+(M^2-1)\delta+M\delta^2,\ N(\theta_4) = 3M,$

$\theta_5 = M^4-M^3-3M^2+2M+1+(M^3-M^2-2M+1)\delta+(M^2-M-1)\delta^2,$
$N(\theta_5) = 3M^2-6M+1,$

$\theta_6 = M^4-3M^2+1+(M^3-2M)\delta+(M^2-1)\delta^2,\ N(\theta_6) = 1.$

As noted above, when D is square-free, the only cases of those values of D given here where $\mathcal{O}\neq\mathcal{O}_{\mathcal{K}}$ are those of $D=M^3+1$ $(3|M)$ and $D=M^3+M$ $(M\equiv 2, 7 \pmod 9))$. In fact, for the case of $D=M^3+1$ $(3|M)$, the continued fraction given in (i) is valid for $\mathcal{O}_{\mathcal{K}}$, but in the other two cases there are several changes. We point out here that $\mathcal{O}_{\mathcal{K}}$ has basis $\{1, \delta, (\delta^2+\sigma\delta+1)/3\}$, where $\sigma=\pm 1$ and $\sigma\equiv D \pmod 9$.

$$\mathcal{O} = \mathcal{O}_{\mathcal{K}},\quad D \text{ square-free.}$$

(ix) $D = M^3+M,\ M \equiv 2 \pmod 9,\ p = 5,$

$\theta_g^{(1)} = M^2+M\delta+\delta^2,\ 3M\theta_g^{(2)} = M^2-3M+(M-1)\delta+\delta^2,$

$(21M^3-19M^2-5M-1)\theta_g^{(3)}/9 = (-5M^3+8M^2+M)/3+$

$+(4M^2-2M)\delta/3+(M+1)\delta^2/3,$

$(3M^3-M^2+4M-1)\theta_g^{(4)}/3 = M^3+2M+(M^2-M+1)\delta+(M+1)\delta^2,$

$M\theta_g^{(5)} = M^2+M\delta+\theta^2,$

$\theta_2 = M^2+M\delta+\delta^2,\ N(\theta_2) = M^2,$

$3\theta_3 = 3M^3-4M^2+2M-1+3(M^4-4M+1)\delta+(3M-4)\delta^2,$

$N(\theta_3) = (21M^3-19M^2-5M-1)/27,$

$3\theta_4 = 3M^3-M^2+2M-1+(3M^2-M+1)\delta+(3M-1)\delta^2,$

$N(\theta_4) = (3M^3-M^2+4M-1)/27,$

$\theta_5 = 3M^3+2M+(3M^2+1)\delta+3M\delta^2,\ N(\theta_5) = M,$

$\theta_6 = 9M^4+9M^2+1+(9M^3+6M)\delta+(9M^2+3)\delta^2,\ N(\theta_6) = 1.$

(x) $D = M^3+M,\ M \equiv 7 \pmod 9),\ p = 6,$

$\theta_g^{(1)} = M^2+M\delta+\delta^2,\ 3M\theta_g^{(2)} = M^2-3M+(M-2)\delta+\delta^2,$

$(21M^3-26M^2-5M-8)\theta_g^{(3)}/9 = (9M^3-2M^2+M-8)/9+$

$+(5M^2-5M)\delta/3+(4-M)\delta^2/3$

$(12M^3-8M^2+13M-8)\theta_g^{(4)}/9 = (4M^2+M+4)/9+(2M^2-M+2)\delta/3+$

$+(2M+1)\delta^2/3,$

$(3M^3-M^2+4M+1)\theta_g^{(5)}/9 = (M^3+2M)/3+(M^2+M+1)\delta/3+$

$+(M-1)\delta^2/3,$

$M\theta_g^{(6)} = M^2+M\delta+\delta^2,$

$\theta_2 = M^2+M\delta+\delta^2,\ N(\theta_2) = M^2,$

$3\theta_3 = 3M^3-5M^2+2M-2+(3M^2-5M+1)\delta+(3M-5)\delta^2,$

$27N(\theta_3) = 21M^3-26M^2-5M-8,$

$3\theta_4 = 3M^3-2M^2+2M-2+(3M^2-2M+1)\delta+(3M-2)\delta^2,$

$27N(\theta_4) = 12M^3-8M^2-13M-8,$

$3\theta_5 = 3M^3+M^2+2M+1+(3M^2+M+1)\delta+(3M+1)\delta^2,$

$27N(\theta_5) = 3M^3+M^2+4M+1,$

$\theta_6 = 3M^3-2M+(3M^2+1)\delta+3M\delta^2,\ N(\theta_6) = M,$

$\theta_7 = 9M^4+9M^2+1+(9M^3+6M)\delta+(9M^2+3)\theta^2,\ N(\delta_7) = 1.$

In the next section we will discuss the simplest case here $(D=M^3+1)$ when $\mathcal{O}=\mathcal{O}_{\mathcal{K}}$, but D is no longer assumed square-free.

6. A special case of $D=M^3+1$. In the previous cases we have dealt only with the order $\mathcal{O}$ with basis $\{1, \delta, \delta^2\}$ or $\{1, \delta, (1+\sigma\delta+\delta^2)/3$, where $\delta^3=D$. As noted above $\mathcal{O}=\mathcal{O}_{\mathcal{K}}$ whenever D is square-free. In fact, we know by a result of NAGELL [4] that for each of the forms of D considered above, there exists an infinitude of values of M such that D is square-free. Hence, we know the Voronoi continued fraction expansion for the maximal order $\mathcal{O}(\sqrt[3]{D})$ for an infinitude of values of D of each form. When D is not square-free the problem of determining the continued fraction for the maximal order becomes much more difficult. In this section we will illustrate how these difficulties arise by examining the simplest of our cases above, that is the case of $D=M^3+1$.

Let $D=M^3+1$ be cube-free and suppose $D=fg^2$ where $gcd(f, g)=1$. Suppose further that $\mathcal{O}$ is the order with basis $\{1, \delta, \bar{\delta}\}$, where $\bar{\delta}^3=\bar{D}=f^2g$. We now prove

Theorem 6.1. *Let D be cube-free, $D=M^3+1$, and let $\mathcal{O}$ have basis $\{1, \delta, \bar{\delta}\}$. If $\sqrt{3}M/(2g)>1+1/(2\sqrt{3}M)$, then the period length of Voronoi's continued fraction for $\mathcal{O}$ is 1; if $g>\delta$, then the period length of Voronoi's continued fraction for $\mathcal{O}$ must exceed 1.*

PROOF. Since $gcd(M, g)=1$, there exist integers x, y such that $0<x<g$, $0<y<M$, and

$$Mx-gy=-1.$$

Put $\mu=M^2+M\delta+g\bar{\delta}$ and $\nu=y\delta+x\bar{\delta}$. Clearly, $\{1, \mu, \nu\}$ is a basis of $\mathcal{O}$. Now $\xi_\mu>\xi_\nu>2$ and $-1/2<\eta_\mu<0$. Also,

$$\eta_\nu=\sqrt{3}\delta(yg-x\delta)/(2g)>\frac{\sqrt{3}\delta}{2g}\left(yg-x\left(M+\frac{1}{3M^2}\right)\right)=$$

$$=\frac{\sqrt{3}\delta}{2g}\left(1-\frac{x}{3M^2}\right)>\frac{\sqrt{3}\delta}{2g}\left(1-\frac{g}{3M^2}\right)>\frac{\sqrt{3}M}{2g}-\frac{1}{2\sqrt{3}M}>0.$$

If $\sqrt{3}M/(2g)-1/(2\sqrt{3}M)>1$, then we can put $\Phi=\mu$, $\psi=\nu$ and we have $|\Phi'|<1$, $|\Phi'-1|>1$, $\eta_\psi>1$, $\xi_\psi>2$. Let $\mathcal{O}$ be the minimum of $\mathcal{L}_1$ (basis $\{1, \delta, \bar{\delta}\}$) adjacent to 1. Since $\eta_\psi>1$, then $|\eta_{\Phi-\psi}|>1$ and $P(\theta)\neq P(\psi)$, $P(\Phi-\psi)$. Since $\xi_\psi>2$, we have $\xi_\theta>2+\xi_\Phi$ when $P(\theta)=P(\Phi+\psi)$, $P(2\Phi+\psi)$. Thus, $P(\theta)\neq P(\Phi+\psi)$, $P(2\Phi+\psi)$. It follows that $\theta=\theta_g^{(1)}=\Phi=\Phi^*$ and it is easy to verify that $\mathcal{L}_2(=(1/\theta_g^{(1)})\mathcal{L}_1)=\mathcal{L}_1$.

If

$$g>\delta \tag{6.1}$$

then $\eta_\nu<\sqrt{3}/2$ and ν^* must exist in $\mathcal{L}_1$. Further, since

$$\xi_\mu-\xi_\nu=3((M-y)\delta+(g-x)\bar{\delta})/2\geqq 3(\delta+\bar{\delta})/2>2,$$

we must have $\nu^*<\mu^*=\mu$. Also, $\nu^*=\xi_\nu+\zeta_{\nu^*}>2-1=1$. Since μ is the fundamental unit ε_0 of $\mathcal{O}$ (see [5]) and $\nu^* \in \mathcal{O}$, $|\nu^{*\prime}| < 1$, $1 < \nu^* < \mu$, we see that either ν^* is a minimum of $\mathcal{L}_1$ or $\mathcal{N}(\nu^*)$ contains a minimum of $\mathcal{L}_1$. In either case we must have some $\gamma \in \mathcal{O}$ such that $1 < \gamma < \mu$ and γ is a minimum of $\mathcal{L}_1$. Thus the period length of Voronoi's algorithm must, in this case, exceed 1. □

Thus, if (6.1) holds, the Voronoi continued fraction for $\mathcal{O}$ must have period length >1. Hence the question which we now need to address is that of whether (6.1) ever happens. We note first that if $g > M$, then (6.1) is certainly true. Thus, if the diophantine equation

$$x^3 + 1 = yz^2 \tag{6.2}$$

has a solution with y, z square-free, $gcd(y, z) = 1$, and $z > x$, we will have a value of D $(= x^3 + 1)$ such that the period length p for Voronoi's continued fraction for $\mathcal{O}$ $(= \mathcal{O}_{\mathcal{K}}$ when $3 \nmid x)$ with basis $\{1, \delta, \bar{\delta}\}$ must exceed 1.

A computer serach for solutions of (6.2) for all $0 < x \leqq 61\,000$ yielded only the solutions

$$2^3 + 1 = 3^2$$

$$293^3 + 1 = 158(399)^2$$

$$11093^3 + 1 = 1742(27993)^2$$

$$16939^3 + 1 = 3605(36718)^2$$

$$20885^3 + 1 = 2654(58587)^2$$

$$60818^3 + 1 = 20273(105339)^2,$$

where $z > x, z, y$ are square-free and $gcd(z, y) = 1$. In fact for $D = 9$, $293^3 + 1 =
= 251\,537\,58$, the value of $p = 3$. In the latter case we get $\theta_g^{(1)} = 137\,70 + 47\delta + 64\bar{\delta}$, $\theta_g^{(2)} = (158\,82 - 22\delta + 111\bar{\delta})/332\,58$, $\theta_g^{(3)} = (462\,25 + 158\delta + 215\bar{\delta})/222\,61$.

7. The problem of when $p = 1$ for $\mathcal{O}_{\mathcal{K}}$. When $\mathcal{K} = \mathcal{Q}(\sqrt[3]{D})$ and $\mathcal{O}$ has basis $\{1, \delta, \delta^2\}$ we know that $p = 1$ for $D = M^3 + 1$. In fact Dubois [3] has pointed out that if $\mathcal{K} = \mathcal{Q}(\sqrt[3]{D})$, $\mathcal{O}$ has basis $\{1, \delta, \delta^2\}$ and $p = 1$, then $D = M^3 + 1$. In fact, it could also occur that $D^2 = M^3 + 1$, but, as this can only occur (non-trivially) when $D = 3$ and $M = 2$ and the period in this case is 3, we need only concern ourselves with the $D = M^3 + 1$ case. The question could also be asked as to what form D must take if the period length p of Voronoi's algorithm for the maximal order $\mathcal{O}_{\mathcal{K}}$ is to be 1. In this case we may assume that D is cube-free and $\delta^3 = D = fg^2$, $\bar{\delta}^3 = \bar{D} = f^2 g$ with f, g square-free and $gcd(f, g) = 1$. We now have the following extension of Dubois' result.

Theorem 1. *If $\mathcal{O}$ has basis $\{1, c_1\delta, c_2\bar{\delta}\}$, where $c_1, c_2 \in \mathbf{Z}$ and the period length of Voronoi's algorithm for $\mathcal{O}$ is 1, then either $c_1^3 D$ or $c_2^3 \bar{D}$ must be of the form $M^3 + 1$ or $M^3 - 1$.*

Proof. Let $\mathcal{L}_1$ be the (reduced) lattice with basis $\{1, c_1\delta, c_2\bar{\delta}\}$ and let p be the period length of Voronoi's algorithm for $\mathcal{O}$. Without loss of generality we may assume that $c_1\delta < c_2\bar{\delta}$. Let

$$c_1^3 D = R^3 + r, \quad \text{where} \quad R < c_1\delta < R + 1.$$

If $\theta=c_1\delta-R$, then $\theta\in\mathcal{O}$ and $N(\theta)=r$. Suppose $r\neq 1$; then $\theta(>0)$ cannot be a unit of $\mathcal{O}$. If θ is a minimum of $\mathcal{L}_1$, then $\varepsilon_0^n\theta$ is a minimum of $\mathcal{L}_1$, where ε_0 is the fundamental unit of $\mathcal{O}$; hence, there must exist some $\gamma=\varepsilon_0^n\theta\in\mathcal{O}$ such that $\gamma>1$, γ is a minimum of $\mathcal{L}_1$, and $N(\gamma)\neq 1$. It follows that in this case $p>1$.

If θ is not a minimum of $\mathcal{L}_1$, there must exist some minimum μ of $\mathcal{L}_1$ such that

$$0<\mu<\theta<1 \quad \text{and} \quad |\mu'|<|\theta'|.$$

Put $\mu=m_0+m_1c_1\delta+m_2c_2\bar{\delta}$, where $m_1, m_2, m_3\in\mathbf{Z}$. If $N(\mu)\neq 1$, we have $p>1$ as above; hence, we suppose that $N(\mu)=1$.

Now

$$|3m_0|,\ |3m_1c_1\delta|,\ |3m_2c_2\bar{\delta}| < \mu+|\mu'|+|\mu''| \leqq \theta+2|\theta'|.$$

Since $|\theta'|^2=r/(c_1\delta-R)=c_1^2\delta^2+c_1\delta R+R^2<(c_1\delta+R)^2$, we get

$$|3m_0|,\ |3m_1c_1\delta|,\ |3m_2c_2\bar{\delta}| < 3c_1\delta+R.$$

Hence

$$(7.1)\qquad |m_0|<c_1\delta+R/3,\ |m_1|<1+R/(3c_1\delta),\ |m_2|<1+R/(3c_2\bar{\delta}).$$

Since $c_2\bar{\delta}>c_1\delta>R$, we find that $|m_1|, |m_2|<2$. If either of m_1 or m_2 is zero, then because $N(\mu)=1$, we must get $\pm c_1^3D$ or $\pm c_2^3\bar{D}=1-m_0^3$ and the theorem follows. Suppose $m_1, m_2\neq 0$; we must have $|m_1|=|m_2|=1$. It now remains to show that no value of μ, subject to the constraints developed here, can exist.

Since $\mu=m_0+c_1m_1\delta+c_2m_2\bar{\delta}$ and $0<\mu<1$, we get $m_0=-[c_1m_1\delta+c_2m_2\bar{\delta}]$. If $m_1=m_2$, then $m_0=-[\pm(c_1\delta+c_2\bar{\delta})]$ and

$$|m_0|>2c_1\delta-1.$$

Since $c_1\delta=\sqrt[3]{R^3+r}\geqq 1+R/3$ $(R\geqq 1)$, we get $|m_0|>c_1\delta+R/3$, which is impossible by (7.1). Thus, $m_1=-m_2$.

Let $\nu=\mu^{-1}$. Since μ is a unit of $\mathcal{L}$, so is ν and ν must be a minimum of $\mathcal{L}_1$ with $\nu>1$ and $|\nu'|<1$. Since $N(\mu)=1$, we get

$$\nu=\mu'\mu''=|\mu'|^2\leqq|\theta'|^2=c_1\delta^2+c_1\delta R+R^2.$$

Further,

$$\nu=\mu^{-1}=m_0^2+c_1c_2fg+\delta(fc_2^2-c_1m_1m_0)+\bar{\delta}(gc_1^2+c_2m_1m_0)=$$
$$=n_0+n_1\delta+n_2\bar{\delta}.$$

Since

$$|3n_3\bar{\delta}|\leqq\nu+|\nu'|+|\nu''|<\nu+2<3c_1^2g\bar{\delta}+2,$$

we get

$$|gc_1^2+c_2m_0m_1|<c_1^2g+1.$$

It follows that $m_0=0$ or $m_1m_0<0$. If $m_0=0$, then from $N(\mu)=1$ we get $c_1^3fg^2-c_2^3f^2g=\pm 1$; hence $D=1$, which is not possible. Thus, $m_0\neq 0$ and, if we put $m=-m_1m_0$, we get $m>0$. We have $\mu=-m_1m+c_1m_1\delta-c_2m_1\bar{\delta}$ and $|\mu|=$ $=|-m+c_1\delta-c_2\bar{\delta}|$. Since $-m\leqq -1$ and $c_1\delta-c_2\bar{\delta}<0$ we get $|\mu|>1$, which is also impossible. □

Now if $D\not\equiv\pm 1 \pmod 9$ and $c_1=c_2=1$, then $\mathcal{O}=\mathcal{O}_{\mathcal{K}}$, the maximal order of $\mathcal{K}$. Further, if D or $\bar{D}$ is of the form M^3-1, then put $\theta=M^2-1+M\delta+\delta^2$ or $M^2-1+M\bar{\delta}+\bar{\delta}^2$, respectively. Note that ε_0, the fundamental unit (>1) of $\mathcal{O}_{\mathcal{K}}$ is

$\theta+1$ and $1<\theta<\varepsilon_0$. Further, it is easy to show that $(\theta')<1$. Thus, either θ is a minimum of $\mathscr{L}_1$, the lattice with basis $\{1, \delta, \bar{\delta}\}$ or a minimum of $\mathscr{L}_1$ occurs in $\mathscr{N}(\theta)$. As we have argued before, this means that p must exceed 1. Thus, if $\mathcal{O}_{\mathscr{K}}$ is the maximal order of $\mathscr{Q}(\delta)$, where $\delta^3=D\not\equiv\pm 1 \pmod 9$ and the period length of Voronoi's algorithm for $\mathcal{O}_{\mathscr{K}}$ is 1, then D or $\bar{D}=M^3+1$. Recall, however, that if D or $\bar{D}=M^3+1$, it is not necessarily the case that the period length for $\mathcal{O}_{\mathscr{K}}$ is 1.

The problem of determining those forms of $D\equiv\pm 1 \pmod 9$ such that the value of p for $\mathcal{O}_{\mathscr{K}}$ is 1 is more complicated. We have not succeeded in solving this problem completely, but we can show that there exists an infinitude of values of D such that neither D nor $\bar{D}$ is of the form M^3+1, but for which $p=1$ for $\mathcal{O}_{\mathscr{K}}$. In order to do this we must first prove two Lemmas.

Lemma 7.1. *Let* $M=3T^2-2$ $(T\geqq 4)$. *If* $\delta^3=(M+9T+9)M^2$ *and* $\gamma=3-3(T+1)/M$, *then*

$$\delta < M+3+3T, \tag{7.2}$$

$$0 < \delta/M-(M+1+3T)/(M+1) < 3(T+1)/\big(M(M+1)^2\big), \tag{7.3}$$

$$\gamma\delta/M > 3, \tag{7.4}$$

$$\frac{\delta}{M} < \frac{M-2-\gamma/M}{N-3T+1}, \tag{7.5}$$

$$\frac{\delta}{M} < 1+\frac{3MT-9T+3/2}{M(M-2)}, \tag{7.6}$$

$$\frac{\delta}{M} < \frac{(M-3T+1)(M+9T+9)}{M^2+3MT-2M-9T-4}. \tag{7.7}$$

Proof. (7.2) can be easily proved by comparing δ^3 to $(M+3T+3)^3$. The left side of (7.3) is easy to show by evaluating $(M+1)^3\delta^3-M^3(M+3T+1)^3=9M^2(M+T+1)>0$. The right side of (7.3) can be proved by first noting that

$$(3M+3T+3)/M-(3M+3T+6)/(M+1) = 3(T+1)/\big(M(M+1)\big).$$

Hence,

$$\frac{3T+3}{M}-\frac{3T}{M+1} = \frac{3M+1+3T}{(M+1)^2}+\frac{3(T+1)}{(M+1)^2M},$$

and we get

$$\frac{9(T+1)}{M} < \frac{9T}{M+1}+\frac{9(3T)^2}{(M+1)^2}+\frac{27T^3}{(M+1)^3}+\frac{9(T+1)}{(M+1)^2M}.$$

Thus,

$$\frac{9(T+1)}{M} < \left(1+\frac{3T}{M+1}+\frac{3(T+1)}{M(M+1)}\right)^3-1.$$

(7.4) follows easily by using the left side of (7.3).

By using

$$\big((3-\gamma)M-\gamma\big)(M+1) > 3T(M+1) > 3(T+1)(M-3T+1)$$

and

$$(3-\gamma)M-\gamma/M = (M+1)(M-2-\gamma/M)-(M+1+3T)(M+1-3T),$$

we get

$$\frac{M-2-\gamma/M}{M-3T+1} > \frac{M+1+3T}{M+1}+\frac{3(T+1)}{M(M+1)^2} > \frac{\delta}{M}.$$

(7.6) can be proved by showing that

$$1+\frac{3MT-9T+3/2}{M(M-2)} > \frac{M-2-\gamma/M}{M-3T+1} > \frac{\delta}{M}.$$

(7.7) can be proved by noting that

$$\frac{(M-3T+1)(M+9T+9)}{M^2+3MT-2M-9T-4}-\frac{M+1+3T}{M+1} = \frac{M+3T-13}{(M+1)(M^2+3MT-2M-9T-4)}$$

and

$$\frac{M+3T-13}{M^2+3MT-2M-9T-4}-\frac{3(T+1)}{M(M+1)} =$$

$$= \frac{M^3-18M^2-4M+30+39T}{M(M+1)(M^2+3MT-2M-9T-4)} > 0;$$

hence,

$$\frac{(M-3T+1)(M+9T+9)}{M^2+3MT-2M-9T-4} > \frac{M+3T+1}{M+1}+\frac{3(T+1)}{M(M+1)^2} > \frac{\delta}{M}. \quad \square$$

Lemma 7.2. *Let D be defined as in Lemma* 7.1 *and put* $\bar{\delta}^3=\bar{D}=(M+9T+9)^2M$. *We have*

(7.8) $$-\sqrt{3} < (M+1+3T)\delta-(M+1)\bar{\delta} < 0,$$

(7.9) $$(M-2)\delta-(M-3T+1)\bar{\delta} > 3,$$

(7.10) $$0 < 2(M^2+3MT-2M-9T)-(M-2)\delta-(M-3T+1)\bar{\delta} < 5.$$

Proof. The right side of (7.8) follows from the left side of (7.3) and $\delta^2=M\bar{\delta}$. The left side of (7.8) can be verified by noting that

$$|(M+1+3T)\delta-(M+1)\bar{\delta}| = \delta|(M+1-3T)M-(M+1)\delta|/M <$$

$$< 3\delta(T+1)/(M(M+1)),$$

by the right side of (7.3), and that $\delta<M+3T+3$. By using (7.5) we get

$$M(M-2)-\delta(M-3T+1) > \gamma;$$

hence,

$$(M-2)\delta-(M-3T+1)\bar{\delta} > \gamma\delta/M > 3$$

by (7.4).

From (7.6) and (7.9) we get

$$2(M^2-2M+3MT-9T) > 2(M-2)\delta-3 > (M-2)\delta+(M-3T+1)\bar{\delta}.$$

From (7.7) we have

$$M^2+3MT-2M-9T-4 < (M-3T+1)\bar{\delta};$$

hence, by (7.9)

$$2(M^2+3MT-2M-9T-4) < 2(M-3T+1)\bar{\delta} <$$

$$< (M-2)\delta+(M-3T+1)\bar{\delta}-3$$

and we get (7.10). □

We now are able to prove

Theorem 7.2. *If $D=M^2(M+9T+9)$, where $M=3T^2-2$ and M and $M+9T+9$ are both square-free, then the period length of Voronoi's continued fraction for $\mathcal{O}_{\mathcal{K}}$ is* 1 *and*

$$3\theta_g^{(1)} = 3\theta_2 = 3\varepsilon_0 = M^2+(6T+4)M+3+(M+3T+1)\delta+(M+1)\bar{\delta}.$$

Proof. The result can be easily verified for $T=2$. If $T=3$, then $M=25$, which is not square-free; hence, we may assume $T\geqq 4$. Now $gcd(M, M+9T+9)=1$ and $D\equiv 1 \pmod 9$; thus, $\mathcal{O}_{\mathcal{K}}$ has basis $\{1, \delta, (1+\delta+\bar{\delta})/3\}$. Also, it is easy to verify that $\mathcal{O}_{\mathcal{K}}$ has basis $\{1, \mu, \nu\}$, where

$$3\mu = M+1+(M+1+3T)\delta+(M+1)\bar{\delta} > 0,$$

$$3\nu = M-3T+1+(M-2)\delta+(M-3T+1)\bar{\delta} > 0.$$

Thus, $\xi_\mu>\xi_\nu>2$ and by (7.8) we find that $-1/2<\eta_\mu<0$. Further, by (7.9) we have $\eta_\nu>\sqrt{3}/2$; thus, we may put $\Phi=\mu, \psi=\nu$.

Put

$$3\Phi_1 = M^2+(6T+4)M+3+(M+1+3T)\delta+(M+1)\bar{\delta},$$

$$3\psi_1 = M^2+(3T-2)M-9T+(M-2)\delta+(M-3T+1)\bar{\delta},$$

and note that $P(\Phi_1)=P(\Phi)$, $P(\psi_1)=P(\psi)$. Now

$$\Phi_1'\Phi_1'' = (-M+3+2\delta-\bar{\delta})/3$$

and

$$N(\Phi_1) = \Phi_1|\Phi_1'|^2 = 1.$$

Since $\Phi_1>1$, we have $|\Phi_1'|<1$. It follows that $\Phi^*=\Phi_1$ or Φ_1-1. We also have

$$N(\Phi_1''+\Phi_1') = (\Phi_1+\Phi_1'+\Phi_1'')(\Phi_1'\Phi_1''+\Phi_1\Phi_1'+\Phi_1\Phi_1'')-N(\Phi_1) =$$

$$= \big(M^2+6(T+4)M+3\big)(-M+3)-1 < 0;$$

hence,

$$2\zeta_{\Phi_1} = \Phi_1''+\Phi_1' < 0 \tag{7.11}$$

and

$$|\Phi_1'-1|^2 = \Phi_1'\Phi_1''-\Phi_1'-\Phi_1''+1 = |\Phi_1|^2+1-2\zeta_{\Phi_1} > 1.$$

Thus, $\Phi^*=\Phi_1$.

By (7.10) we have

(7.12) $$0 < \zeta_{\psi_1} < 5/6;$$

hence, if ψ_1^* exists, we must have $\psi_1^* = \psi_1$ or $\psi_1 - 1$. Now

$$N(\psi_1) = M^2 + 6MT - 18T - M$$

and

$$3\psi_1 < 3(M^2 + 3MT - 2M - 9T)$$

by the left side of (7.10); hence, $\psi_1 < N(\psi_1)$ and $|\psi_1'| > 1$. Also,

$$N(\psi_1 - 1) = M^2 + 9MT + 6M - 27T - 19$$

and

$$\psi_1 - 1 < N(\psi_1 - 1);$$

thus ψ_1^* does not exist.

Put $\omega = \Phi_1 - \psi_1$. By (7.11) and (7.12) we have $-2 < \zeta_\omega < 0$. It follows that if ω^* exists, then $\omega^* = \omega + 2$, $\omega + 1$, or ω. Since

$$\omega = TM + 2M + 3T + 1 + (T+1)\delta + T\bar{\delta},$$

we have

$$\omega'\omega'' = -MT - M + 6T + 7 + (2T+3)\delta - (T+2)\bar{\delta}$$

and

$$N(\omega) = 3MT + 9M + 27T + 19.$$

Since $\zeta_\omega > -2$, we also have

$$(T+1)\delta + T\bar{\delta} < 2(TM + 2M + 3T + 3);$$

thus,

$$\omega < 3TM + 6M + 9T + 6 < N(\omega).$$

Since $N(\omega+1) = N(\omega) + 3M + 27T + 25$ and $N(\omega+2) = N(\omega) + 6TM + 18M + 72T + 62$, we get $\omega + 1 < N(\omega+1)$, $\omega + 2 < N(\omega+2)$ and ω^* does not exist.

Thus, if θ is the minimum adjacent to 1 in $\mathscr{L}_1$, the lattice with basis $\{1, \delta, (1+\delta+\bar{\delta})/3\}$, then $P(\theta) \neq P(\psi)$, $P(\Phi - \psi)$. Also, since Φ^* exists and $\xi_\psi > 2$, we cannot have $P(\theta) = P(\Phi + \psi)$ or $P(2\Phi + \psi)$ by Lemma 3.1. It follows that $\theta_g^{(1)} = \theta = \Phi^* = \Phi_1$; and, since $N(\Phi_1) = 1$, we must have $p = 1$.

We mention here that since $(3T^2 - 2)(3T^2 + 9T + 7)$ is square-free infinitely often for $T \geqq 2$ [4], there must exist an infinitude of values of D of the form $M^2(M + 9T + 9)$ such that M and $M + 9T + 9$ are both square-free. Further, since for such values of D we get

$$\varepsilon_0^{-1} = (-M + 3 + 2\delta - \bar{\delta})/3$$

and [5]

$$\varepsilon_0^{-1} = N - \delta$$

for $D = N^3 + 1$, no D of the form given by Theorem 7.2 can also be of the form $N^3 + 1$. Thus, we have an infinitude of new values of D for which $p = 1$ for $\mathcal{O}_{\mathscr{K}}$. It is not known whether the numbers D or $\bar{D}$ of the form $N^3 + 1$ or of the form given in Theorem 7.2 are the only values of D for which $p = 1$ for $\mathcal{O}_{\mathscr{K}}$. A computer search over all $D \equiv \pm 1 \pmod 9$ with $D < 10^7$ revealed no other values of D for which $p = 1$.

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