On Darboux points and the property of Świątkowski of transformations with closed graphs

By HELENA PAWLAK and RYSZARD JERZY PAWLAK (Łódź)

Abstract. In this paper conditions are presented under which a function with closed graph is continuous, or possesses Darboux property.

It is known, that if Y is a Hausdorff space and $f: X \to Y$ is a continuous function, then the graph of f $W(f)=\{(x, y): x \in X \wedge y=f(x)\}$ is a closed subset of $X \times Y$ (see, for example [2, Corollary 2.3.22, p. 114]). Since the inverse theorem is not true, many authors have required additional assumptions under which a function with closed graph is continuous ([6, Theorem 1], [5, Theorem 4]) or possesses properties "near" to continuity (see, for example [1], [5], [6], [7]). Analogous problems where studied for other classes of functions connected with closedness (see, [4], [9], [10], [13]).

In this paper we shall consider similar problems and whenever possible, we shall use assumptions weaker than the closedness of the graph.

We shall use the standard notations and terminology as for example in [2] and [11]. To specify some terms we remind of some definitions and symbols. An arc with endpoints a, b we denote by $L(a, b)$. For $Ł=L(a, b)$, by $Ł^0$ we understand the open arc $Ł \setminus \{a, b\}$. Let $Ł$ be an arc and $a, b \in Ł$, then the symbol $L_Ł(a, b)$ denotes the arc with the endpoints a and b contained in $Ł$. We assume the definitions of cutting and quasi-cutting as in [11]. Moreover, we say that *a set A arcwise cuts a topological space X between the points x, y, if A is open and A quasi-cuts X between x, y in this way that there exists an* arc $Ł=L(x, y)$ *such that* $Ł^0 \subset A$.

The symbol $C(f, A)$ denotes the set $f(A \cap C_f)$, where by C_f we understand the set of all continuity points of f. The cluster set of f at x_0 we denote by $L(f, x_0)$ (i.e. $\alpha \in L(f, x_0)$ if there exists a net $\{x_\sigma\}_{\sigma \in \Sigma}$, such that $x_0 \in \lim_{\sigma \in \Sigma} x_\sigma$ and $\alpha \in \lim_{\sigma \in \Sigma} f(x_\sigma)$, if we additionally require $\{x_\sigma\}_{\sigma \in \Sigma} \subset A$ then we write $\alpha \in L_A(f, x_0)$. Let A be some set. Put $L(f, A)=\bigcup_{x \in A} L(f, x)$.

We say that a function $f: X \to Y$ has an $Ł$-closed graph (at x_0) if the restriction $f_{|Ł}$ possesses a closed graph, for every arc $Ł$ $(Ł=L(x_0, t))$.

In 1977 T. MAŃK and T. ŚWIĄTKOWSKI defined a new class of functions (see [8]). The elements of this class we shall call Świątkowski functions or the functions with the property of Świątkowski. In the original definition of Świątkowski function the natural order of the real line is used, in topological terms this definition may be formulated as follows:

We say that $f\colon R\to R$ possesses the property of Świątkowski if for every $x, y\in R$ such that $f(x)\neq f(y)$ and for every arc $Ł=L(x, y)$ and for every set A such that A arcwise cuts R between $f(x)$ and $f(y)$, $A\cap C(f, Ł^0)\neq\emptyset$.

Now, it is not difficult to generalize the definition of Świątkowski function to the case of a transformation from a topological space to a topological space. In our considerations it is sufficient to adopt a weaker definition:

Definition. We say, that a function $f\colon X\to Y$, where X, Y are topological spaces, possesses the weak property of Świątkowski if for every $x, y\in X$ such that $f(x)\neq f(y)$ and for every arc $Ł=L(x, y)$ and for every set A such that A arcwise cuts Y between $f(x)$ and $f(y)$, $A\cap L(f, Ł^0)\neq\emptyset$.

The definitions of Darboux point and D-point are the same as in [10], [11], [12].

The symbol $\overline{T}$ denotes the closure of T in some topological space X. If A is a subspace of X and $T\subset A$ then the closure of T in A (as a subspace), we denote by $\mathrm{cl}_A\, T$.

By a Fréchet space we understand a topological space X such that for every $A\subset X$ if $x\in\overline{A}$ then there exists a sequence $\{x_n\}_{n=1}^{\infty}\subset A$ such that $x=\lim\limits_{n\to\infty} x_n$ (there exist Fréchet spaces that are not first-countable, see [2, Example 1.6.18, p. 79]. A Hausdorff space X is called a Mazurkiewicz—Moore space if every open and connected set in X is arcwise connected (for example, every metric, locally connected and complete space is a Mazurkiewicz—Moore space, see [3, Theorem 13.5.17, p. 223]). A Hausdorff space X is called an m-dimensional almost-manifold if for every $x\in X$ there exists a neighbourhood of x which is a finite union of the subspaces of X homeomorphic with R^m and including x. A Hausdorff space X^* is called an m-dimensional almost-manifold with boundary, if X^* contains an open and dense subspace X which is an m-dimensional almost-manifold. Then $X^*\setminus X$ is called the boundary of X^*. We say that the boundary is discrete if it is a discrete subspace of X^*.

The fundamental theorems of this paper will be preceded by two lemmas.

Lemma 1. *Every m-dimensional almost-manifold X is a Fréchet space.*

Proof. Let $x_0\in\overline{A}\subset X$ and let U_0 be a neighbourhood of x_0 such that $U_0=\bigcup\limits_{k=1}^{n} U_k$, where U_k is homeomorphic with R^m and $x_0\in U_k$ for $k=1, \dots, n$. We can assume that $x_0\notin A$. Thus there exists $k\in\{1, \dots, n\}$ such that $x_0\in\overline{A\cap U_k}$ and so $x_0\in\mathrm{cl}_{U_k}(A\cap U_k)$. Consequently, there exists a sequence $\{x_n\}\in A\cap U_k$ which is convergent to x_0 in U_k as a subspace of X. It is easy to see that $x_0=\lim\limits_{n\to\infty} x_n$ in X.

It is not difficult to give an example of an almost-manifold which is not first countable.

Lemma 2. *Let X be an m-dimensional almost-manifold and let $\{x_n\}$ be a sequence converging to x_0. Then there exists an arc $Ł$ with endpoint x_0 such that the set of elements of $\{x_n\}$ belonging to $Ł$ is infinite.*

Proof. Let V be a neighbourhood of x_0 such that V is a finite union of subspaces homeomorphic to R^m, containing x_0. Of course, Lemma 2 is true in the case if $\{x_n\}$ contains a constant subsequence, and so we consider the opposite case. Let V^* be an element of the above union such that V^* includes some subsequence $\{x_{k_n}\}$ of

$\{x_n\}$ and let $h\colon V^*\to R^m$ be a homeomorphism. Put $v^k=h(v)$ for every $v\in V^*$. We may assume that if $n_1<n_2$ then $\varrho(x^h_{k_{n_1}}, x^h_0)>\varrho(x^h_{k_{n_2}}, x^h_0)$.

For $m=1$ we moreover assume that all elements of $\{x^h_{k_n}\}$ lie on the same side of x^h_0.

Let r_{k_n} be a positive integer such that $x^h_{k_n}\notin\overline{K(x^h_0, r_{k_n})}$ and $x^h_{k_{n+1}}\in K(x^h_0, r_{k_n})$ $(n=1, 2, \ldots)$.

By the symbol $Ł_0$ we denote the segment joining $x^h_{k_1}$ with the sphere $\operatorname{Fr} K(x^h_0, r_{k_1})$ of length equal to $\operatorname{dist}\,(x^h_{k_1}, K(x^h_0, r_{k_1}))$. Thus $Ł_0=L(x^h_{k_1}, z_{k_1})$.

Let $Ł^1=L(x^h_{k_2}, z'_{k_1})$ be the segment joining $x^h_{k_2}$ with the sphere $\operatorname{Fr} K(x^h_0, r_{k_1})$ of length equal to $\operatorname{dist}\,(x^h_{k_2}, \operatorname{Fr} K(x^h_0, r_{k_1}))$. Let $Ł^2=L(x^h_{k_2}, z_{k_2})$ be the segment joining $x^h_{k_2}$ with the sphere $\operatorname{Fr} K(x^h_0, r_{k_2})$ of length equal to $\operatorname{dist}\,(x^h_{k_2}, K(x^h_0, r_{k_2}))$. Moreover, let $Ł^3=\{z_{k_1}\}$ if $z_{k_1}=z'_{k_1}$ and $Ł^3=L(z_{k_1}, z'_{k_1})\subset\operatorname{Fr} K(x^h_0, r_{k_1})$ if $z_{k_1}\neq z'_{k_1}$. Put $Ł_1=Ł^1\cup Ł^2\cup Ł^3$. Then $Ł_1$ is an arc such that $Ł_1\cap Ł_0=\{z_{k_1}\}$.

Analogously we construct arcs $Ł_2, Ł_3, \ldots$ such that $x^h_{k_n}\in Ł_{n-1}$, $Ł_n\cap Ł_{n-1}=\{p\}$, where p is the endpoint of $Ł_n$ and $Ł_{n-1}$, and $Ł_n\subset\overline{K(x^h_0, r_{k_n})}$ for $n=1, 2, \ldots$. It is easy to see that $Ł_h=\{x^h_0\}\cup\bigcup_{n=0}^{\infty}Ł_n$ is an arc and $Ł=h^{-1}(Ł_h)$ is an arc fulfilling the required conditions.

Now, we present Proposition 3 which gives the connection between continuity at some point and the Darboux property at this point of a transformation with a closed graph. Moreover, *this Proposition is a preparatory step forwards Theorem 4.*

Proposition 3. *Let X be an m-dimensional almost-manifold, Y a locally compact and connected space. Let $f\colon X\to Y$ possess an Ł-closed graph at $x_0\in X$. Then the following conditions are equivalent:*

(i) *x_0 is a continuity point of f,*
(ii) *x_0 is a Darboux point of the first kind of f,*
(iii) *x_0 is a Darboux point of the second kind of f,*
(iv) *x_0 is a Darboux point of the third kind of f,*
(v) *x_0 is a D-point of f.*

PROOF. According to the considerations contained in [12] (see also [10], [11]) it is easy to see that:

(iv)$\Leftarrow$(v)$\Leftarrow$(i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv).

To prove this Proposition it is sufficient to show the following implication: (iv)$\Rightarrow$(i).

Let $f(x_0)=y_0$ and let T be a neighbourhood of y_0 such that $\overline{T}$ is a compact set. Assume the contrary i.e. that f is not continuous at x_0. According to Lemma 1 there exists a sequence $\{x_n\}_{n=1}^{\infty}$ and a neighbourhood $U\subset T$ of y_0 such that $x_0=\lim_{n\to\infty}x_n$ and $f(x_n)\notin\overline{U}$ for $n=1, 2, \ldots$. By Lemma 2 we infer that there exists an arc $Ł_0=$ $=L(x_0, b)$ containing all elements of some subsequence of $\{x_n\}_{n=1}^{\infty}$. To simplify, we may assume that $\{x_n\colon n=1, 2, \ldots\}\subset Ł_0$. Let V be a neighbourhood of y_0, such that $\overline{V}\subset U$ and let $\mathscr{W}$ be the family of the neighbourhoods W of $\overline{V}$ such that $W\subset U$. By the connectedness of Y and the supposition that x_0 is a Darboux point of the third kind, we have

$$(1)\qquad f^{-1}(\operatorname{Fr} W)\cap L_{Ł_0}(x_0, x_n)\neq\emptyset\quad\text{for}\quad W\in\mathscr{W}\quad\text{and}\quad n=1, 2, \ldots.$$

Let $\Sigma=\{(\operatorname{Fr} W, n): W\in\mathscr{W}\wedge n\in N\}$. In Σ we define the directedness relation $\rightarrow$ as follows

$$(\operatorname{Fr} W_1, n_1) \rightarrow (\operatorname{Fr} W_2, n_2) \Leftrightarrow W_2 \subset W_1 \quad n_2 \geqq n_1.$$

Let for every $\sigma=(\operatorname{Fr} W, n)\in\Sigma$, be y_σ an arbitrary element of the intersection $f^{-1}(\operatorname{Fr} W)\cap L_{Ł_0}(x_0, x_n)$ (such an element exists according to (1)). Of course $x_0=\lim_{\sigma\in\Sigma} y_\sigma$. Let $\{z_{\sigma'}\}_{\sigma'\in\Sigma'}$ be a subnet of $\{y_\sigma\}_{\sigma\in\Sigma}$ such that $\{f(z_{\sigma'})\}_{\sigma'\in\Sigma'}$ converges to some z. It is not difficult to see that $z\in\operatorname{Fr} V$. Since $f_{|Ł_0}$ possesses a closed graph and $\{(z_{\sigma'}, f(z_{\sigma'}))\}$ converges to (x_0, z) we have $z=y_0$. This contradiction ends the proof.

Theorem 4. *Let X^* be an m-dimensional almost-manifold with a discrete boundary. Let Y be an arcwise connected and locally compact T_σ-space. If $f\colon X^*\to Y$ possesses an Ł-closed graph and the weak property of Świątkowski then:*

(a) *x_0 is a continuity point for every x_0 belonging to the complement of the boundary of X^**

and

(b) *x_0 is a Darboux point of first kind for every x_0 belonging to the boundary of X^*.*

Proof. Let X be an open almost-manifold such that $\overline{X}=X^*$ and $X^*\setminus X$ is discrete.

Let $x_0\in X^*$. We shall show that x_0 is a Darboux point of first kind. Let $Ł=L(x_0, y_0)$ be some arc with the endpoint x_0. Of course, it fulfils Condition 3° of the Definition of a Darboux point of the first kind (see [10], [11], [12]). Now, we shall show that Condition 1° of this definition also takes place. Assume the contrary, i.e. that there exists $p\in Ł$ such that $\overline{f(L_Ł(x_0, p))}=Y\neq f(L_Ł(x_0, p))$. Denote $\hat{Ł}=L_Ł(x_0, p)$. Let $y\in Y\setminus f(\hat{Ł})$ and let $\{y_\sigma\}_{\sigma\in\Sigma}\subset f(\hat{Ł})$ be a net such that $y=\lim_{\sigma\in\Sigma} y_\sigma$. Let x_σ be an element of $\hat{Ł}$ such that $f(x_\sigma)=y_\sigma$ (for $\sigma\in\Sigma$) and let $\{z_{\sigma'}\}_{\sigma'\in\Sigma'}$ be a subnet of $\{x_\sigma\}_{\sigma\in\Sigma}$ such that it is convergent to some $x^0\in\hat{Ł}$. Then $f(x^0)=y$. The obtained contradiction ends the proof of Condition 1°.

Now, we shall establish Condition 2° of the Definition of a Darboux point of the first kind. Let K be a set such that, for some net $\{x_\sigma\}_{\sigma\in\Sigma}\subset Ł$ such that $x_0=\lim_{\sigma\in\Sigma} x_\sigma$, K quasi-cuts the set $A=f(Ł)\cup\operatorname{acp}_{\sigma\in\Sigma} f(x_\sigma)$ (where $\operatorname{acp}_{\sigma\in\Sigma} f(x_\sigma)$ denotes the set of all the accumulation points of $\{f(x_\sigma)\}$) between the sets $\{f(x_0)\}$ and $\{f(x_\sigma)\colon \sigma\in\Sigma\}\cup\operatorname{acp}_{\sigma\in\Sigma} f(x_\sigma)$.

Put

$$A\setminus K = U\cup V, \tag{1}$$

where U, V are separated sets, $f(x_0)\in U$ and $\{f(x_\sigma)\colon \sigma\in\Sigma\}\cup\operatorname{acp}_{\sigma\in\Sigma} f(x_\sigma)\subset V$. Let $\sigma_0\in\Sigma$ and $Ł^*=L_Ł(x_0, x_{\sigma_0})$. It is sufficient to consider the case if $Ł^*\subset X\cup\{x_0\}$. We shall show that

$$K\cap f(Ł^*) \neq \emptyset. \tag{2}$$

Suppose, on the contrary, that

$$K\cap f(Ł^*) = \emptyset. \tag{3}$$

First, we prove that:

(4) $P=f(Ł^*)\cap U$ and $Q=f(Ł^*)\cap V$ are closed sets.

Of course, these sets are separated. According to (1) and (3) we infer that:

$$f(Ł^*)=P\cup Q. \tag{5}$$

We prove the closedness of P, for Q the proof is similar. Assume, to the contrary that there exists $z\in\overline{P}\setminus P$. Let $\{z_\sigma\}_{\sigma\in\Sigma}\subset P$ be a net which is convergent to z. Since $P\subset f(Ł^*)$, we may infer that $z\in f(Ł^*)$ (see the proof of Condition 1°), which is impossible.

Let $\hat{Ł}$ be an arc joining $f(x_0)$ and $f(x_{\sigma_0})$. Let h: $[0, 1]\xrightarrow{\text{onto}}\hat{Ł}$ be a homeomorphism such that $h(0)=f(x_0)$ and $h(1)=f(x_{\sigma_0})$. Let $P_1=h^{-1}(P\cap\hat{Ł})$. Thus there exists $a'\in[0, 1)$ such that $(a', 1]\cap P_1=\emptyset$. Let $a_0=\sup P_1$, according to (4) $a_0\in P_1$. Put $a=h(a_0)$. Define $b_0=\inf\{x>a_0\colon x\in h^{-1}(Q\cap\hat{Ł})\}\in h^{-1}(Q\cap\hat{Ł})$ and put $b=h(b_0)$. Of course, $b_0>a_0$ and $L_{\hat{Ł}}(a, b)\cap P=\{a\}$, $L_{\hat{Ł}}(a, b)\cap Q=\{b\}$. Put $A_0=Y\setminus(P\cup Q)$. Then A_0 is an open set and A_0 quasi-cuts Y between a and b, moreover the arc $Ł'==L_{\hat{Ł}}(a, b)$ joining a and b fulfils the following condition: $Ł'^0\subset A_0$. Let a_1, b_1 be elements of $Ł^*$ such that $f(a_1)=a$ and $f(b_1)=b$. Thus, according to the weak property of Świątkowski, we have

$$\underset{\alpha\in A_0}{\exists}\ \underset{c\in L^0_{Ł^*}(a_1, b_1)}{\exists}\ \alpha\in L(f, c). \tag{6}$$

Of course, $c\in X$. Let T be a neighbourhood of α such that $\alpha\in T\subset\overline{T}\subset A_0$ and $\overline{T}$ is compact ([2], Theorem 3.32 p. 196). According to (6) and the Lemmas 1 and 2 we can prove that there exist an arc $Ł_*=L(c, t)$ and a sequence $\{c_n\}_{n=1}^{\infty}$ such that $\{c_n\colon n=1, 2, \ldots\}\subset Ł_*$, $\lim_{n\to\infty} c_n=c$ and $\{f(c_n)\colon n=1, 2, \ldots\}\subset T$. Let $\{t_\sigma\}_{\sigma\in\Sigma}$ be a subnet of $\{f(c_n)\}_{n=1}^{\infty}$ which converges to some $\alpha_0\in\overline{T}$ and let $\{d_\sigma\}_{\sigma\in\Sigma}$ be a subnet of $\{c_n\}_{n=1}^{\infty}$ such that $f(d_\sigma)=t_\sigma$ (for $\sigma\in\Sigma$). Since $f_{|Ł_*}$ has a closed graph and $\{d_\sigma\colon\sigma\in\Sigma\}\subset Ł_*$, we have $f(c)=\alpha_0$. This, according to (5), contradicts the fact that $\alpha_0\in A_0$. The obtained contradiction ends the proof of (2).

We have shown that an arbitrary point from X^* is a Darboux point of the first kind of f, which according to Proposition 3 ends the proof.

Remark. If we only assume that Y is arcwise connected and locally compact then of course, every point from the boundary of X^* is a Darboux point of the second kind. The remaining part of the theorem is the same as above.

It is easy to see that if f fulfils the conditions of Theorem 4 then not every point of some almost-manifold with boundary X^* is a continuity point of f. The following theorem holds:

Theorem 5. *Let X be a locally connected Mazurkiewicz—Moore space and let Y be a locally compact and arcwise connected space. Then, if in some neighbourhood of x_0 a function $f\colon X\to Y$ possesses a closed graph and the weak property of Świątkowski then f is continuous at x_0.*

Proof. For simplicity sake, we assume that f possesses a closed graph and the weak property of Świątkowski in the whole space.

Let $y_0=f(x_0)$, and let $B(x_0)$ be a base of X and x_0 consisting of the connected and open sets. Assume to the contrary that x_0 is not a continuity point of f. Thus there exists a neighbourhood T of y_0, such that $\overline{T}\neq Y$ and $\overline{T}$ is compact and moreover

$$(1) \qquad f(U)\setminus\overline{T}\neq\emptyset \quad \text{for every} \quad U\in B(x_0).$$

Let V be a neighbourhood of y_0 such that $\overline{V}\subset T$ (see [2, Th. 3.32 p. 196]).
Let $\mathscr{W}$ be the family of all neighbourhoods W of $\overline{V}$ such that $W\subset T$. Put $W^*=W\setminus\overline{V}$ for every $W\in\mathscr{W}$. We shall show that

$$(2) \qquad U\cap f^{-1}(W^*)\neq\emptyset \quad \text{for every} \quad U\in B(x_0) \quad \text{and} \quad W\in\mathscr{W}.$$

Let $U\in B(x_0)$, $W\in\mathscr{W}$. According to (1) there exists $z_0\in U$ such that $f(z_0)\in\in Y\setminus W=\hat{W}$. Let $Ł^*=L(x_0,z_0)\subset U$. In order to prove (2) it is sufficient to show that $f(Ł^*)\cap W^*\neq\emptyset$. Suppose on the contrary, that $f(Ł^*)\cap W^*=\emptyset$. Then $f(Ł^*)\subset\overline{V}\cup\hat{W}$ and so analogously to (4) in the proof of Theorem 4 we may prove that $f(Ł^*)\cap\overline{V}$ and $f(Ł^*)\cap\hat{W}$ are closed and as in the proof of Theorem 4 there exist $a\in f(Ł^*)\cap\overline{V}$ and $b\in f(Ł^*)\cap\hat{W}$ and an arc $Ł'=L(a,b)$ such that $Ł'^{\circ}\subset W^*$. Let a_1,b_1 be elements of $Ł^*$, such that $f(a_1)=a$ and $f(b_1)=b$. Therefore, according to the weak property of Świątkowski and the closedness of $W(f)$ we get in contradiction to the supposition that $f(Ł^*)\cap W^*=\emptyset$.

In the family $\Sigma=\{(U,W)\colon U\in B(x_0),\ W\in\mathscr{W}\}$ we define a directed relation $\leqq$ in the following way:

$$(U_1,W_1)\leqq(U_2,W_2)\Leftrightarrow U_1\supset U_2\wedge W_1\supset W_2.$$

For every $\sigma=(U,W)\in\Sigma$, let x_σ be an arbitrary element of the intersection $U\cap f^{-1}(W^*)$. It is easy to see that $\{x_\sigma\}_{\sigma\in\Sigma}$ tends to x_0. Let $\{d_{\sigma'}\}_{\sigma'\in\Sigma'}$ be a subnet of $\{x_\sigma\}_{\sigma\in\Sigma}$ such that $\{f(d_{\sigma'})\}_{\sigma'\in\Sigma'}$ tends to some $\alpha\in\operatorname{Fr}V$. By the closedness of the graph of f, we infer that $f(x_0)=\alpha$, which is impossible. The obtained contradiction ends the proof of this theorem.

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INSTITUTE OF MATHEMATICS
ŁÓDŹ UNIVERSITY
BANACHA 22, 90-238 ŁÓDŹ
POLAND

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