π-holomorphically planar curves and π-holomorphically projective transformations

GEORGY MARKOV (Plovdiv, Bulgaria)

and

MILEVA PRVANOVIĆ (Novi Sad, Yugoslavia)

1. Introduction. Let A_{2n} be an affinely connected manifold of dimension $2n$ whose components of the affine connection are Γ^i_{jk}. Let A_{2n} be endowed with a tensor field $f^i_j \neq \delta^i_j$ satisfying

$$f^i_s f^s_j = \omega \delta^i_j, \quad \omega = +1 \quad \text{or} \quad -1. \tag{1.1}$$

If $\omega = -1$, the manifold is an almost complex space, and if $\omega = +1$, it is an almost product space. In the adapted coordinate systems, the matrix of f has the form:

$$(f^i_j) = \begin{pmatrix} \sqrt{\omega} f^a_b & 0 \\ 0 & -\sqrt{\omega} f^a_b \end{pmatrix},$$

where the indices a, b run over the range $1, 2, \ldots, n$, while $i, j, k, \ldots$, run over the range $1, 2, \ldots, 2n$. Therefore

$$f^i_i = 0. \tag{1.2}$$

An affine connection Γ is called an f-connection if

$$\nabla_k f^i_j = 0 \tag{1.3}$$

where ∇_k denotes the covariant derivative with respect to the connection Γ.

A curve $x^i = x^i(t)$ satisfying the differential equations

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = \alpha(t) \frac{dx^i}{dt} + \beta(t) f^i_s \frac{dx^s}{dt} \tag{1.4}$$

is a holomorphically planar curve. We see directly from (1.4) that a curve is a holomorphically planar curve if and only if the plane elements determined by $\frac{dx^i}{dt}$ and $f^i_s \frac{dx^s}{dt}$ are parallel along the curve itself.

Two f-connections Γ and $\bar{\Gamma}$ are said to be H-projectively (holomorphically projectively) related to each other if they have all holomorphically planar curves in common.

H-planar curves and H-projective transformations have been studied by many authors ([1], [2], [3], [4], [5], [6]). So it is known that two symmetric f-connections Γ and $\bar{\Gamma}$ are H-projectively related if and only if

$$\bar{\Gamma}^i_{ks} = \Gamma^i_{ks} + \psi_k \delta^i_s + \psi_s \delta^i_k + \omega \psi_p f^p_k f^i_s + \omega \psi_p f^p_s f^i_k, \tag{1.5}$$

where ψ_i is an arbitrary covector field. Equations (1.5) give H-projective transformations of the symmetric f-connection.

The object of the present paper is to generalize the notion of holomorphically planar curves and holomorphically projective transformations using the notion of π-geodesic vector field given by RADZISZEWSKI [7].

Let be given a fixed symmetric differentiable tensor field π of type (0, 2) in A_{2n}, satisfying the condition $\det(\pi_{ij}) \neq 0$. In this paper we suppose that π_{ij} is a pure tensor, i.e.

$$^*O^{pq}_{ij} \pi_{pq} = 0 \tag{1.6}$$

where

$$^*O^{pq}_{ij} = \frac{1}{2}(\delta^p_i \delta^q_j - \omega f^p_i f^q_j).$$

The condition (1.6) is equivalent with

$$f^p_k \pi_{pj} = f^p_j \pi_{pk}. \tag{1.7}$$

In fact, (1.6) means

$$\pi_{ij} = \omega f^p_i f^q_j \pi_{pq},$$

from which, transvecting by f^i_k and taking into account the symmetry of π_{ij}, we get (1.7).

It is easy to prove that the tensor field $\tilde{\pi}^{ij}$, determined by

$$\pi_{sj} \tilde{\pi}^{si} = \delta^i_j$$

is also a pure tensor field, that is

$$f^i_s \tilde{\pi}^{sj} = f^j_s \tilde{\pi}^{is}. \tag{1.8}$$

According to the definition given by RADZISZEWSKI [7], a vector field w is called π-geodesic if

$$\nabla_k (\pi_{is} w^s) w^k = \lambda \pi_{is} w^s. \tag{1.9}$$

Integral curves of a π-geodesic vector field are called π-geodesics. Their differential equations are [7], [8]:

$$\frac{d^2 x^i}{dt^2} + [\Gamma^i_{ks} + (\nabla_k \pi_{ps}) \tilde{\pi}^{pi}] \frac{dx^k}{dt} \frac{dx^s}{dt} = \lambda \frac{dx^i}{dt};$$

In this paper we consider a vector field w satisfying, instead of (1.9), the condition

$$\nabla_k(\pi_{ip}w^p)w^k = \pi_{ip}\big(\alpha(t)w^p+\beta(t)f_q^p w^q\big). \tag{1.10}$$

We shall define, in § 2, πH-planar curves and πH-projective transformations and we shall prove in §§ 3 and 4 th eexistence of two different πH-projective curvature tensors, i.e. the existence of two different tensors each of which is invariant with respect to πH-projective transformations. One of them, called the second πH-projective curvature tensor, can be obtained directly from the H-projective curvature tensor.

2. πH-projective transformation. We have from (1.10)

$$(\nabla_k\pi_{ip})w^p w^k+\pi_{ip}\left(\frac{\partial w^p}{\partial x^k}+\Gamma_{ks}^p w^s\right)w^k = \pi_{ip}(\alpha w^p+\beta f_q^p w^q),$$

so that, transvecting with $\tilde{\pi}^{ih}$, we get

$$\frac{\partial w^h}{\partial x^k}+[\Gamma_{ks}^h+(\nabla_k\pi_{ip})\tilde{\pi}^{ph}]w^s w^k = \alpha w^h+\beta f_q^h w^q. \tag{2.1}$$

Definition. We say that the integral curves of the vector field w satisfying (2.1) are *π-holomorphically planar curves*, abbreviated to *πH-planar curves*.

If $x^i=x^i(t)$ are πH-planar curves, then $w^h=\dfrac{dx^h}{dt}$, and we obtain from (2.1) that

$$\frac{d^2x^h}{dt^2}+[\Gamma_{ks}^h+(\nabla_k\pi_{ps})\tilde{\pi}^{pk}]\frac{dx^s}{dt}\frac{dx^k}{dt} = \alpha\frac{dx^h}{dt}+\beta f_p^h\frac{dx^p}{dt} \tag{2.2}$$

are their differential equations.

Definition. A πH-projective transformation is a change of connection Γ which does not change the system of πH-planar curves.

To find a πH-projective transformation, we first notice that (1.4) and (2.2) show that a πH-planar curve with respect to the connection Γ is a holomorphically planar curve with respect to the connection

$$G_{ks}^i = \Gamma_{ks}^i+(\nabla_k\pi_{ps})\tilde{\pi}^{pi}. \tag{2.3}$$

If we suppose that G_{ks}^i is a symmetric f-connection, a change of G_{ks}^i which des not change the system of holomorphically planar curves has the form (1.5). Thus we have

$$\bar{G}_{ks}^i = G_{ks}^i+\psi_k\delta_s^i+\psi_s\delta_k^i+\omega(\psi_p f_k^p f_s^i+\psi_p f_s^p f_k^i). \tag{2.4}$$

As for $\bar{G}_{ks}^i$, it must be a symmetric f-connection too, and must also determine πH-planar curves, i.e. it must have the form

$$\bar{G}_{ks}^i = \bar{\Gamma}_{ks}^i+(\bar{\nabla}_k\pi_{ps})\tilde{\pi}^{pi}, \tag{2.5}$$

where $\bar{\nabla}$ denotes the covariant derivative with respect to the connection $\bar{\Gamma}$.

From (2.3) and (2.5), we have

$$G^i_{ks} = \frac{\partial \pi_{as}}{\partial x^k}\,\tilde{\pi}^{ai} - \Gamma^p_{ka}\pi_{ps}\tilde{\pi}^{ai},$$

$$\bar{G}^i_{ks} = \frac{\partial \pi_{as}}{\partial x^k}\,\tilde{\pi}^{ai} - \bar{\Gamma}^p_{ka}\pi_{ps}\tilde{\pi}^{ai}.$$

Consequently

$$\bar{G}^i_{ks} - G^i_{ks} = -(\bar{\Gamma}^p_{ka} - \Gamma^p_{ka})\pi_{ps}\tilde{\pi}^{ai},$$

from which, transvecting with $\tilde{\pi}^{st}\pi_{ij}$, we obtain

$$\bar{\Gamma}^t_{kj} = \Gamma^t_{kj} - (\bar{G}^i_{ks} - G^i_{ks})\tilde{\pi}^{st}\pi_{ij}.$$

Substituting $\bar{G}^i_{ks} - G^i_{ks}$ from (2.4), we get

(2.6) $$\bar{\Gamma}^t_{kj} = \Gamma^t_{kj} - \psi_s\tilde{\pi}^{st}\pi_{kj} - \delta^t_j\psi_k - \omega\psi_a(f^a_s f^i_k + f^a_k f^i_s)\tilde{\pi}^{st}\pi_{ij}.$$

Conversely, let us suppose that (2.6) holds good. Then we have

$$\bar{G}^i_{kj} = \frac{\partial \pi_{aj}}{\partial x^k}\,\tilde{\pi}^{ai} - \bar{\Gamma}^p_{ka}\pi_{pj}\tilde{\pi}^{ai} =$$

$$= \frac{\partial \pi_{aj}}{\partial x^k}\,\tilde{\pi}^{ai} - \Gamma^p_{ka}\pi_{pj}\tilde{\pi}^{ai} + \psi_j\delta^i_k + \psi_k\delta^i_j + \omega\psi_b(f^b_k f^i_j + f^b_j f^i_k) =$$

$$= G^i_{kj} + \psi_j\delta^i_k + \psi_k\delta^i_j + \omega\psi_b(f^b_k f^i_j + f^b_j f^i_k),$$

that is, we can express the connection (2.5) in the form (2.4). Thus, we have the following

Theorem. *The condition* (2.6) *is necessary and sufficient for the connections* $\bar{\Gamma}$ *and* Γ *to have all* πH*-planar curves in common.*

Taking into account (1.7), the relation (2.6) can be rewritten in the following form

(2.7) $$\bar{\Gamma}^t_{kj} = \Gamma^t_{kj} - \psi_s\tilde{\pi}^{st}\pi_{kj} - \delta^t_j\psi_k - \omega\psi_a f^a_k f^t_j - \omega\psi_a f^a_s f^i_k\tilde{\pi}^{st}\pi_{ij}.$$

Definition. The correspondence (2.7) is called a πH*-projective transformation.*

It was already mentioned that (1.5) takes place for two symmetric *f*-connections. This means that both connections $\bar{G}$ and G in (2.4) are symmetric *f*-connections. But the symmetry of the connection G implies

(2.8) $$\Gamma^i_{ks} - \Gamma^i_{sk} + (\nabla_k\pi_{as} - \nabla_s\pi_{ak})\tilde{\pi}^{ai} = 0,$$

because of (2.3). In the case when Γ is a symmetric connection too, this condition reduces to

(2.9) $$\nabla_k\pi_{as} - \nabla_s\pi_{ak} = 0,$$

and the connection $\bar{\Gamma}$ cannot be symmetric. Therefore, it must satisfy the condition

$$\bar{\Gamma}^i_{ks} - \bar{\Gamma}^i_{sk} + (\bar{\nabla}_k\pi_{as} - \bar{\nabla}_s\pi_{ak})\tilde{\pi}^{ai} = 0.$$

On the other hand, both connections $\bar{\Gamma}$ and Γ are f-connections. This statement follows from the following

Theorem. *If G is a f-connection, the connection Γ is an f-connection, too.*

PROOF. If G is an f-connection, we have

$$\frac{\partial f_r^i}{\partial x^k}+G_{ks}^i f_r^s-G_{kr}^s f_s^i=0,$$

or, substituting (2.3) and taking into account (1.8),

$$\nabla_k f_r^i+(\nabla_k \pi_{as})\tilde{\pi}^{ai} f_r^s-(\nabla_k \pi_{ra})\tilde{\pi}^{is} f_s^a=0. \tag{2.10}$$

On the other hand

$$\nabla_k(\pi_{sa} f_r^s)\tilde{\pi}^{ai}=[(\nabla_k \pi_{sa}) f_r^s+(\nabla_k f_r^s)\pi_{sa}]\tilde{\pi}^{ai},$$

from which, using (1.7), we get

$$(\nabla_k \pi_{sa})\tilde{\pi}^{ai} f_r^s=\nabla_k(\pi_{sr} f_a^s)\tilde{\pi}^{ai}-\nabla_k f_r^i=(\nabla_k \pi_{sr}) f_a^s \tilde{\pi}^{ai}+(\nabla_k f_a^s)\pi_{sr}\tilde{\pi}^{ai}-\nabla_k f_r^i.$$

Substituting this into (2.10), we obtain

$$(\nabla_k f_a^s)\pi_{sr}\tilde{\pi}^{ai}=0,$$

from which, transvecting with $\pi_{ij}\,\tilde{\pi}^{rt}$, we get

$$\nabla_k f_j^t=0. \tag{2.11}$$

This proves the theorem.

3. The first πH-projective curvature tensor. Now, we shall consider the curvature tensor

$$\bar{R}^i{}_{rkj}=\frac{\partial \bar{\Gamma}_{jr}^i}{\partial x^k}-\frac{\partial \bar{\Gamma}_{kr}^i}{\partial x^j}+\bar{\Gamma}_{kq}^i \bar{\Gamma}_{jr}^q-\bar{\Gamma}_{jq}^i \bar{\Gamma}_{kr}^q.$$

After some computation, using (2.7), we get

$$\begin{aligned}\bar{R}^i{}_{rkj}=R^i{}_{rkj}-\delta_r^i\left(\frac{\partial \psi_j}{\partial x^k}-\frac{\partial \psi_k}{\partial x^j}\right)-\omega f_r^i\left[\frac{\partial}{\partial x^k}(\psi_a f_j^a)-\frac{\partial}{\partial x^j}(\psi_a f_k^a)\right]-\\ -\pi_{jr}[\nabla_k(\psi_a \tilde{\pi}^{ai})-\psi_a \tilde{\pi}^{ai}\psi_k-\omega\psi_a f_k^a f_q^i \tilde{\pi}^{bq}\psi_b]+\\ +\pi_{kr}[\nabla_j(\psi_a \tilde{\pi}^{ai})-\psi_a \tilde{\pi}^{ai}\psi_j-\omega\psi_a f_j^a f_q^i \tilde{\pi}^{bq}\psi_b]-\\ -\omega f_j^t \pi_{tr}[\nabla_k(\psi_a f_p^a \tilde{\pi}^{pi})-\psi_b f_s^b \tilde{\pi}^{si}\psi_k-\omega\psi_a f_k^a f_q^i \tilde{\pi}^{sq}\psi_b f_s^b]+\\ +\omega f_k^t \pi_{tr}[\nabla_j(\psi_a f_p^a \tilde{\pi}^{pi})-\psi_b f_s^b \tilde{\pi}^{si}\psi_j-\omega\psi_a f_j^a f_q^i \tilde{\pi}^{sq}\psi_b f_s^b]+\\ +\psi_a \tilde{\pi}^{ai}[\nabla_j \pi_{kr}-\nabla_k \pi_{jr}+(\Gamma_{jk}^q-\Gamma_{kj}^q)\pi_{qr}]+\\ +\omega\psi_a f_p^a \tilde{\pi}^{pi}[\nabla_j(f_k^t \pi_{tr})-\nabla_k(f_j^t \pi_{tr})+(\Gamma_{jk}^q-\Gamma_{kj}^q) f_q^t \pi_{tr}],\end{aligned}$$

where $R^i{}_{rkj}$ are the components of the curvature tensor of the connection Γ.

Transvecting (2.8) with π_{ij}, we find

$$(\Gamma^i_{ks}-\Gamma^i_{sk})\pi_{ij}+\nabla_k\pi_{js}-\nabla_s\pi_{jk}=0.$$

On the other hand, transvecting (2.8) with $f^b_i\pi_{br}$ and taking into account (1.7) and (1.8), we have

$$(\Gamma^i_{ks}-\Gamma^i_{sk})f^b_i\pi_{br}+\nabla_k(f^a_r\pi_{as})-\nabla_s(f^a_r\pi_{ak})=0.$$

Therefore

$$\begin{aligned}(3.1)\qquad \bar{R}^i{}_{rkj}=R^i{}_{rkj}-\delta^i_r\left(\frac{\partial\psi_j}{\partial x^k}-\frac{\partial\psi_k}{\partial x^j}\right)-\omega f^i_r\left[\frac{\partial}{\partial x_k}(\psi_a f^a_j)-\frac{\partial}{\partial x^j}(\psi_a f^a_k)\right]-\\ -\pi_{jr}\nabla_k(\psi_a\tilde{\pi}^{ai})-\psi_a\tilde{\pi}^{ai}\psi_k-\omega\psi_a f^a_k f^i_q\tilde{\pi}^{bq}\psi_b]+\\ +\pi_{kr}[\nabla_j(\psi_a\tilde{\pi}^{ai})-\psi_a\tilde{\pi}^{ai}\psi_j-\omega\psi_a f^a_j f^i_q\tilde{\pi}^{bq}\psi_b]-\\ -\omega f^t_j\pi_{tr}[\nabla_k(\psi_a f^a_p\tilde{\pi}^{pi})-\psi_b f^b_s\tilde{\pi}^{si}\psi_k-\omega\psi_a f^a_k f^i_q\tilde{\pi}^{sq}\psi_b f^b_s]+\\ +\omega f^t_k\pi_{tr}[\nabla_j(\psi_a f^a_p\tilde{\pi}^{pi})-\psi_b f^b_s\tilde{\pi}^{si}\psi_j-\omega\psi_a f^a_j f^i_q\tilde{\pi}^{sq}\psi_b f^b_s].\end{aligned}$$

Let us put

$$(3.2)\qquad \theta^i_k=\nabla_k(\psi_a\tilde{\pi}^{ai})-\psi_a\tilde{\pi}^{ai}\psi_k-\omega\psi_a f^a_k f^i_q\tilde{\pi}^{bq}\psi_b,$$

$$(3.3)\qquad \varphi_{jk}=\frac{\partial\psi_k}{\partial x^j}-\frac{\partial\psi_j}{\partial x^k},$$

$$(3.4)\qquad \eta_{jk}=\frac{\partial}{\partial x^j}(\psi_a f^a_k)-\frac{\partial}{\partial x^k}(\psi_a f^a_j).$$

Then, taking into account (1.8) and (2.11), we have

$$\nabla_k(\psi_a f^a_p\tilde{\pi}^{pi})-\psi_b f^b_s\tilde{\pi}^{si}\psi_k-\omega\psi_a f^a_k f^i_q\tilde{\pi}^{sq}\psi_b f^b_s=f^i_a\theta^a_k.$$

This means that (3.4) may be written in the form

$$\begin{aligned}(3.5)\qquad \bar{R}^i{}_{rkj}=R^i{}_{rkj}+\delta^i_r\varphi_{jk}+\omega f^i_r\eta_{jk}-\\ -\pi_{jr}\theta^i_k+\pi_{kr}\theta^i_j-\omega f^a_j\pi_{ar}f^i_p\theta^p_k+\omega f^a_k\pi_{ar}f^i_p\theta^p_j.\end{aligned}$$

It is easily seen that

$$-\pi_{ja}\theta^a_k=-\nabla_k\psi_j-\psi_b\pi_{ja}\nabla_k\tilde{\pi}^{ba}-\psi_j\psi_k-\omega\psi_a\psi_b f^a_k f^b_j.$$

From $\pi_{ja}\tilde{\pi}^{ba}=\delta^b_j$, we have

$$-\pi_{ja}\nabla_k\tilde{\pi}^{ba}=(\nabla_k\pi_{ja})\tilde{\pi}^{ba}.$$

Therefore

$$-\pi_{ja}\theta^a_k=-\frac{\partial\psi_j}{\partial x^k}+\Gamma^s_{kj}\psi_s+\psi_b\tilde{\pi}^{ba}(\nabla_k\pi_{ja})-\psi_j\psi_k-\omega\psi_a\psi_b f^a_k f^b_j$$

and

$$-\pi_{ja}\theta^a_k+\pi_{ka}\theta^a_j=\frac{\partial\psi_k}{\partial x^j}-\frac{\partial\psi_j}{\partial x^k}+[\Gamma^s_{kj}-\Gamma^s_{jk}+\tilde{\pi}^{sa}(\nabla_k\pi_{ja}-\nabla_j\pi_{ka})]\psi_s,$$

from which

(3.6) $$\pi_{ka}\theta^a_j-\pi_{ja}\theta^a_k=\varphi_{jk},$$

because of (2.8) and (3.3).

Similarly

$$-\pi_{js}f^s_a\theta^a_k=-\frac{\partial}{\partial x^k}(\psi_b f^b_j)+\psi_b\frac{\partial f^b_j}{\partial x^k}+f^b_j\Gamma^s_{kb}\psi_s+$$
$$+\psi_b f^s_j(\nabla_k\pi_{as})\tilde{\pi}^{ba}+\psi_b f^b_j\psi_k+\psi_b f^b_k\psi_j.$$

From (2.11) we obtain

$$\psi_b\frac{\partial f^b_j}{\partial x^k}=-\psi_s\Gamma^s_{kb}f^b_j+\psi_b f^b_s\Gamma^s_{kj}.$$

Substituting this into the preceding relation, we get

$$-\pi_{js}f^s_a\theta^a_k=-\frac{\partial}{\partial x^k}(\psi_b f^b_j)+\psi_b f^b_s\Gamma^s_{kj}+\psi_b f^s_j(\nabla_k\pi_{as})\tilde{\pi}^{ba}+\psi_b f^b_j\psi_k+\psi_b f^b_k\psi_j.$$

Thus

(3.7) $$\pi_{ks}f^s_a\theta^a_j-\pi_{js}f^s_a\theta^a_k=\eta_{jk}$$

because of (2.8) and (3.4).

Now we shall determine the functions θ, φ and η.

Contracting (3.5) with respect to i and r and taking into account (1.2), (1.7) and (3.6), we find

(3.8) $$\varphi_{jk}=\frac{1}{2(n+1)}(\bar{R}^a{}_{akj}-R^a{}_{akj}).$$

Also, we have from (3.5)

$$\bar{R}^a{}_{rkj}f^i_a=R^a{}_{rkj}f^i_a+f^i_r\varphi_{jk}+\delta^i_r\eta_{jk}+\pi_{kr}f^i_a\theta^a_j-\pi_{jr}f^i_a\theta^a_k+f^a_k\pi_{ar}\theta^i_j-f^a_j\pi_{ar}\theta^i_k,$$

from which, contracting with respect to i and r and taking into account (1.2), (1.7) and (3.4), we get

(3.9) $$\eta_{jk}=\frac{1}{2(n+1)}(\bar{R}^a{}_{skj}-R^a{}_{skj})f^s_a.$$

Contracting (3.5) with respect to i and j, we obtain

(3.10) $$\bar{R}_{rk}=R_{rk}+\varphi_{rk}+\omega f^a_r\eta_{ak}-2\pi_{ar}\theta^a_k+\pi_{kr}\theta^a_a+\omega f^a_k\pi_{ar}f^s_p\theta^p_s,$$

from which, transvecting with $\tilde{\pi}^{rk}$, we find

$$\bar{R}_{rk}\tilde{\pi}^{rk}=R_{rk}\tilde{\pi}^{rk}+\varphi_{rk}\tilde{\pi}^{rk}+\omega f^a_r\tilde{\pi}^{rk}\eta_{ak}+2(n-1)\theta^a_a.$$

Here, R_{rk} and $\bar{R}_{rk}$ are Ricci tensors of connections Γ and $\bar{\Gamma}$ respectively.

Since φ_{rk} is a skew-symmetric tensor, while $\tilde{\pi}^{rk}$ is symmetric, $\varphi_{rk}\tilde{\pi}^{rk}=0$. Similarly $f^a_r\tilde{\pi}^{rk}\eta_{ak}=0$. Therefore, the preceding relation reduces to

(3.11) $$\theta^a_a=\frac{1}{2(n-1)}(\bar{R}_{rk}-R_{rk})\tilde{\pi}^{rk}.$$

Also, we find from (3.10)

$$\bar{R}_{rb} f_k^b = R_{rb} f_k^b + \varphi_{rb} f_k^b + \omega f_r^a f_k^b \eta_{ab} + \pi_{br} f_k^b \theta_a^a - 2\pi_{ar} f_k^b \theta_b^a + \pi_{kr} f_b^a \theta_a^b.$$

Transvecting with $\tilde{\pi}^{rk}$, we get

$$f_a^b \theta_b^a = \frac{1}{2(n-1)} (\bar{R}_{rb} - R_{rb}) f_k^b \tilde{\pi}^{rk}. \tag{3.12}$$

Transvecting (3.10) with $\tilde{\pi}^{ir}$, we find

$$\theta_k^i = \frac{1}{2} (-\bar{R}_{rk} \tilde{\pi}^{ri} + R_{rk} \tilde{\pi}^{ri} + \varphi_{ak} \tilde{\pi}^{ai} + \omega f_b^a \tilde{\pi}^{bi} \eta_{ak} + \delta_k^i \theta_a^a + \omega f_k^i f_b^a \theta_a^b)$$

so that, substituting from (3.8), (3.9), (3.11) and (3.12), we get

$$\begin{aligned} \theta_k^i = {} & \frac{1}{2} (-\bar{R}_{ar} + R_{ar}) \tilde{\pi}^{ai} + \frac{1}{4(n+1)} (\bar{R}^s{}_{ska} - R^s{}_{ska}) \tilde{\pi}^{ai} + \\ & + \frac{\omega}{4(n+1)} (\bar{R}^t{}_{ska} - R^t{}_{ska}) f_t^s f_b^a \tilde{\pi}^{bi} + \\ & + \frac{1}{4(n-1)} [\delta_k^i (\bar{R}_{ab} - R_{ab}) \tilde{\pi}^{ab} + \omega f_k^i (\bar{R}_{as} - R_{as}) f_b^s \tilde{\pi}^{ab}]. \end{aligned} \tag{3.13}$$

If we put

$$O_{ij}^{ab} = \frac{1}{2} (\delta_i^a \delta_j^b + \omega f_i^a f_j^b), \tag{3.14}$$

we can express (3.5) in the form

$$\bar{R}^i{}_{rkj} = R^i{}_{rkj} + \delta_r^i \varphi_{jk} + \omega f_r^i \eta_{jk} + 2O_{kp}^{ai} \pi_{ar} \theta_j^p - 2O_{jp}^{ai} \tilde{\pi}_{ar} \theta_k^p.$$

Thus, substituting (3.8), (3.9) and (3.13), we find

$$\begin{aligned} & \bar{R}^i{}_{rkj} - \frac{1}{2(n+1)} (\delta_r^i \bar{R}^s{}_{skj} + \omega f_r^i \bar{R}^b{}_{skj} f_b^s) + \\ & + O_{kp}^{qi} \pi_{qr} \left[\bar{R}_{aj} \tilde{\pi}^{ap} - \frac{1}{n+1} \tilde{\pi}^{ap} \bar{R}^t{}_{sjb} O_{ta}^{sb} - \frac{1}{n-1} \tilde{\pi}^{ab} \bar{R}_{as} O_{jb}^{ps} \right] - \\ & - O_{jp}^{qi} \pi_{qr} \left[\bar{R}_{ak} \tilde{\pi}^{ap} - \frac{1}{n+1} \tilde{\pi}^{ap} \bar{R}^t{}_{skb} O_{ta}^{sb} - \frac{1}{n-1} \tilde{\pi}^{ab} \bar{R}_{as} O_{kb}^{ps} \right] = \\ & = R^i{}_{rkj} - \frac{1}{2(n+1)} (\delta_r^i R^s{}_{skj} + \omega f_r^i R^b{}_{skj} f_b^s) + \\ & + O_{kp}^{qi} \pi_{qr} \left[R_{aj} \tilde{\pi}^{ap} - \frac{1}{n+1} \tilde{\pi}^{ap} R^t{}_{sjb} O_{ta}^{sb} - \frac{1}{n-1} \tilde{\pi}^{ab} R_{as} O_{jb}^{ps} \right] - \\ & - O_{jp}^{qi} \pi_{qr} \left[R_{ak} \tilde{\pi}^{ap} - \frac{1}{n+1} \tilde{\pi}^{ap} R^t{}_{skb} O_{ta}^{sb} - \frac{1}{n-1} \tilde{\pi}^{ab} R_{as} O_{kb}^{ps} \right]. \end{aligned}$$

Thus, we have the following

Theorem. *The tensor*

$$P_{1}{}^{i}{}_{rkj} = R^{i}{}_{rkj} - \frac{1}{2(n+1)}(\delta^{i}_{r}R^{s}{}_{skj} + \omega f^{i}_{r}R^{b}{}_{skj}f^{s}_{b}) + \tag{3.15}$$

$$+O^{qi}_{kp}\pi_{qr}\left[R_{aj}\tilde{\pi}^{ap} - \frac{1}{n+1}\tilde{\pi}^{ap}R^{t}{}_{sjb}O^{sb}_{ta} - \frac{1}{n-1}\tilde{\pi}^{ab}R_{as}O^{ps}_{jb}\right] -$$

$$-O^{qi}_{jp}\pi_{qr}\left[R_{ak}\tilde{\pi}^{ap} - \frac{1}{n+1}\tilde{\pi}^{ap}R^{t}{}_{skb}O^{sb}_{ta} - \frac{1}{n-1}\tilde{\pi}^{ab}R_{as}O^{ps}_{kb}\right]$$

is invariant with respect to the πH-projective transformation (2.7).

We shall call this tensor the first *πH-projective curvature tensor.*

4. The second πH-projective curvature tensor. Taking into account (3.6) and (3.7), we can express (3.5) in the form

$$\overline{R}^{i}{}_{rkj} = R^{i}{}_{rkj} + \delta^{i}_{r}(\pi_{ka}\theta^{a}_{j} - \pi_{ja}\theta^{a}_{k}) + \omega f^{i}_{r}(\pi_{ks}f^{s}_{a}\theta^{a}_{j} - \pi_{js}f^{s}_{a}\theta^{a}_{k}) + \tag{4.1}$$

$$+\pi_{kr}\theta^{i}_{j} - \pi_{jr}\theta^{i}_{k} + \omega f^{a}_{k}\pi_{ar}f^{i}_{p}\theta^{p}_{j} - \omega f^{a}_{j}\pi_{ar}f^{i}_{p}\theta^{p}_{k}.$$

Therefore

$${}^{*}\overline{R}^{i}{}_{k} = {}^{*}R^{i}{}_{k} - (2n+1)\theta^{i}_{k} + \tilde{\pi}^{ib}\pi_{ka}\theta^{a}_{b} + \omega\tilde{\pi}^{ab}f^{i}_{a}\pi_{ks}f^{s}_{p}\theta^{p}_{b} + \omega f^{b}_{k}f^{i}_{a}\theta^{a}_{b}, \tag{4.2}$$

where we have put

$${}^{*}\overline{R}^{i}{}_{k} = \tilde{\pi}^{ab}\overline{R}^{i}{}_{akb}, \quad {}^{*}R^{i}{}_{k} = \tilde{\pi}^{ab}R^{i}{}_{akb}. \tag{4.3}$$

From (4.2), we get

$${}^{*}\overline{R}^{a}{}_{k}\pi_{ai} = {}^{*}R^{a}{}_{k}\pi_{ai} - (2n+1)\theta^{a}_{k}\pi_{ai} + \pi_{ka}\theta^{a}_{i} + \omega f^{a}_{i}f^{b}_{k}(\pi_{bs}\theta^{s}_{a} + \pi_{as}\theta^{s}_{b}),$$

which may be written as

$${}^{*}\overline{R}^{a}{}_{k}\pi_{ai} = {}^{*}R^{a}{}_{k}\pi_{ai} - 2(n+1)\theta^{a}_{k}\pi_{ai} + 2O^{ab}_{ik}(\pi_{as}\theta^{s}_{b} + \pi_{bs}\theta^{s}_{a}). \tag{4.4}$$

If we put in (4.4) q instead of k, p instead of i, transvect with O^{pq}_{ik} and take into account that $O.O=O$, we get

$$({}^{*}\overline{R}^{a}{}_{q}\pi_{ap} - {}^{*}R^{a}{}_{q}\pi_{ap})O^{pq}_{ik} = -2(n+1)\theta^{a}_{q}\pi_{ap}O^{pq}_{ik} + 2O^{ab}_{ik}(\pi_{as}\theta^{s}_{b} + \pi_{bs}\theta^{s}_{a}).$$

Changing indices and adding, we obtain

$$-2(\theta^{a}_{q}\pi_{ap} + \theta^{a}_{p}\pi_{aq})O^{pq}_{ik} =$$

$$= \frac{1}{n-1}[({}^{*}\overline{R}^{a}{}_{q}\pi_{ap} + {}^{*}\overline{R}^{a}{}_{p}\pi_{aq}) - ({}^{*}R^{a}{}_{q}\pi_{ap} + {}^{*}R^{a}{}_{p}\pi_{aq})]O^{pq}_{ik}.$$

Substituting this into (4.4), we get

$$\theta^{a}_{k}\pi_{ai} = -\overline{S}_{ki} + S_{ki} \quad \text{and} \quad \theta^{i}_{k} = (-\overline{S}_{ka} + S_{ka})\tilde{\pi}^{ai}, \tag{4.5}$$

where

$$\bar{S}_{ki} = \frac{1}{2(n+1)}\left[{}^*\bar{R}^a{}_k\pi_{ai} + \frac{1}{n-1}({}^*\bar{R}^a{}_q\pi_{ap} + {}^*\bar{R}^a{}_p\pi_{aq})O^{pq}_{ik}\right], \tag{4.6}$$

$$S_{ki} = \frac{1}{2(n+1)}\left[{}^*R^a{}_k\pi_{ai} + \frac{1}{n-1}({}^*R^a{}_q\pi_{ap} + {}^*R^a{}_p\pi_{aq})O^{pq}_{ik}\right].$$

Substituting (4.5) into (4.1), we find

$$\begin{aligned} &\bar{R}^i{}_{rkj} - \delta^i_r(\bar{S}_{kj} - \bar{S}_{jk}) + \omega f^i{}_r(f^a{}_k\bar{S}_{ja} - f^a_j\bar{S}_{ka}) + \\ &+ (\pi_{kr}\bar{S}_{ja} - \pi_{jr}\bar{S}_{ka})\tilde{\pi}^{ai} + \omega f^i_p\tilde{\pi}^{pb}\pi_{ar}(f^a_k\bar{S}_{jb} - f^a_j\bar{S}_{kb}) = \\ &= R^i{}_{rkj} - \delta^i_r(S_{kj} - S_{jk}) + \omega f^i_r(f^a{}_kS_{ja} - f^a_jS_{ka}) + \\ &+ (\pi_{kr}S_{ja} - \pi_{jr}S_{ka})\tilde{\pi}^{ai} + \omega f^i_p\tilde{\pi}^{pb}\pi_{ar}(f^a{}_kS_{jb} - f^a_jS_{kb}). \end{aligned} \tag{4.7}$$

Thus we get this

Theorem. *The tensor*

$$\begin{aligned} \underset{2}{P}{}^i_{rkj} &= R^i{}_{rkj} - \delta^i_r(S_{kj} - S_{jk}) + \omega f^i_r(f^a{}_kS_{ja} - f^a_jS_{ka}) + \\ &+ (\pi_{kr}S_{ja} - \pi_{jr}S_{ka})\tilde{\pi}^{ai} + \omega f^i_p\tilde{\pi}^{pb}\pi_{ar}(f^a{}_kS_{jb} - f^a_jS_{kb}). \end{aligned} \tag{4.8}$$

is invariant with respect to the πH-projective transformations (2.7).

We obtained the tensors (3.15) and (4.8) starting from the same relation (3.5). Thus, it is natural to ask: are they two different tensors? To answer the question, we shall discuss the case of a special space, namely the space satisfying $R^a{}_{akj}=0$. Then

$$\underset{2}{P}{}^a{}_{akj} = {}^*R^a{}_j\pi_{ak} - {}^*R^a{}_k\pi_{aj} = (R^a{}_{pjq}\pi_{ak} - R^a{}_{pkq}\pi_{aj})\tilde{\pi}^{pq}.$$

On the other hand, we find from (3.15) that

$$\underset{1}{P}{}^a{}_{akj} = R_{kj} - R_{jk} + \frac{1}{2(n+1)}\omega(R^t{}_{skb}f^b_j - R^t{}_{sjb}f^b_k)f^s_t.$$

Thus

$$\underset{2}{P}{}^a{}_{akj} \neq \underset{1}{P}{}^a{}_{akj}.$$

But if

$$\underset{2}{P}{}^i{}_{rkj} = \underset{1}{P}{}^i{}_{rkj}$$

it would be

$$\underset{2}{P}{}^a{}_{akj} = \underset{1}{P}{}^a{}_{akj}.$$

This proves that $\underset{1}{P}$ and $\underset{2}{P}$ are two different tensors.

We shall call the tensor (4.8) the second *πH-projective curvature tensor.*

The relation (4.7) can also be obtained in the following way.

Let $G^i{}_{rkj}$ indicate the components of the curvature tensor of the connection G. By straightforwar computation, using (2.3), we find

$$G^i{}_{rkj} = -R^b{}_{akj}\pi_{br}\tilde{\pi}^{ai}, \tag{4.9}$$

from which

(4.10) $$G_{rk} = G^a{}_{rka} = -{}^*R^b{}_k\pi_{br}.$$

Similarly

(4.11) $$\bar{G}^i{}_{rkj} = -\bar{R}^b{}_{akj}\pi_{br}\tilde{\pi}^{ai},$$

and

(4.12) $$\bar{G}_{rk} = -{}^*\bar{R}^b{}_k\pi_{br},$$

where $\bar{G}^i{}_{rkj}$ are the components of the curvature tensor of the connection $\bar{G}$.

On the other hand, we know ([2], [3], [4] p. 263, [5]) that two symmetric f-connections, $\bar{G}$ and G, which are related to each other as in (2.4), have the H-projective curvature tensor in common, that is

$$\bar{G}^i{}_{rkj} + \delta^i_r(\bar{P}_{kj} - \bar{P}_{jk}) + \omega f^i_r(f^a_j\bar{P}_{ka} - f^a{}_k\bar{P}_{ja}) +$$
$$+ \delta^i_j\bar{P}_{kr} - \delta^i_k\bar{P}_{jr} + \omega f^i_j f^a_r\bar{P}_{ka} - \omega f^i_k f^a{}_r\bar{P}_{ja} =$$
$$= G^i{}_{rkj} + \delta^i_r(P_{kj} - P_{jk}) + \omega f^i{}_r(f^a_j P_{ka} - f^a{}_k P_{ja}) +$$
$$+ \delta^i_j P_{kr} - \delta^i_k P_{jr} + \omega f^i_j f^a{}_r P_{ka} - \omega f^i_k f^a{}_r P_{ja},$$

where

$$P_{kr} = -\frac{1}{2(n+1)}\left[G_{rk} + \frac{1}{n-1}(G_{ab} + G_{ba})O^{ab}_{rk}\right],$$

$$\bar{P}_{kr} = -\frac{1}{2(n+1)}\left[\bar{G}_{rk} + \frac{1}{n-1}(\bar{G}_{ab} + \bar{G}_{ba})O^{ab}_{rk}\right].$$

(Let us remember that the dimension of our space is $2n$.)

Substituting from (4.9) and (4.11), we find.

$$R^b{}_{akj}\pi_{br}\tilde{\pi}^{ai} - \delta^i_r(P_{kj} - P_{jk}) - \omega f^i{}_r(f^a_j P_{ka} - f^a{}_k P_{ja}) -$$
$$- \delta^i_j P_{kr} + \delta^i_k P_{jr} - \omega f^i_j f^a_r P_{ka} + \omega f^i_k f^a{}_r P_{ja} =$$
$$= \bar{R}^b{}_{akj}\pi_{br}\tilde{\pi}^{ai} - \delta^i_r(\bar{P}_{kj} - \bar{P}_{jk}) - \omega f^i{}_r(f^a_j\bar{P}_{ka} - f^a{}_k\bar{P}_{ja}) -$$
$$- \delta^i_j\bar{P}_{kr} + \delta^i_k\bar{P}_{jr} - \omega f^i_j f^a_r\bar{P}_{ka} + \omega f^i_k f^a{}_r\bar{P}_{ja},$$

or

$$R^i{}_{rkj} - \delta^i_r(P_{kj} - P_{jk}) - \omega f^i{}_r(f^a_j P_{ka} - f^a{}_k P_{ja}) -$$

(4.13) $$- \pi_{jr}\tilde{\pi}^{ai}P_{ka} + \pi_{kr}\tilde{\pi}^{ai}P_{ja} - \omega f^a_j\pi_{ar}f^b_p\tilde{\pi}^{pi}P_{kb} + \omega f^a{}_k\pi_{ar}f^b_p\tilde{\pi}^{pi}P_{jb} =$$

$$= \bar{R}^i{}_{rkj} - \delta^i_r(\bar{P}_{kj} - \bar{P}_{jk}) - \omega f^i_r(f^a_j\bar{P}_{ka} - f^a{}_k\bar{P}_{ja}) -$$
$$- \pi_{jr}\tilde{\pi}^{ai}\bar{P}_{ka} + \pi_{kr}\tilde{\pi}^{ai}\bar{P}_{ja} - \omega f^a_j\pi_{ar}f^b_p\tilde{\pi}^{pi}\bar{P}_{kb} + \omega f^a{}_k\pi_{ar}f^b_p\tilde{\pi}^{pi}\bar{P}_{jb}.$$

As for the tensors P_{rk} and $\bar{P}_{rk}$ they can be, taking into account (4.10) and (4.12), expressed as follows

$$P_{kr} = \frac{1}{2(n+1)}\left[{}^*R^b{}_k\pi_{br} + \frac{1}{n-1}({}^*R^s{}_a\pi_{sb} + {}^*R^s{}_b\pi_{sa})O^{ab}_{rk}\right],$$

$$\bar{P}_{kr} = \frac{1}{2(n+1)}\left[{}^*\bar{R}^b{}_k\pi_{br} + \frac{1}{n-1}({}^*\bar{R}^s{}_a\pi_{sb} + {}^*\bar{R}^s{}_b\pi_{sa})O^{ab}_{rk}\right].$$

Now, it can be seen at once that (4.13) is the same relation as (4.7).

References

[1] S. Ishihara, Holomorphically projective changes and their groups in an almost complex manifold, *Tôhoku Math. J.* **9** (1959), 273—297.
[2] T. Otsuki and Y. Tashiro, On curves in Kählerian spaces, *Math. J. Okayama Univ.* **4** (1954), 57—78.
[3] Y. Tashiro, On holomorphically projective correspondences in an almost complex space, *Math. J. Okayama Univ.* **6** (1957), 147—152.
[4] K. Yano, Differential geometry on complex and almost complex spaces, *Pergamon Press*, 1965.
[5] M. Prvanović, Holomorphically projective transformations in a locally product space, *Mathematica Balkanica* **1** (1971), 195—213.
[6] Г. Марков—А. П. Норден, О голоморфно—проективных преобразованиях, *Изв. вузов, Матем.*, 1975, Но. **6**, 82—87
[7] K. Radziszewski, π-Geodesics and lines of shadow, *Colloquium Mathematicum* **26** (1972), 157—163.
[8] A. Bucki, π-Geodesics on hypersurfaces, *Ann. Univ. Mariae Curie-Sklodowska, Sec. A* **33** (1979), 29—44.

(Received April 22, 1987)