A result on the neutrix convolution product of distributions

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The convolution product of two distributions satisfying certain conditions was defined by GELFAND and SHILOV [3] as follows.

Definition 1. Let f and g be distributions satisfying either of the following conditions:
(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side.
Then the convolution product $f*g$ is defined by

$$\big((f*g)(x),\ \varphi(x)\big) = \big(g(y),\ (f(x), \varphi(x+y))\big)$$

for arbitrary test function φ in the space K of infinitely differentiable functions with compact support.

Note that if f has bounded support then $\big(f(x), \varphi(x+y)\big)$ is in K and so $\big(g(y), (f(x), \varphi(x+y))\big)$ is meaningful. On the other hand, if g has bounded support or the supports of f and g are bounded on the same side, then the intersection of the supports of $g(y)$ and $\big(f(x), \varphi(x+y)\big)$ is bounded and so $\big(g(y), (f(x), \varphi(x+y))\big)$ is again meaningful.

It follows that if the convolution product $f*g$ exists by this definition then

$$f*g = g*f, \tag{1}$$

$$(f*g)' = f*g' = f'*g.$$

This definition of the convolution product is rather restrictive and the next definition was introduced in [2] in order to extend the convolution product to a larger class of distributions. In this definition, we denote the convolution product by $f*g$ to distinguish it from the convolution product given in definition 1.

Definition 2. Let f and g be distributions and let $f_n = f\tau_n$ for $n=1, 2, \ldots,$ where τ is an infinitely differentiable function satisfying the following conditions:

(i) $\tau(x) = \tau(-x)$,

(ii) $0 \leqq \tau(x) \leqq 1$,

(iii) $\tau(x) = 1 \quad \text{for} \quad |x| \leqq \dfrac{1}{2}$,

(iv) $\tau(x) = 0 \quad \text{for} \quad |x| \geqq 1$

and the function τ_n is defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leqq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n. \end{cases}$$

Then the convolution product $f \boxed{*} g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, providing the limit h exists in the sense that

$$N-\lim_{n\to\infty} (f_n * g, \varphi) = (h, \varphi)$$

for all test functions φ in K, where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, ..., n, ...\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n \quad (\lambda > 0,\ r = 1, 2, ...)$$

and all functions $E(n)$ for which $\lim_{n\to\infty} E(n) = 0$.

Note that in this definition the convolution product $f_n * g$ is in the sense of definition 1, the distribution f_n having bounded support, since the support of τ_n is contained in the interval $(-n-n^{-n}, n+n^{-n})$. Note also that because the definition is nonsymmetric, the convolution product $f \boxed{*} g$ is not always commutative.

The following theorem was proved in [2] and shows that definition 2 is a generalization of definition 1.

Theorem 1. *Let f and g be distributions satisfying either condition* (a) *or condition* (b) *of definition 1. Then the convolution product $f \boxed{*} g$ exists and*

$$f \boxed{*} g = f * g.$$

The next two theorems were also proved in [2].

Theorem 2. *Let f and g be distributions and suppose that the convolution product $f \boxed{*} g$ exists. Then the convolution product $f \boxed{*} g'$ exists and*

$$(f \boxed{*} g)' = f \boxed{*} g'.$$

Theorem 3. *The convolution product $x_+^\lambda \boxed{*} x^s$ exists and*

$$x_+^\lambda \boxed{*} x^s = 0$$

for $\lambda > -1$ and $s = 0, 1, 2, ...$

In the following, $E(n)$ denotes any function such that $\lim_{n\to\infty} E(n) = 0$ and $0(n)$ denotes any function such that $N\text{-}\lim_{n\to\infty} 0(n) = 0$. We will therefore write

$$0(n) + 0(n) = 0(n) + E(n) = 0(n).$$

We now prove the following extension of Theorem 3.

Theorem 4. *The convolution product $x_+^\lambda \boxed{*} x^s$ exists and*

$$x_+^\lambda \boxed{*} x^s = 0$$

for $\lambda < -1$, $\lambda \neq -2, -3, ...$ and $s = 0, 1, 2, ...$

PROOF. We will suppose that $-r-1<\lambda<-r$. Then the distribution x_+^λ is defined by

$$x_+^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r+1)}(x_+^{\lambda+r})^{(r)},$$

where $\lambda+r>-1$.

The convolution product $(x_+^\lambda)_n * x^s$, where

$$(x_+^\lambda)_n = x_+^\lambda \tau_n(x),$$

exists by definition 1 and so

$$((x_+^\lambda)_n * x^s, \varphi(x)) = ((y_+^\lambda)_n, (x^s, \varphi(x+y)))$$

for arbitrary test function φ in K. If the support of φ is contained in the interval (a, b), we have

$$(x^s, \varphi(x+y)) = \int_{a-y}^{b-y} x^s\varphi(x+y)\,dx = \int_a^b (t-y)^s\varphi(t)\,dt =$$

$$= \sum_{i=0}^{s} a_i y^i = p(y),$$

where

$$a_i = \binom{s}{i}(-1)^i(t^{s-i}, \varphi(t))$$

for $i=0, 1, \ldots, s$ and $\binom{s}{i}$ denotes the binomial coefficients.

Thus

$$((x_+^\lambda)_n * x^s, \varphi(x)) = ((y_+^\lambda)_n, p(y)) = (y_+^\lambda, (\tau_n p)(y)) =$$

$$= \frac{(-1)^r\Gamma(\lambda+1)}{\Gamma(\lambda+r+1)}(y_+^{\lambda+r}, (\tau_n p)^{(r)}(y)) =$$

$$= \frac{(-1)^r\Gamma(\lambda+1)}{\Gamma(\lambda+r+1)}\Big[\int_0^n y^{\lambda+r}p^{(r)}(y)\,dy + \int_n^{n+n^{-n}} y^{\lambda+r}(\tau_n p)^{(r)}(y)\,dy\Big].$$

It is easily seen that

$$\int_0^n y^{\lambda+r}p^{(r)}(y)\,dy = 0(n),$$

where $0(n)=0$ for $r>s$, and on noting that

$$\tau_n^{(i)}(n) = \begin{cases} 1, & i=0, \\ 0, & i \geqq 1, \end{cases}$$

$$\tau_n^{(i)}(n+n^{-n}) = 0, \quad i \geqq 0$$

we have

$$\int_n^{n+n^{-n}} y^{\lambda+r}(\tau_n p)^{(r)}(y)\,dy = \int_n^{n+n^{-n}} y^{\lambda+r}\,d(\tau_n p)^{(r-1)}(y) =$$

$$= [y^{\lambda+r}(\tau_n p)^{(r-1)}(y)]_n^{n+n^{-n}} - (\lambda+r)\int_n^{n+n^{-n}} y^{\lambda+r-1}(\tau_n p)^{(r-1)}(y)\,dy =$$

$$= 0(n) + \frac{(-1)^r \Gamma(\lambda+r+1)}{\Gamma(\lambda+1)} \int_n^{n+n^{-n}} y^{\lambda}(\tau_n p)(y)\,dy.$$

Now

$$\Big|\int_n^{n+n^{-n}} y^{\lambda}(\tau_n p)(y)\,dy\Big| \leq (s+1)\max\{|a_i|:\, 0 \leq i \leq s\}(n+n^{-n})^{s+\lambda} n^{-n} = E(n)$$

and so

$$\int_n^{n+n^{-n}} y^{\lambda+r}(\tau_n p)^{(r)}(y)\,dy = 0(n).$$

It follows that

$$((x_+^{\lambda})_n * x^s, \varphi(x)) = 0(n)$$

and so

$$N\!-\!\lim_{n\to\infty} ((x_+^{\lambda})_n * x^s, \varphi(x)) = 0 = (0, \varphi(x))$$

for $s=0, 1, 2, \ldots$. The result of the theorem follows.

Corollary 1. *The convolution products* $x_-^{\lambda} \boxed{*}\, x^s$, $|x|^{\lambda} \boxed{*}\, x^s$ *and* $(\operatorname{sgn} x \cdot |x|^{\lambda}) \boxed{*}\, x^s$ *exist and*

$$x_-^{\lambda} \boxed{*}\, x^s = |x|^{\lambda} \boxed{*}\, x^s = (\operatorname{sgn} x \cdot |x|^{\lambda}) \boxed{*}\, x^s = 0$$

for $\lambda < -1$, $\lambda \neq -2, -3, \ldots$ and $s=0, 1, 2, \ldots$.

Proof. The results follow from the theorem on noting that

$$x_-^{\lambda} = (-x)_+^{\lambda}, \; |x|^{\lambda} = x_+^{\lambda} + x_-^{\lambda}, \quad \operatorname{sgn} x \cdot |x|^{\lambda} = x_+^{\lambda} - x_-^{\lambda}.$$

Corollary 2. *The convolution product* $x_+^{\lambda} \boxed{*}\, x_-^s$ *exists and*

$$x_+^{\lambda} \boxed{*}\, x_-^s = (-1)^{s+1} B(\lambda+1, s+1) x_+^{\lambda+s+1}$$

for $\lambda < -1$, $\lambda \neq -2, -3, \ldots$ *and* $s=0, 1, 2, \ldots$, *where B denotes the beta function.*

Proof. We will again suppose that $-r-1 < \lambda < -r$ and then

$$x_+^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r+1)} (x_+^{\lambda+r})^{(r)}.$$

The convolution product $x_+^{\lambda+r} * x_+^s$ exists by definition 1 and

$$x_+^{\lambda+r} * x_+^s = \int_{-\infty}^{\infty} (x-t)_+^{\lambda+r} t_+^s \, dt = x_+^{\lambda+r+s+1} \int_0^1 (1-u)^{\lambda+r} u^s dt =$$

$$= B(\lambda+r+1, s+1) x_+^{\lambda+r+s+1}.$$

Equation (1) holds and it follows that

$$\frac{\Gamma(\lambda+r+1)}{\Gamma(\lambda+1)} x_+^\lambda * x_+^s = \frac{\Gamma(\lambda+r+1, s+2)}{\Gamma(\lambda+s+2)} B(\lambda+r+1, s+1) x_+^{\lambda+s+1}.$$

Thus

$$x_+^\lambda * x_+^s = B(\lambda+1, s+1) x_+^{\lambda+s+1}$$

for $\lambda < -1$, $\lambda \neq -2, -3, \ldots$ and $s = 0, 1, 2, \ldots$. The result of the corollary now follows on noting that

$$x_-^s = (-1)^s (x^s - x_+^s).$$

Corollary 3. *The convolution products* $x_-^\lambda \boxed{*} x_+^s$, $|x|^\lambda \boxed{*} x_+^s$ *and* $(\operatorname{sgn} x \cdot |x|^\lambda) \boxed{*} x_+^s$ *exists and*

$$x_-^\lambda \boxed{*} x_+^s = (-1)^{s+1} B(\lambda+1, s+1) x_+^{\lambda+s+1},$$

$$|x|^\lambda \boxed{*} x_+^s = \begin{cases} B(\lambda+1, s+1) \operatorname{sgn} x \cdot |x|^{\lambda+s+1}, & \textit{even } s, \\ B(\lambda+1, s+1) |x|^{\lambda+s+1}, & \textit{odd } s, \end{cases}$$

$$(\operatorname{sgn} x \cdot |x|^\lambda) \boxed{*} x_+^s = \begin{cases} B(\lambda+1, s+1) |x|^{\lambda+s+1}, & \textit{even } s, \\ B(\lambda+1, s+1) \operatorname{sgn} x \cdot |x|^{\lambda+s+1}, & \textit{odd } s \end{cases}$$

for $\lambda < -1$, $\lambda \neq -2, -3, \ldots$ and $s = 0, 1, 2, \ldots$.

The results of this corollary are immediate.

References

[1] J. G. van der Corput, Introduction to the neutrix calculus, *J. Analyse Math.*, **7** (1959—60), 291—398.

[2] B. Fisher, Neutrices and the convolution of distributions, *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.*, **17** (1987), 107—118.

[3] I. M. Gelfand and G. E. Shilov, Generalized functions, Vol. I, *Academic Press* (1964).

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