On realizations of Fourier series operators

By JOUKO TERVO (Jyväskylä)

Abstract. The paper deals with linear operators $L(x, D)$ defined by

(1) $$\big(L(x, D)\varphi\big)(x) = (2\pi)^{-n} \sum_{l\in \mathbf{Z}^n} L(x, l)\varphi_l e^{i(l,x)},$$

where φ lies in the space C_π^∞ of all smooth periodic functions, φ_l is the Fourier coefficient of φ and $L(\cdot, \cdot)$ is a mapping $\mathbf{R}^n\times\mathbf{Z}^n\to\mathbf{C}$ such that $L(\cdot, l)\in C_\pi^\infty$ and that

(2) $$\sup_{x\in W} |D_x^\alpha L(x, l)| \leqq C_\alpha(1+|l|^2)^{\mu_\alpha/2}.$$

Here W is the cube $\{x\in\mathbf{R}^n | x_j\in]-\pi, \pi[.\}$ The assumption (2) quarantees that $L(x, D)$ maps C_π^∞ continuously into C_π^∞.

In the case when $\mu_\alpha=\mu+\delta|\alpha|$ with $\delta<1$, the formal transpose $L'(x, D)$. $C_\pi^\infty\to C_\pi^\infty$ of $L(x, D)$ exists. Denote by $L_{p,k}^{\sim}$ (and $L_{p,k}^{\prime\#}$) the minimal realization (the maximal realization, resp.) of $L(x, D)$ in the appropriate weighted spaces $\mathscr{B}_{p,k}^{\pi}$. Conditions for the bijectivity of $L_{p,k}^{\sim}+aI$ and for the essential maximality of $L(x, D)$, that is, for the equality of operators $L_{p,k}^{\sim}$ and $L_{p,k}^{\prime\#}$ are established.

1. Introduction

Let $L(x, D)$ be a linear operator defined on the space C_π^∞ of all smooth periodic functions $\varphi\colon \mathbf{R}^n\to\mathbf{C}$ by the requirement

(1.1) $$\big(L(x, D)\varphi\big)(x) = (2\pi)^{-n} \sum_{l\in \mathbf{Z}^n} L(x, l)\varphi_l e^{i(l,x)}.$$

Here $L(\cdot, \cdot)\colon \mathbf{R}^n\times\mathbf{Z}^n\to\mathbf{C}$ is a mapping such that $L(\cdot, l)\in C_\pi^\infty$ for $l\in\mathbf{Z}^n$ and that with some constants $C_\alpha>0$ and $\mu_\alpha\in\mathbf{R}$ one has

(1.2) $$\sup_{x\in W} |D_x^\alpha L(x, l)| \leqq C_\alpha(1+|l|^2)^{\mu_\alpha/2} =: C_\alpha k_{\mu_\alpha}(l) \quad \text{for} \quad l\in\mathbf{Z}^n.$$

The constant $\varphi_l:=\int_W \varphi(x)e^{-i(l,x)}\,dx$ is the Fourier coefficient of φ. Every linear partial differential operator with C_π^∞-coefficients can be presented in the form (1.1). For the theory of periodic partial differential operators cf. [3], [4], [2], [1] and [8]. In [11] it is showed some boundedness and compactness criteria for the operators (1.1) in the appropriate spaces and in [10] one-sided invertibility of the operators (1.1) is considered. For the related topic we also refer to [7].

The operators (1.1) for which (1.2) is valid form an algebra (cf. Corollary 2.2). In the case when μ_α is of the form $\mu_\alpha=\mu+\delta|\alpha|$ with $\delta<1$, the formal transpose

$L'(x, D)$ of $L(x, D)$ exists and $L'(x, D)$ is of the form (1.1) (cf. Remark 2.6 and Theorem 2.8). Supposing that $L'(x, D)$ exists we can construct the minimal and maximal realizations, say $L_{p,k}^{\sim}$ and $L_{p,k}'^{\#}$, of $L(x, D)$ in the weighted spaces $\mathscr{B}_{p,k}^{\pi}$ (cf. Section 2.4). In the case when $L(D):=L(x, D)$ is a translation invariant operator, one has $L_{p,k}^{\sim}=L_{p,k}'^{\#}$ (cf. Theorem 3.1). The spectrum $\sigma(L_{p,k}^{\sim})$ of the minimal realization of the translation invariant operator $L(D)$ is given by $\sigma(L_{p,k}^{\sim})=\overline{\{L(l)|l\in \mathbf{Z}^n\}}$ (cf. Theorem 3.4). This fact implies a criterion for the bijectivity of $L_{p,k}^{\sim}+aI$ when a is large enough.

In the fourth chapter we pertuebate the translation invariant operator and obtain a condition for the bijectivity of $L_{p,k}^{\sim}+aI$, where $L_{p,k}^{\sim}$ is the minimal realization of the perturbed operator $L(x, D)$. A condition for the essential maximality, that is, for the equality $L_{p,k}^{\sim}=L_{p,k}'^{\#}$, is also established. The fifth chapter considers operators (1.1), which are semi-bounded (from below) in appropriate spaces. Criterions for the bijectivity of $L_{p,k}^{\sim}+aI$ and for the equality $L_{p,k}^{\sim}=L_{p,k}'^{\#}$ are proved. Our theory especially implies that for elliptic periodic partial differential operators one has $L_{2,k_t}^{\sim}=L_{2,k_t}'^{\#}$, $N(L_{2,k_t}'^{\#}+aI)=\{0\}$ and that $R(L_{2,k_t}^{\sim}+aI)=\mathscr{B}_{2,k_t}^{\pi}$ for a large enough, where t lies in $\mathbf{Z}$.

2. The algebra of Fourier series operators

2.1. Denote by C_π^∞ the linear space of all $C^\infty(\mathbf{R}^n)$-functions φ such that each function $x_j\to(x_1, \dots, x_j, \dots, x_n)$ is 2π-periodic. Eguip the space C_π^∞ with the (standard Frechet space) topology defined by the semi-norms $\varphi\to\sup_{x\in W}|(D^\sigma\varphi)(x)|$, $\sigma\in\mathbf{N}_0^n$. Here W denotes the cube $W:=\{x\in\mathbf{R}^n|x_j\in]-\pi, \pi[, \ j=1, \dots, n\}$. The dual of C_π^∞ is denoted by D_π', in other words, D_π' is the linear space of all periodic distributions. We recall that for $T\in D_\pi'$ and for $\varphi\in C_\pi^\infty$ one has

$$T(\varphi)=(2\pi)^{-n}\sum_{l\in\mathbf{Z}^n}T_l\varphi_{-l}, \tag{2.1}$$

where $T_h\in\mathbf{C}$ is defined by

$$T_l:=T(e^{-i(l,\cdot)}), \tag{2.2}$$

and where $\varphi_l\in\mathbf{C}$ is the Fourier coefficient of φ, that is,

$$\varphi_l:=\int_W\varphi(x)e^{-i(l,x)}\,dx=:\varphi(e^{-i(l,\cdot)}). \tag{2.3}$$

Furthermore, for each $T\in D_\pi'$ there exist constants $C>0$ and $s\in\mathbf{R}$ such that

$$|T_l|\le C(1+|l|^2)^{s/2}=:Ck_s(l)\quad\text{for all}\quad l\in\mathbf{Z}^n. \tag{2.4}$$

A distribution $T\in D_\pi'$ lies in C_π^∞ if and only if for each $N\in\mathbf{N}$ there exists a constant $C>0$ such that

$$|T_l|\le Ck_{-N}(l)\quad\text{for all}\quad l\in\mathbf{Z}^n. \tag{2.5}$$

The product $\psi T\in D_\pi'$ of $T\in D_\pi'$ and $\psi\in C_\pi^\infty$ is defined through the relation

$$(\psi T)(\varphi)=T(\psi\varphi)\quad\text{for}\quad\varphi\in C_\pi^\infty. \tag{2.6}$$

Then one has

(2.7) $$(\psi T)_l = (2\pi)^{-n} \sum_{z \in \mathbf{Z}^n} T_z \psi_{l-z}.$$

For the definition of Banach spaces $\mathscr{B}^{\pi}_{p,k}$ we refer to [11]. Here p lies in the interval $[1, \infty[$ and k belongs to an appropriate class $\mathscr{K}_\pi$ of weight functions $k\colon \mathbf{Z}^n \to \mathbf{R}$. We have the topological inclusions $C^\infty_\pi \subset \mathscr{B}^\pi_{p,k} \subset D'_\pi$, when D'_π is equipped with the weak dual topology. The relations

$$C^\infty_\pi = \bigcap_{k \in \mathscr{K}_\pi} \mathscr{B}^\pi_{p,k}$$

and

$$D'_\pi = \bigcup_{k \in \mathscr{K}_\pi} \mathscr{B}^\pi_{p,k}$$

are also valid. Let $p' \in [1, \infty]$ such that $1/p + 1/p' = 1$ and let $k^{\vee} \in \mathscr{K}_\pi$ such that $k^{\vee}(l) = k(-l)$. Denote by $\mathscr{B}^{\pi *}_{p,k}$, $p \in [1, \infty[$ the dual space of $\mathscr{B}^\pi_{p,k}$. Then for each $t \in \mathscr{B}^{\pi *}_{p,k}$ there exists an unique $T \in \mathscr{B}^\pi_{p',1/k^{\vee}}$ such that

(2.1) $$t\varphi = T(\varphi) \quad \text{for} \quad \varphi \in C^\infty_\pi.$$

On the other hand, the linear form

(2.9) $$\varphi \to T(\varphi), \; T \in \mathscr{B}^\pi_{p',1/k^{\vee}}$$

has an unique continuous extension from $\mathscr{B}^\pi_{p,k}$ to $\mathbf{C}$. Furthermore, the mapping $\mathscr{L}\colon \mathscr{B}^\pi_{p',1/k^{\vee}} \to \mathscr{B}^{\pi *}_{p,k}$ defined by

(2.10) $$\ell(T) = t$$

is an isometrical isomorphism. For some additional properties of $\mathscr{B}^\pi_{p,k}$-spaces we refer to [11], [10] and [7].

2.2. In the sequel $L(\cdot, \cdot)$ denotes a mapping $\mathbf{R}^n \times \mathbf{Z}^n \to \mathbf{C}$ such that

(2.11) $$L(\cdot, l) \in C^\infty_\pi \quad \text{for each} \quad l \in \mathbf{Z}^n.$$

We shall deal with linear operators $L(x, D)$ defined by the requirement

(2.12) $$\big(L(x, D)\varphi\big)(x) = (2\pi)^{-n} \sum_{l \in \mathbf{Z}^n} L(x, l)\varphi_l e^{i(l,x)} \quad \text{for} \quad \varphi \in C^\infty_\pi.$$

Supposing that with some $C > 0$ and $M \in \mathbf{R}$ one has

(2.13) $$\sup_{x \in W} |L(x, l)| \leqq C k_M(l) \quad \text{for all} \quad l \in \mathbf{Z}^n$$

one sees that $L(x, D)\varphi$ is a mapping $\mathbf{R}^n \to \mathbf{C}$, which, in addition is 2π-periodic with respect to each variable.

Furthermore, we have

Theorem 2.1. *Suppose that for each* $\alpha \in \mathbf{N}^n_0$ *there exist constants* $C_\alpha > 0$ *and* $\mu_\alpha \in \mathbf{R}$ *such that*

(2.14) $$\sup_{x \in W} |D^\alpha_x L(x, l)| \leqq C_\alpha k_{\mu_\alpha}(l) \quad \textit{for all} \quad l \in \mathbf{Z}^n.$$

Then the operator $L(x, D)$ *defined by* (2.12) *maps* C^∞_π *continuously into* C^∞_π.

On the contrary, suppose that a linear operator $L\colon C_\pi^\infty \to C_\pi^\infty$ *is continuous. Then there exists a mapping* $L(\cdot,\cdot)\colon \mathbf{R}^n\times\mathbf{Z}^n\to\mathbf{C}$ *such that* (2.11) *and* (2.14) *are valid and that one has*

$$L = L(x, D). \tag{2.15}$$

PROOF. A) The proof of the first part of the assertion is established in [11], Theorem 3.1; cf. also [10].

B) We show the last part of the assertion (for an alternative proof cf. also [10]). Let L be a continuous operator $C_\pi^\infty\to C_\pi^\infty$. Then for each $\alpha\in\mathbf{N}_0^n$ there exist constants $C_\alpha>0$ and $N_\alpha\in\mathbf{N}$ such that

$$\sup_{x\in W}|D^\alpha(L\varphi)(x)| \leqq C_\alpha \sum_{|\beta|\leqq N_\alpha}\sup_{x\in W}|(D^\beta\varphi)(x)| \quad \text{for all} \quad \varphi\in C_\pi^\infty \tag{2.16}$$

(cf. [12], p. 42). Define a mapping $L(\cdot,\cdot)\colon\mathbf{R}^n\times\mathbf{Z}^n\to\mathbf{C}$ by

$$L(x, l) = [L(e^{i(l,\cdot)})](x)e^{-i(l,x)}. \tag{2.17}$$

Then $L(\cdot, l)$ belongs to C_π^∞ for each $l\in\mathbf{Z}^n$ and by (2.16) one has

$$\begin{aligned} |D_x^\alpha L(x,l)| &= \left|\sum_{\gamma\leqq\alpha}\binom{\alpha}{\gamma} D^{\alpha-\gamma}\big(L(e^{i(l,\cdot)})\big)(x)D^\gamma(e^{-i(l,\cdot)})(x)\right| \leqq \\ &\leqq \sum_{\gamma\leqq\alpha}\binom{\alpha}{\gamma}C_{\alpha-\gamma}\sum_{|\beta|\leqq N_{\alpha-\gamma}}\sup_{x\in W}|D^\beta(e^{i(l,\cdot)})(x)|\,|(-l)^\gamma e^{i(l,x)}| \leqq \\ &\leqq \sum_{\gamma\leqq\alpha}\binom{\alpha}{\gamma}C_{\alpha-\gamma}\sum_{|\beta|\leqq N_{\alpha-\gamma}}|l^{\beta+\gamma}| \leqq \\ &\leqq \sum_{\gamma\leqq\alpha}\binom{\alpha}{\gamma}C_{\alpha-\gamma}\sum_{|\beta|\leqq N_{\alpha-\gamma}}k_{N_{\alpha-\gamma}+|\gamma|}(l) \leqq C'_\alpha k_{\mu_\alpha}(l) \end{aligned} \tag{2.18}$$

for all $l\in\mathbf{Z}^n$ and $x\in W$, where C'_α is a suitable constant and where $\mu_\alpha := \max_{\gamma\leqq\alpha}\{N_{\alpha-\gamma}\} + |\alpha|$. Hence the estimate (2.14) is valid.

We have to show that $L=L(x, D)$. Let $\varphi\in C_\pi^\infty$ and let $\varphi_n\in C_\pi^\infty$ such that

$$\varphi_n(x) = (2\pi)^{-n}\sum_{|l|\leqq n}\varphi_l e^{i(l,x)}$$

(that is, $(\varphi_n)_l=\varphi_l$ for $|l|\leqq n$ and $(\varphi_n)_l=0$ for $|l|>n$). Then one sees that $\varphi_n\to\varphi$ in C_π^∞. Hence the continuity of L implies that

$$L\varphi_n \to L\varphi \quad \text{in} \quad C_\pi^\infty. \tag{2.19}$$

Let $L(x, D)$ be the operator (2.12) with the symbol (2.17). In virtue of the first part of the assertion $L(x, D)\colon C_\pi^\infty\to C_\pi^\infty$ is continuous, as well. Hence one has

$$L(x, D)\varphi_n \to L(x, D)\varphi \quad \text{in} \quad C_\pi^\infty. \tag{2.20}$$

Noting that

$$\begin{aligned} (L\varphi_n)(x) &= \Big[(2\pi)^{-n}\sum_{|l|\leqq n}\varphi_l L(e^{i(l,\cdot)})\Big](x) = \\ &= (2\pi)^{-n}\sum_{|l|\leqq n}\big([L(e^{i(l,\cdot)})](x)e^{-i(l,x)}\big)\varphi_l e^{i(l,x)} = \big(L(x, D)\varphi_n\big)(x), \end{aligned} \tag{2.21}$$

one obtains $L\varphi_n = L(x, D)\varphi_n$. Hence by (2.19)—(2.20) one has $L\varphi = L(x, D)\varphi$, as required. □

Denote by S^π the linear space of all mappings $L(\cdot, \cdot)\colon \mathbf{R}^n \times \mathbf{Z}^n \to \mathbf{C}$ such that $L(\cdot, l) \in C_\pi^\infty$ for each $l \in \mathbf{Z}^n$ and that

$$\sup_{x \in W} |D_x^\alpha L(x, l)| \leqq C_\alpha k_{\mu_\alpha}(l) \quad \text{for all} \quad l \in \mathbf{Z}^n. \tag{2.14}$$

In virtue of Theorem 2.1 for each $L(\cdot, \cdot) \in S^\pi$ there exists the continuous linear operator $L(x, D)\colon C_\pi^\infty \to C_\pi^\infty$ defined by (2.12) and, in vice versa, every continuous linear operator $L\colon C_\pi^\infty \to C_\pi^\infty$ is of the form $L(x, D)$ with $L(\cdot, \cdot) \in S^\pi$. We denote by L^π the linear space $\{L(x, D) | L(\cdot, \cdot) \in S^\pi\}$ of linear operators $C_\pi^\infty \to C_\pi^\infty$. Then the mapping $\varkappa\colon S^\pi \to L^\pi$ such that

$$\varkappa(L(\cdot, \cdot)) = L(x, D)$$

is a well-defined linear bijection.

Let $\circ$ denote the composition of two operators $L(x, D)$ and $K(x, D) \in L^\pi$. Since (due to Theorem 2.1) L^π is exactly the linear space of all continuous linear operators $C_\pi^\infty \to C_\pi^\infty$ one has (for the definition of an algebra cf. [5], p. 140, for example)

Corollary 2.2. *The pair* $(L^\pi, \circ)$ *forms an algebra with unit.* □

Remark 2.3. Since one has

$$\varphi(x) = (2\pi)^{-n} \sum_{l \in \mathbf{Z}^n} \varphi_l e^{i(l, x)}, \tag{2.22}$$

the symbol $I(\cdot, \cdot)$ of the identical operator (which is the unit of the algebra $(L^\pi, \circ)$) is defined by $I(x, l) = 1$.

For the symbol $(L \circ K)(\cdot, \cdot)$ of the composed operator $L(x, D) \circ K(x, D)$ we obtain

Theorem 2.4. *Let* $L(\cdot, \cdot)$ *and* $K(\cdot, \cdot)$ *be in* S^π. *Then* $(L \circ K)(\cdot, \cdot)$ *has the expression*

$$(L \circ K)(x, l) = (2\pi)^{-n} \sum_{z \in \mathbf{Z}^n} L(x, z) \big(K(\cdot, l)\big)_{l-z} e^{i(l-z, x)}. \tag{2.23}$$

Proof. A) At first we verify that $(L \circ K)(\cdot, \cdot)$ lies in S^π. For each $\gamma \in \mathbf{N}_0^n$ one has with suitable constants $C_\gamma > 0$ and $\mu_\gamma \in \mathbf{R}$

$$\begin{aligned} &\big|(l-z)^\gamma \big(K(\cdot, l)\big)_{l-z}\big| = \big|\big(D_x^\gamma K(\cdot, l)\big)_{l-z}\big| \leqq \\ &\leqq \sup_{x \in W} |D_x^\gamma K(x, l)| \int_W 1\, dx = (2\pi)^n C'_\gamma k_{\mu'_\gamma}(l) \quad \text{for} \quad l, z \in \mathbf{Z}^n. \end{aligned} \tag{2.24}$$

Here we used the fact that by the periodicity the relation

$$l^\gamma \varphi_l = (D^\gamma \varphi)_l \tag{2.25}$$

holds. Due to (2.24) for each $m \in \mathbf{N}$ there exist constants $C_m > 0$ and $\mu_m \in \mathbf{R}$ such that

$$\big|\big(K(\cdot, l)\big)_{l-z}\big| \leqq C_m k_{\mu_m}(l) k_{-m}(z). \tag{2.26}$$

For each $\alpha \in \mathbf{N}_0^n$ we get

$$(2.27)\qquad \left|D_x^\alpha\big(L(\cdot, z)e^{i(l-z,\cdot)}\big)(x)\right| \leqq \sum_{v \leqq \alpha} \binom{\alpha}{v} |D_x^{\alpha-v} L(x, z)|\, |(l-z)^v| \leqq$$
$$\leqq \sum_{v \leqq \alpha} \binom{\alpha}{v} C_{\alpha-v} k_{\mu_{\alpha-v}}(z) k_{|v|}(l-z) \leqq$$
$$\leqq C_\alpha k_{\mu''_\alpha+|\alpha|}(z) k_{|\alpha|}(l),$$

where C_α is a suitable constant and where $\mu''_\alpha := \max_{v \leqq \alpha} \{\mu_{\alpha-v}\}$. Choosing in (2.26) $m = n+1+\mu''_\alpha+|\alpha|$ one sees by (2.27) that the series

$$\sum_{z \in \mathbf{Z}^n} D_x^\alpha\big(L(\cdot, z)e^{i(l,\cdot)}\big)(x)\big(K(\cdot, l)\big)_{l-z}$$

is absolutely and uniformly convergent in $\mathbf{R}^n$. Hence $D_x^\alpha(L \circ K)(x, l)$ exists, $D_x^\alpha(L \circ K)(\cdot, l)$ is continuous and

$$(2.28)\qquad D_x^\alpha(L \circ K)(x, l) = (2\pi)^{-n} \sum_{z \in \mathbf{Z}^n} D_x^\alpha\big(L(\cdot, z)e^{i(l,\cdot)}\big)(x)\big(K(\cdot, l)\big)_{l-z}.$$

Thus $(L \circ K)(\cdot, l) \in C_\pi^\infty$ and by (2.26)—(2.27) one has

$$\sup_{x \in W} |D_x^\alpha(L \circ K)(x, l)| \leqq C_\alpha C_m \Big(\sum_{z \in \mathbf{Z}^n} k_{-(n+1)}(z)\Big) k_{|\alpha|+\mu_m}(l)$$

where we chosed $m = n+1+\mu''_\alpha+|\alpha|$. Hence $(L \circ K)(\cdot, \cdot) \in S^\pi$.

B) Let $\varphi \in C_\pi^\infty$ and let $\varphi_n \in C_\pi^\infty$ be defined by $\varphi_n(x) = (2\pi)^{-n} \sum_{|l| \leqq n} \varphi_l e^{i(l,x)}$. Since $\varphi_n \to \varphi$ in C_π^∞ one has

$$(2.29)\qquad \big(L(x, D) \circ K(x, D)\big)\varphi_n \to L(x, D) \circ K(x, D)\varphi$$

and

$$(2.30)\qquad \big((L \circ K)(x, D)\big)\varphi_n \to (L \circ K)(x, D)\varphi.$$

To complete the proof it sufficies to note that

$$(2.31)\qquad \big(L(x, D) \circ K(x, D)\big)\varphi_n = (2\pi)^{-n} L(x, D)\Big(\sum_{|l|=n} K(\cdot, l)e^{i(l,\cdot)}\varphi_l\Big) =$$
$$= (2\pi)^{-n} \sum_{|l| \leqq n} \big[L(x, D)\big(K(\cdot, l)e^{i(l,\cdot)}\big)e^{-i(l,\cdot)}\big] \varphi_l e^{i(l,\cdot)} =$$
$$= (2\pi)^{-2n} \sum_{|l| \leqq n} \Big[\sum_{z \in \mathbf{Z}^n} L(x, z)\big(K(\cdot, l)e^{i(l,\cdot)}\big)_z e^{i(z-l,\cdot)}\Big] \varphi_l e^{i(l,\cdot)} =$$
$$= (2\pi)^{-2n} \sum_{|l| \leqq n} \Big[\sum_{z \in \mathbf{Z}^n} L(x, z)\big(K(\cdot, l)\big)_{z-l} e^{i(z-l,\cdot)}\Big] \varphi_l e^{i(l,\cdot)} =$$
$$= (L \circ K)\varphi_n. \quad \square$$

Remark 2.5. A) The symbol $(L \circ K)(\cdot, \cdot)$ can also be expressed in the form

$$(2.32)\qquad (L \circ K)(x, l) = (2\pi)^{-n} \sum_{z \in \mathbf{Z}^n} \Big(\int_W L(x, z)K(y, l)e^{i(z-l, x-y)}\, dy\Big).$$

B) Let S_δ^π; $\delta\in\mathbf{R}$ be the linear subspace of S^π such that $L(\cdot,\cdot)\in S_\delta^\pi$ if the estimate

$$\sup_{x\in W}|D_x^\alpha L(x,l)| < C_\alpha k_{\mu+\delta|\alpha|}(l) \quad \text{for all} \quad l\in\mathbf{Z}^n$$

holds. Equip the space S_δ^π with a locally convex topology defined by the semi-norms

$$L(\cdot,\cdot)\to\sup_{l\in\mathbf{Z}^n}\big(\sup_{x\in W}|D_x^\alpha L(x,l)|\big)/k_{\mu+\delta|\alpha|}(l),\ \alpha\in\mathbf{N}_0^n.$$

Since C_π^∞ is complete, one sees that S_δ^π is a Frechet space.

2.3. Let L be a linear operator $C_\pi^\infty\to C_\pi^\infty$. We say that the formal transpose of L exists, if one is able to find a linear operator $L'\colon C_\pi^\infty\to C_\pi^\infty$ such that

$$(2.33)\qquad (L\varphi)(\psi):=\int_W (L\varphi)(x)\psi(x)\,dx=\varphi(L'\psi) \quad \text{for all} \quad \varphi,\psi\in C_\pi^\infty.$$

The operator L' is the formal transpose of L. One sees that L' is unique and that $(L')'=L$.

Remark 2.6. Suppose that for each $\alpha\in\mathbf{N}_0^n$ there exists a constant $C_\alpha>0$ such that

$$(2.34)\qquad \sup_{x\in W}|D_x^\alpha L(x,l)|\le C_\alpha k_{\mu+\delta|\alpha|}(l) \quad \text{for} \quad l\in\mathbf{Z}^n,$$

where $\mu\in\mathbf{R}$ and where $\delta<1$. Then the formal transpose $L'(x,D)\colon C_\pi^\infty\to C_\pi^\infty$ of $L(x,D)$ exists (cf. [11]).

Lemma 2.7. *Let L be a linear operator* $C_\pi^\infty\to C_\pi^\infty$ *such that the formal transpose* $L'\colon C_\pi^\infty\to C_\pi^\infty$ *exists. Then L (and L′) is continuous.*

Proof. Since C_π^∞ is a Frechet space we have only to verify that $L\colon C_\pi^\infty\to C_\pi^\infty$ is closed (cf. the Closed Graph Theorem given e.g. in [12], p. 79). Let $\{\varphi_n\}\subset C_\pi^\infty$ be a sequence such that $\varphi_n\to\varphi$ and $L\varphi_n\to\psi$ in C_π^∞ with some $\varphi,\psi\in C_\pi^\infty$. Then one has for each $l\in\mathbf{N}_0^n$

$$(2.35)\qquad \psi_l=\psi(e^{-i(l,\cdot)})=\lim_{n\to\infty}(L\varphi_n)(e^{-i(l,\cdot)})$$

$$=\lim_{n\to\infty}\varphi_n\big(L'(e^{-i(l,\cdot)})\big)=\varphi\big(L'(e^{-i(l,\cdot)})\big)=(L\varphi)_l.$$

This shows that $L\varphi=\psi$ and then L is closed. □

Denote by $L^{t\pi}$ the linear subspace of L^π defined by

$$(2.36)\qquad L^{t\pi}=\{L\in L^\pi\,|\,\text{the formal transpose } L' \text{ exists}\}.$$

If $L\in L^{t\pi}$ then $L'\in L^{t\pi}$ (cf. Lemma 2.7). Furthermore, one has

Theorem 2.8. *The pair* $(L^{t\pi},\circ)$ *forms a subalgebra (with unit) of* $(L^\pi,\circ)$.

Proof. Let $L=L(x,D)$ and $K=K(x,D)$ be in $L^{t\pi}$. Then one has

$$(2.37)\qquad ((L\circ K)\varphi)(\psi)=(K\varphi)(L'\psi)=\varphi\big((K'\circ L')\psi\big) \quad \text{for} \quad \varphi,\psi\in C_\pi^\infty.$$

Hence $(L\circ K)'$ exists and $(L\circ K)'=K'\circ L'\in L^\pi$. Clearly I' exists and $I'=I\in L^\pi$. This completes the proof. □

Let $S^{t\pi}$ be a subclass of S^π defined by $S^{t\pi}=\varkappa^{-1}(L^{t\pi})$. We have

Theorem 2.9. *Let $L(\cdot,\cdot)$ be in $S^{t\pi}$. Then the symbol $L'(\cdot,\cdot)$ of $L'(x,D)$ has the expression*

$$L'(x,z)=(2\pi)^{-n}\sum_{l\in\mathbf{Z}^n}\big(L(\cdot,-(z+l))\big)_l e^{i(l,x)} \quad\text{for}\quad z\in\mathbf{Z}^n,\ x\in\mathbf{R}^n. \tag{2.38}$$

Proof. Since $L'(x,D)\in L^\pi$ there exists $L'(\cdot,\cdot)\ S^\pi$ such that

$$\big(L'(x,D)\psi\big)(x)=(2\pi)^{-n}\sum_{l\in\mathbf{Z}^n}L'(x,l)\psi_l e^{i(l,x)}$$

and the one has

$$L'(x,z)=[L'(x,D)(e^{i(z,\cdot)})](x)e^{-i(z,x)} \quad\text{for}\quad z\in\mathbf{Z}^n. \tag{2.39}$$

Furthermore, we obtain for each $z\in\mathbf{Z}^n$

$$\begin{aligned}\big(L'(\cdot,z)\big)_l&=\big(L'(\cdot,z)\big)(e^{-i(l,\cdot)})=\\&=\big([L'(x,D)(e^{i(z,\cdot)})]e^{-i(z,\cdot)}\big)(e^{-i(l,\cdot)})=\\&=(e^{i(z,\cdot)})\big(L(x,D)(e^{-i(z+l,\cdot)})\big)=\big(L(\cdot,-(z+l)\big)_l\end{aligned}$$

and then one has

$$\begin{aligned}L'(x,z)&=(2\pi)^{-n}\sum_{l\in\mathbf{Z}^n}\big(L'(\cdot,z)\big)_l e^{i(l,x)}=\\&=(2\pi)^{-n}\sum_{l\in\mathbf{Z}^n}\big(L(\cdot,-(z+l))\big)_l e^{i(l,x)},\end{aligned}$$

as required. □

Remark 2.10. In virtue of (2.40) the symbols $L(\cdot,\cdot)$ and $L'(\cdot,\cdot)$ satisfy the relation

$$\big(L'(\cdot,z)\big)_l=\big(L(\cdot,-(z+l))\big)_l \quad\text{for}\quad z,l\in\mathbf{Z}^n. \tag{2.40}$$

2.4. Let $L=L(x,D)$ be in $L^{t\pi}$. Then we are able to construct the linear operators $L'^{\#}_{p,k}$ and $L_{p,k}\colon \mathscr{B}^\pi_{p,k}\to\mathscr{B}^\pi_{p,k}$ as follows

$$\begin{cases}D(L'^{\#}_{p,k})=\{u\in\mathscr{B}^\pi_{p,k}\mid \exists f\in\mathscr{B}^\pi_{p,k}\text{ such that } u(L'\varphi)=f(\varphi)\text{ for }\varphi\in C^\infty_\pi\}\\ L'^{\#}_{p,k}u=f,\end{cases} \tag{2.41}$$

and

$$\begin{cases}D(L_{p,k})=C^\infty_\pi\\ L_{p,k}\varphi=L\varphi\quad\text{for}\quad\varphi\in C^\infty_\pi.\end{cases} \tag{2.42}$$

One sees that $L'^{\#}_{p,k}$ and $L_{p,k}$ are densily defined. Furthermore, $L'^{\#}_{p,k}$ is closed and $L_{p,k}$ is closable (cf. [12], pp. 77 and use (2.6) of [11]). Let $L^{\sim}_{p,k}\colon\mathscr{B}^\pi_{p,k}\to\mathscr{B}^\pi_{p,k}$ be the smallest closed extension of $L_{p,k}$. One always has $L^{\sim}_{p,k}\subset L'^{\#}_{p,k}$, that is, $L'^{\#}_{p,k}$ is the extension of $L^{\sim}_{p,k}$.

3. The translation invariant operator $L(D)$

3.1. Let $L(\cdot)\colon \mathbf{Z}^n \to \mathbf{C}$ be a mapping such that

$$|L(l)| \leqq Ck_\mu(l) \quad \text{for} \quad l \in \mathbf{Z}^n. \tag{3.1}$$

Then by Theorem 2.1, by Remark 2.6 and by Theorem 2.9 the operator $L(D)$ defined by

$$\big(L(D)\varphi\big)(x) = (2\pi)^{-n} \sum_{l \in \mathbf{Z}^n} L(l)\varphi_l e^{i(l,x)} \tag{3.2}$$

maps C_π^∞ continuously into C_π^∞, the formal transpose $L'(D)$ of $L(D)$ exists and $L'(1)=L(-1)$. In addition, we have for $p \in [1, \infty]$

Theorem 3.1. *Suppose that* $L(\cdot)$ *obeys* (3.1). *Then the identity*

$$L_{p,k}^{\sim} = L_{p,k}'^{\#} \tag{3.3}$$

holds.

Proof. Let u be in $D(L_{p,k}'^{\#})$ and let $L_{p,k}'^{\#}u=f$. Then for each $l \in \mathbf{Z}^n$ we get

$$\begin{aligned} f_l = (L_{p,k}'^{\#}u)(e^{-i(l,\cdot)}) &= u\big(L'(D)(e^{-i(l,\cdot)})\big) \\ &= u\big(L(l)e^{-i(l,\cdot)}\big) = L(l)u_l. \end{aligned} \tag{3.4}$$

Let $\{u_n\} \subset C_\pi^\infty$ be a sequence such that

$$u_n(x) = (2\pi)^{-n} \sum_{|l| \leqq n} u_l e^{i(l,x)}.$$

Let $p \in [1, \infty[$. Then one sees that $\|u_n - u\|_{p,k} = \big((2\pi)^{-n} \sum_{|l|>n} |u_l k(l)|^p\big)^{1/p} \to 0$ and that by (3.4)

$$\|L_{p,k}'^{\#}u_n - f\|_{p,k} = \big((2\pi)^{-n} \sum_{l \in \mathbf{Z}^n} \big|(L(l)(u_n)_l - f_l)k(l)\big|^p\big)^{1/p} = \big((2\pi)^{-n} \sum_{|l|>n} |f_l k(l)|^p\big)^{1/p} \to 0$$

with $n \to \infty$. Hence $u \in D(L_{p,k}^{\sim})$ and $L_{p,k}^{\sim}u=f$, as required. □

3.2. Let p be in the interval $[1, \infty[$ and let k and $k^{\sim}$ be in in $\mathscr{K}_\pi$. We have

Theorem 3.2. *Suppose that* $L(\cdot)$ *obeys* (3.1). *Then the inequality*

$$\|L(D)\varphi\|_{p,k} \geqq C_1 \|\varphi\|_{p,kk^{\sim}} - C_2 \|\varphi\|_{p,k} \quad \text{for all} \quad \varphi \in C_\pi^\infty \tag{3.5}$$

holds with $C_1 > 0$ *and* $C_2 \geqq 0$ *if and only if there exists a constant* $C > 0$ *such that*

$$k^{\sim}(l) \leqq C\big(1 + |L(l)|\big) \quad \textit{for all} \quad l \in \mathbf{Z}^n. \tag{3.6}$$

Proof. Suppose that (3.5) holds. For each $k \in \mathscr{K}_\pi$, $p \in [1, \infty[$ and $l \in \mathbf{Z}^n$ one has

$$\begin{aligned} \|L(D)(e^{i(l,\cdot)})\|_{p,k} &= \big((2\pi)^{-n} \sum_{z \in \mathbf{Z}^n} |(L(D)(e^{i(l,\cdot)}))_z k(z)|^p\big)^{1/p} = \\ &= \big((2\pi)^{-n} \sum_{z \in \mathbf{Z}^n} |L(l)(e^{i(l,\cdot)})_z k(z)|^p\big)^{1/p} = |L(l)|\, k(l) \end{aligned} \tag{3.7}$$

and

$$\|e^{i(l,\cdot)}\|_{p,k} = k(l).$$

Hence (3.6) is valid.

Multiplying (3.6) by $|\varphi_l k(l)|$ and taking into account that $(L(D)\varphi)_l = L(l)\varphi_l$ one sees easily that (3.6) implies (3.5). □

Suppose that $p=2$. Then the space $\mathscr{H}_k^\pi := \mathscr{B}_{2,k}^\pi$ is a Hilbert space with the inner product

$$(3.8)\qquad (u, v)_k = (2\pi)^{-n} \sum_{l\in\mathbf{Z}^n} u_l v_l k^2(l).$$

Similarly as Theorem 3.2 one has obviously

Theorem 3.3. *Suppose that* $L(\cdot)$ *obeys* (3.1). *Then the inequality*

$$(3.9)\qquad \operatorname{Re}(L(D)\varphi, \varphi)_k \geqq C_1\|\varphi\|_{kk^\sim}^2 - C_2\|\varphi\|_k^2 \quad \text{for} \quad \varphi\in C_\pi^\infty$$

holds with $C_1>0$ *and* $C_2\geqq 0$ *if and only if there exists a constant* $C>0$ *such that*

$$(3.10)\qquad k^\sim(l) \leqq C(\operatorname{Re} L(l)+1) \quad \text{for} \quad l\in\mathbf{Z}^n.$$

3.3. Denote by $\sigma(L_{p,k}^\sim)$ the spectrum of $L_{p,k}^\sim$. Let $L_{p,k}^{\sim *}: \mathscr{B}_{p,k}^{\pi *} \to \mathscr{B}_{p,k}^{\pi *}$ be the dual operator of $L_{p,k}^\sim$. Because $L_{p,k}^\sim$ is a closed operator, the scalar λ lies in the resolvent set $\varrho(L_{p,k}^\sim) := \mathbf{C} \setminus \sigma(L_{p,k}^\sim)$ of $L_{p,k}^\sim$ if and only if the relations

$$(3.11)\qquad N(L_{p,k}^\sim - \lambda I) = \{0\}$$

and

$$(3.12)\qquad R(L_{p,k}^\sim - \lambda I) = \mathscr{B}_{p,k}^\pi$$

hold (cf. [9], p. 14). We obtain for $p\in[1, \infty[$

Theorem 3.4. *Suppose that* $L(\cdot)$ *obeys* (3.1). *Then one has*

$$(3.13)\qquad \sigma(L_{p,k}^\sim) = \overline{\{L(l) \mid l\in\mathbf{Z}^n\}}.$$

Proof. A) Suppose that $\lambda\notin\sigma(L_{p,k}^\sim)$, that is, $\lambda\in\varrho(L_{p,k}^\sim)$. Then by (3.11)—(3.12) then inverse operator $(L_{p,k}^\sim - \lambda I)^{-1}: \mathscr{B}_{p,k}^\pi \to \mathscr{B}_{p,k}^\pi$ exists and $D((L_{p,k}^\sim - \lambda I)^{-1}) = \mathscr{B}_{p,k}^\pi$. Since the operator $(L_{p,k}^\sim - \lambda I)^{-1}$ is closed, the Closed Graph Theorem implies that $(L_{p,k}^\sim - \lambda I)^{-1}$ is bounded. Hence one has

$$(3.14)\qquad \|u\|_{p,k} \leqq \|(L_{p,k}^\sim - \lambda I)^{-1}\| \, \|(L_{p,k}^\sim - \lambda I)u\|_{p,k} \quad \text{for} \quad u\in D(L_{p,k}^\sim).$$

Choosing $u = e^{i(l,\cdot)}$ one obtains that

$$(3.15)\qquad |L(l) - \lambda| \geqq 1/\|(L_{p,k}^\sim - \lambda I)^{-1}\| \quad \text{for all} \quad l\in\mathbf{Z}^n,$$

and then $\lambda\notin\overline{\{L(l) \mid l\in\mathbf{Z}^n\}}$.

B) Conversely, suppose that $\lambda\notin\overline{\{L(l) \mid l\in\mathbf{Z}^n\}}$. Then there exists a constant $c>0$ such that

$$(3.16)\qquad |L(l) - \lambda| \geqq c \quad \text{for all} \quad l\in\mathbf{Z}^n.$$

Hence one has for all $\varphi\in C_\pi^\infty$

$$\big\|(L(D)-\lambda I)\varphi\big\|_{p,k} \geqq c\|\varphi\|_{p,k}, \tag{3.17}$$

which implies that $R(L_{p,k}^{\sim}-\lambda I)$ is closed and that

$$N(L_{p,k}^{\sim}-\lambda I) = \{0\}. \tag{3.18}$$

We establish that

$$N(L_{p,k}^{\sim *}-\lambda I) = \{0\}. \tag{3.19}$$

Choose U from $N(L_{p,k}^{\sim *}-\lambda I)\subset \mathscr{B}_{p,k}^{\pi *}$. Let $u\in\mathscr{B}_{p',1/k^{\vee}}^{\pi}$ be defined by $u=\mathscr{L}^{-1}(U)$ (cf. Section 2.1). Then one has for all $\varphi\in C_\pi^\infty$

$$u\big((L(D)-\lambda I)\varphi\big) = U\big((L(D)-\lambda I)\varphi\big) = U\big((L_{p,k}^{\sim}-I)\varphi\big) = 0 \tag{3.20}$$

and then $u\in N(L_{p',1/k^{\vee}}^{\#}-\lambda I)$. In virtue of Theorem 3.1 one has $L_{p',1/k^{\vee}}^{\#}=L'^{\sim}_{p',1/k^{\vee}}$. Due to (3.16) we also have

$$\big\|(L'(D)-\lambda I)\varphi\big\|_{p',1/k^{\vee}} \geqq c\|\varphi\|_{p',1/k^{\vee}}$$

and then $N(L_{p',1/k^{\vee}}^{\#}-\lambda I)=N(L'^{\sim}_{p',1/k^{\vee}}-\lambda I)=\{0\}$. This shows that $U=\mathscr{L}(u)=0$.

Since $R(L_{p,k}^{\sim}-\lambda I)$ is closed, the relation (3.19) implies that $R(L_{p,k}^{\sim}-\lambda I)= =\mathscr{B}_{p,k}^{\pi}$. Hence $\lambda\in\varrho(L_{p,k}^{\sim})$, that is, $\lambda\notin\sigma(L_{p,k}^{\sim})$. This completes the proof. □

Remark 3.5. Suppose that $|L(l)|\to\infty$ with $|l|\to\infty$. Then the set $\{L(l)|l\in\mathbf{Z}^n\}$ is closed and so one has

$$\sigma(L_{p,k}^{\sim}) = \{L(l)\,|\,l\in\mathbf{Z}^n\}. \tag{3.21}$$

For $p\in[1,\infty[$ we obtain

Corollary 3.6. *Suppose that* $L(\cdot)$ *obeys* (3.1). *Furthermore, suppose that there exists a positive constant* $c>0$ *such that*

$$\operatorname{Re} L(l) \geqq -c \quad \textit{for all} \quad l\in\mathbf{Z}^n. \tag{3.22}$$

Then the relations

$$N(L_{p,k}^{\sim}+aI) = \{0\} \quad \text{and} \quad R(L_{p,k}^{\sim}+aI) = \mathscr{B}_{p,k}^{\pi} \tag{3.23}$$

hold for $a>c$.

Proof. In virtue of (3.22) one has

$$|L(l)+a| \geqq \operatorname{Re} L(l)+a = a-c > 0 \quad \text{for all} \quad l\in\mathbf{Z}^n. \tag{3.24}$$

Hence $-a$ does not belong to $\overline{\{L(l)|l\in\mathbf{Z}^n\}}$ and then by Theorem 3.4 $-a$ belongs to $\varrho(L_{p,k}^{\sim})$. This proves the assertion. □

Remark 3.7. A) The proof of Corollary 3.6 shows that the condition (with $\lambda\in\mathbf{C}$)

$$|L(l)+\lambda I| \geqq E > 0 \quad \text{for all} \quad l\in\mathbf{Z}^n \tag{3.25}$$

implies the relations (3.11)—(3.12).

B) The elements k of $\mathscr{K}_\pi$ satisfy (3.1). In the Chapter 5 we shall denote the corresponding operators (3.2) by $k(D)$.

4. The perturbed operator

In the sequel we assume that the operator $L=L(x, D)$ belongs to $L^{t\pi}$ and that $p\in[1, \infty[$. At first we establish

Theorem 4.1. *Suppose that* $P(\cdot)\colon \mathbf{Z}^n\to\mathbf{C}$ *obeys the estimates*

$$|P(l)| \leqq Ck_M(l) \tag{4.1}$$

and

$$\operatorname{Re} P(l) \geqq -c \quad \textit{for all} \quad l\in\mathbf{Z}^n, \tag{4.2}$$

where C, M *and* c *are positive constants. Furthermore, suppose that* $L\in L^{t\pi}$ *such that*

$$\|(L-P)\varphi\|_{p,k} \leqq \alpha\|P\varphi\|_{p,k}+\beta\|\varphi\|_{p,k} \quad \textit{for all} \quad \varphi\in C_\pi^\infty, \tag{4.3}$$

where $\alpha\in]0, 1[$, $\beta\geqq0$ *and* $P=P(D)$.

Then there exists $R\geqq0$ *such that the relations*

$$N(\tilde{L}_{p,k}+aI) = \{0\} \quad \textit{and} \quad R(\tilde{L}_{p,k}+aI) = \mathscr{B}_{p,k}^\pi \tag{4.4}$$

hold for $a\geqq R$.

Proof. Due to Corollary 3.6 the operator $\tilde{P}_{p,k}+aI$ is a Fredholm operator with

$$\operatorname{ind}(\tilde{P}_{p,k}+aI) = 0 \quad \text{for} \quad a > c. \tag{4.5}$$

In virtue of (4.2) we obtain

$$|P(l)+a| \geqq a-c \quad \text{for} \quad l\in\mathbf{Z}^n$$

and then one has

$$\|(P+aI)\varphi\|_{p,k} \geqq (a-c)\|\varphi\|_{p,k} \quad \text{for all} \quad \varphi\in C_\pi^\infty. \tag{4.6}$$

Furthermore, one has for $a\geqq 2c$

$$|P(l)+a|^2 = |P(l)|^2+2a\operatorname{Re} P(l)-a^2 \geqq$$

$$\geqq |P(l)|^2-2ac+a^2 \geqq |P(l)|^2 \quad \text{for all} \quad l\in \mathbf{Z}^n$$

and then

$$(4.7) \qquad \|(P+aI)\varphi\|_{p,k} \geqq \|P\varphi\|_{p,k} \quad \text{for all} \quad \varphi\in C_\pi^\infty.$$

In view of (4.3), (4.6) and (4.7) we obtain for a large enough, say $a\geqq R$ $(\geqq 2c)$

$$(4.8) \qquad \|(L_{p,k}+aI)\varphi-(P_{p,k}+aI)\varphi\|_{p,k} = \|(L_{p,k}-P_{p,k})\varphi\|_{p,k} \leqq$$

$$\leqq \big(\alpha+\beta/(a-c)\big)\|(P_{p,k}+aI)\varphi\|_{p,k} \leqq \alpha'\|(P_{p,k}+aI)\varphi\|_{p,k} \quad \text{for all} \quad \varphi\in C_\psi^\infty,$$

where $\alpha'\in]0, 1[$.

The inequality (4.8) implies that $D(L_{p,k}^{\sim})=D(P_{p,k}^{\sim})$ and that for $a\geqq R$

$$(4.9) \qquad \|(L_{p,k}^{\sim}+aI)u-(P_{p,k}^{\sim}+aI)u\|_{p,k} \leqq \alpha'\|(P_{p,k}^{\sim}+aI)u\|_{p,k}$$

for all $u\in D(P_{p,k}^{\sim})$. Since $P_{p,k}^{\sim}+aI$ is a Fredholm operator with $\operatorname{ind}(P_{p,k}^{\sim}+aI)=0$ and since by (4.9) the operator $L_{p,k}^{\sim}+aI$ is $(P_{p,k}^{\sim}+aI)$-bounded with relative bound α' smaller than one, one sees that $L_{p,k}^{\sim}+aI$ is a Fredholm operator with

$$(4.10) \qquad \operatorname{ind}(L_{p,k}^{\sim}+aI) = 0$$

(cf. [6], p. 236). Due to (4.9) and (4.6) the range $R(L_{p,k}^{\sim}+aI)$ is closed and the kernel $N(L_{p,k}^{\sim}+aI)=\{0\}$. Thus by (4.10) $R(L_{p,k}^{\sim}+aI)=\mathscr{B}_{p,k}^{\pi}$. This finishes the proof. □

For $p\in]1, \infty[$ we obtain

Corollary 4.2. *Suppose that* $P(\cdot)$ *and* $Q(\cdot)\colon \mathbf{Z}^n\to\mathbf{C}$ *obey the estimates*

$$(4.11) \qquad |P(l)| \leqq Ck_M(l) \quad \text{and} \quad |Q(l)| \leqq Ck_M(l)$$

and

$$(4.12) \qquad \operatorname{Re} P(l) \geqq -c \quad \textit{and} \quad \operatorname{Re} Q(l) \geqq -c \quad \textit{for all} \quad l\in\mathbf{Z}^n,$$

where C, M *and* c *are positive constants. Furthermore, suppose that* $L\in L^{t\pi}$ *such that*

$$(4.3) \qquad \|(L-P)\varphi\|_{p,k} \leqq \alpha\|P\varphi\|_{p,k}+\beta\|\varphi\|_{p,k}$$

and

$$(4.13) \qquad \|(L'-Q)\varphi\|_{p',1/k^{\vee}} \leqq \alpha\|Q\varphi\|_{p',1/k^{\vee}}+\beta\|\varphi\|_{p',1/k^{\vee}}$$

for all $\varphi\in C_\pi^\infty$, *where* $\alpha\in]0, 1[$, $\beta\geqq 0$ *and* $Q=Q(D)$.

Then the equation

$$(4.14) \qquad L_{p,k}^{\sim} = L_{p,k}'^{\#}$$

is valid.

Proof. In virtue of Theorem 4.1 one has

$$(4.15) \qquad R(L_{p,k}^{\sim}+aI) = \mathscr{B}_{p,k}^{\pi} \quad \text{and} \quad R(L_{p',1/k^{\vee}}'^{\sim}+aI) = \mathscr{B}_{p',1/k^{\vee}}^{\pi}$$

for a large enough, say $a\geqq R$. We show that

$$(4.16) \qquad N(L_{p,k}'^{\#}+aI) = \{0\}.$$

Let u be in $N(L_{p,k}'^{\#}+aI)$ and let $U=\mathscr{L}(u)$. Then one has

$$U\big((L_{p',1/k^{\vee}}'+aI)\varphi\big) = u\big((L'+aI)\varphi\big) = 0 \quad \text{for all} \quad \varphi\in C_\pi^\infty$$

and thus

$$U\big((L'^{\sim}_{p',1/k^{\sim}}+aI)v\big)=0 \quad \text{for all} \quad v\in D(L'^{\sim}_{p',1/k^{\sim}}).$$

This shows that $U\in\big(R(L'^{\sim}_{p',1/k^{\sim}}+aI)\big)^{\perp}=\{0\}$. Hence (4.16) holds. Let u be in $D(L'^{\#}_{p,k})$ and let $L'^{\#}_{p,k}u=f$. Choose $w\in D(L^{\sim}_{p,k})$ with $(L^{\sim}_{p,k}+aI)w=f+au=$ $=(L'^{\#}_{p,k}+aI)u$. Since $(L^{\sim}_{p,k}+aI)w=(L'^{\#}_{p,k}+aI)w$ one sees by (4.16) that $u=$ $=w\in D(L^{\sim}_{p,k})$ and that $L^{\sim}_{p,k}u=f$. This shows (4.14). □

5. The semi-bounded operator

5.1. We assume further that $L=L(x,D)$ lies in $L^{t\pi}$. In addition, we in this chapter assume that p lies in the interval $]1,\infty[$. Then the spaces $\mathscr{B}^{\pi}_{p,k}$ are reflexive and so one has $L^{\sim}_{p,k}=L^{**}_{p,k}$ (cf. [6], p. 168).

Let k and $k^{\sim}$ be in $\mathcal{K}_{\pi}$. We show

Theorem 5.1. *Suppose that $L\in L^{t\pi}$ such that with some constants $C>0$, $a\in\mathbf{C}$ and $c>0$ the estimates*

(5.1) $$\|(L+aI)\varphi\|_{p,k}\leqq C\|\varphi\|_{p,kk^{\sim}}$$

(5.2) $$\|(L+aI)\varphi\|_{p,k}\geqq c\|\varphi\|_{p,k}$$

and

(5.3) $$\|(L'+aI)\varphi\|_{p',1/(kk^{\sim})^{\sim}}\geqq c\|\varphi\|_{p',1/(kk^{\sim})^{\sim}} \quad \textit{for} \quad \varphi\in C^{\infty}_{\pi}$$

hold.

Then the relations

(5.4) $$N(L^{\sim}_{p,k}+aI)=\{0\} \quad \text{and} \quad R(L^{\sim}_{p,k}+aI)=\mathscr{B}^{\pi}_{p,k}$$

are valid.

Proof. A) In virtue of (5.2) and (5.3) one has

(5.5) $$\|(L^{\sim}_{p,k}+aI)u\|_{p,k}\geqq c\|u\|_{p,k} \quad \text{for} \quad u\in D(L^{\sim}_{p,k})$$

and

(5.6) $$\|(L'^{\sim}_{p',1/(kk^{\sim})^{\sim}}+aI)v\|_{p',1/(kk^{\sim})^{\sim}} \quad \text{for} \quad v\in D(L'^{\sim}_{p',1/(kk^{\sim})^{\sim}}).$$

Thus one sees that $R(L^{\sim}_{p,k}+aI)$ (and $R(L'^{\sim}_{p',1/(kk^{\sim})^{\sim}}+aI)$) is closed in $\mathscr{B}^{\pi}_{p,k}$ (and in $\mathscr{B}^{\pi}_{p',1/(kk^{\sim})^{\sim}}$, resp.). In addition, the relations

(5.7) $$N(L^{\sim}_{p,k}+aI)=\{0\} \quad \text{and} \quad N(L'^{\sim}_{p',1/(kk^{\sim})^{\sim}}+aI)=\{0\}$$

hold. Hence we get

(5.8) $$R(L'^{*}_{p',1/(kk^{\sim})^{\sim}}+aI)=R(L'^{\sim *}_{p',1/(kk^{\sim})^{\sim}}+aI)=$$
$$=N(L'^{\sim}_{p',1/(kk^{\sim})^{\sim}}+aI)^{\perp}=\{0\}^{\perp}=\mathscr{B}^{\pi *}_{p',1/(kk^{\sim})^{\sim}}.$$

B) We show that

(5.9) $$R(L'^{\#}_{p,kk^{\sim}}+aI)=\mathscr{B}^{\pi}_{p,kk^{\sim}}.$$

Let $f\in\mathscr{B}^{\pi}_{p,kk^{\sim}}$ and let $F=\mathscr{L}(f)\in\mathscr{B}^{\pi *}_{p',1/(kk^{\sim})^{\sim}}$. Due to (5.8) we find an element U

from $\mathscr{B}^{\pi *}_{p',1/(kk^{\sim})^{\vee}}$ such that

(5.10) $$(L'^{*}_{p',1/(kk^{\sim})^{\vee}}+aI)U=F$$

and then one has with $u:=\mathscr{L}^{-1}(U)$

(5.11) $$u\big((L'+aI)\varphi\big)=U\big((L'_{p',1/(kk^{\sim})^{\vee}}+aI)\varphi\big)$$
$$=F(\varphi)=f(\varphi)\quad \text{for all}\quad \varphi\in C^{\infty}_{\pi}.$$

This shows that $f=(L'^{\#}_{p,kk^{\sim}}+aI)u\in R(L'^{\#}_{p,kk^{\sim}}+aI)$, as required.

C) It remains to show that

(5.12) $$R(L^{\sim}_{p,k}+aI)=\mathscr{B}^{\pi}_{p,k}.$$

Let ψ be in C^{∞}_{π}. Due to (5.9) we find an element $w\in\mathscr{B}^{\pi}_{p,kk^{\sim}}$ such that $(L'^{\#}_{p,kk^{\sim}}+aI)w=\psi$. From (5.1) and (5.2) one sees that

(5.13) $$\mathscr{B}^{\pi}_{p,kk^{\sim}}\subset D(L^{\sim}_{p,k})$$

and then $w\in D(L^{\sim}_{p,k})$ and $(L^{\sim}_{p,k}+aI)w=\psi$. Hence the inclusion

(5.14) $$C^{\infty}_{\pi}\subset R(L^{\sim}_{p,k}+aI)$$

is valid. Since C^{∞}_{π} is dense in $\mathscr{B}^{\pi}_{p,k}$ and since $R(L^{\sim}_{p,k}+aI)$ is closed, the relation (5.12) holds. This completes the proof. □

Let k, $k_1^{\sim}$ and $k_2^{\sim}$ be in $\mathscr{K}_{\pi}$. The following corollary is obtained

Corollary 5.2. *Suppose that* $L\in L^{t\pi}$ *such that with some constants* $C>0$, $a\in\mathbf{C}$ *and* $c>0$ *the estimates*

(5.1) $$\|(L+aI)\varphi\|_{p,k}\leqq C\|\varphi\|_{p,kk_1^{\sim}}$$

(5.2) $$\|(L+aI)\varphi\|_{p,k}\geqq c\|\varphi\|_{p,k}$$

(5.3) $$\|(L'+aI)\varphi\|_{p',1/(kk_1^{\sim})^{\vee}}\geqq c\|\varphi\|_{p',1/(kk_1^{\sim})}$$

(5.15) $$\|(L'+aI)\varphi\|_{p',1/k^{\vee}}\leqq C\|\varphi\|_{p',k_2^{\sim}/k^{\vee}}$$

(5.16) $$\|(L'+aI)\varphi\|_{p',1/k^{\vee}}\geqq c\|\varphi\|_{p',1/k^{\vee}}$$

and

(5.17) $$\|(L+aI)\varphi\|_{p',k/(k_2^{\sim})^{\vee}}\geqq c\|\varphi\|_{p',k/(k_2^{\sim})^{\vee}}\quad \textit{for all}\quad \varphi\in C^{\infty}_{\pi}$$

hold.

Then the equality

(5.18) $$L^{\sim}_{p,k}=L'^{\#}_{p,k}$$

is valid.

Proof. In virtue of Theorem 5.1 one has $R(L^{\sim}_{p,k}+aI)=\mathscr{B}^{\pi}_{p,k}$ and $R(L'^{\sim}_{p',1/k^{\vee}}+aI)=\mathscr{B}^{\pi}_{p',1/k^{\vee}}$. Thus the assertion follows as in the proof of Corollary 4.2. □

5.2. In the sequel we assume that $p=2$. As we mentioned above, the spaces $\mathcal{H}_k^\pi=\mathcal{B}_{2,k}^\pi$ are Hilbert spaces with the inner product

$$(u, v)_k := (2\pi)^{-n} \sum_{l\in\mathbf{Z}^n} u_l \bar{v}_l k^2(l).$$

In the case when $k\equiv 1$ we denote $(u, v)_k=(u, v)_0$.

Theorem 5.3. *Suppose that $L\in L^{t\pi}$ such that with some constants $C>0$ and $c>0$ the estimates*

$$\|L\varphi\|_k \leqq C\|\varphi\|_{kk^\sim} \tag{5.1}$$

$$\operatorname{Re}\big((L+cI)\varphi,\ k^2(D)\varphi\big)_0 \geqq 0 \tag{5.19}$$

and

$$\operatorname{Re}\big((L+cI)\varphi, (kk^\sim)^2(D)\varphi\big)_0 \geqq 0 \quad \text{for all} \quad \varphi\in C_\pi^\infty, \tag{5.20}$$

where $k^\sim \geqq 1$.

Then the relations

$$N(L_{2,k}^\sim+aI) = \{0\} \quad \text{and} \quad R(L_{2,k}^\sim+aI) = \mathcal{H}_k^\pi \tag{5.4}$$

are valid for $a>c$.

Proof. In virtue of (5.19) one has for $a>c$

$$\begin{aligned}\operatorname{Re}\big((L+aI)\varphi, \varphi\big)_k &= \operatorname{Re}\big((L+aI)\varphi, k^2(D)\varphi\big)_0 \geqq \\ &\geqq -c\big(\varphi, k^2(D)\varphi\big)_0+a\big(\varphi, k^2(D)\varphi\big)_0 = (a-c)\|\varphi\|_k^2\end{aligned} \tag{5.21}$$

and then

$$\big\|(L+aI)\varphi\big\|_k \geqq (a-c)\|\varphi\|_k \quad \text{for all} \quad a>c. \tag{5.22}$$

Similarly we obtain

$$\operatorname{Re}\big((L+aI)\varphi, (kk^\sim)^2(D)\varphi\big)_0 \geqq (a-c)\|(kk^\sim)(D)\varphi\|_0^2 \tag{5.23}$$

and then

$$\operatorname{Re}\big((L+aI)(kk^\sim)^{-2}(D)\varphi, \varphi\big)_0 \geqq (a-c)\|(kk^\sim)^{-1}(D)\varphi\|_0^2 = (a-c)\|\varphi\|_{1/(kk^\sim)}^2. \tag{5.24}$$

One has

$$\begin{aligned}\big|\operatorname{Re}\big((L+aI)(kk^\sim)^{-2}(D)\varphi, \varphi\big)_0\big| &= \big|\operatorname{Re}\big((L+aI)(kk^\sim)^{-2}(D)\varphi\big)(\bar{\varphi})\big| = \\ &= \big|\operatorname{Re}\big((kk^\sim)^{-2}(D)\varphi\big)\big((L'+aI)\bar{\varphi}\big)\big| = \\ &= \big|\operatorname{Re}\big((kk^\sim)^{-2}(D)\varphi, \overline{(L'+aI)\bar{\varphi}}\big)_0 \leqq \\ &\leqq \|\varphi\|_{1/kk^\sim}\big\|(kk^\sim)^{-1}(D)\big(\overline{(L'+aI)\bar{\varphi}}\big)\big\|_0 = \\ &= \|\varphi\|_{1/kk^\sim}\|(L'+aI)\varphi\|_{1/(kk^\sim)^\vee},\end{aligned} \tag{5.25}$$

where we used the fact that

$$\begin{aligned}\|\bar{\varphi}\|_k^2 &= (2\pi)^{-n}\sum_{l\in\mathbf{Z}^n}|(\bar{\varphi})_l k(l)|^2 = (2\pi)^{-n}\sum_{l\in\mathbf{Z}^n}|\bar{\varphi}_{-l}k(l)|^2 = \\ &= (2\pi)^{-n}\sum_{l\in\mathbf{Z}^n}|\varphi_l k(-l)|^2 = \|\varphi\|_{k^\vee}^2.\end{aligned} \tag{5.26}$$

Hence by (5.24) we obtain

(5.27) $$\|(L'+aI)\varphi\|_{1/(kk^{\sim})^{\vee}} \geqq (a-c)\|\bar{\varphi}\|_{1/kk^{\sim}} = (a-c)\|\varphi\|_{1/(kk^{\sim})^{\vee}}$$

for $a>c$.

Finally we note by (5.1) that

(5.28) $$\|(L+aI)\varphi\|_k \leqq (C+a)\|\varphi\|_{kk^{\sim}}$$

since $k^{\sim} \geqq 1$. Hence all of the assumptions of Theorem 5.1 hold. This completes the proof. □

Corollary 5.4. *Suppose that* $L \in L^{t\pi}$ *such that with some constants* $C>0$ *and* $c>0$ *the estimates*

(5.1) $$\|L\varphi\|_k \leqq C\|\varphi\|_{kk^{\sim}}$$

(5.29) $$\|L\varphi\|_{k/k^{\sim}} \leqq C\|\varphi\|_k$$

(5.19) $$\mathrm{Re}\big((L+cI)\varphi, k^2(D)\varphi\big)_0 \geqq 0$$

(5.20) $$\mathrm{Re}\big((L+cI)\varphi, (kk^{\sim})^2(D)\varphi\big)_0 \geqq 0$$

and

(5.30) $$\mathrm{Re}\big((L+cI)(k^{\sim})^2(D)\varphi, k^2(D)\varphi\big)_0 \geqq 0 \quad \textit{for all} \quad \varphi \in C_\pi^\infty$$

hold, where $k^{\sim} \geqq 1$.

Then the equality

(5.31) $$L_{2\,k}^{\sim} = L_{2,k}'^{\sharp}$$

is valid.

Proof. In virtue of (5.1) we obtain

$$|(L'\varphi)(\psi)| = |\varphi(L\psi)| \leqq \|\varphi\|_{(k^{\sim}/k)^{\vee}}\|L\psi\|_{k/k^{\sim}} \leqq C\|\varphi\|_{(k^{\sim}/k)^{\vee}}\|\psi\|_k$$

and then

(5.32) $$\|L'\varphi\|_{1/k^{\vee}} \leqq C\|\varphi\|_{k^{\sim\vee}/k^{\vee}} \quad \text{for all} \quad \varphi \in C_\pi^\infty.$$

Furthermore, one sees by (5.19) that

$$\mathrm{Re}\big((L+cI)(k^{-2}(D)\psi), \psi\big)_0 \geqq 0 \quad \text{for all} \quad \psi \in C_\pi^\infty$$

(choose $\varphi = k^{-2}(D)\psi \in C_\pi^\infty$ in (5.19)). Hence one has

(5.33) $$\mathrm{Re}\big((L'+cI)\psi, (k)^{-2}(D)\psi\big)_0 =$$

$$= \mathrm{Re}\big((L'+cI)\psi\big)\big(\overline{(k^{\vee})^{-2}(D)\psi}\big) = \mathrm{Re}\big((L'+cI)\psi\big)\big(k^{-2}(D)\bar{\psi}\big) =$$

$$= \mathrm{Re}\,\psi\big((L+cI)k^{-2}(D)\bar{\psi}\big) = \mathrm{Re}\,\big((L+cI)(k^{-2}(D)\bar{\psi}), \bar{\psi}\big)_0 \geqq 0$$

for $\psi \in C_\pi^\infty$. Similarly one sees by (5.30) that (at first choose $\varphi = k^{-2}(D)\psi$ and then evaluate as in (5.33))

(5.34) $$\mathrm{Re}\big((L'+cI)\psi, ((k^{\sim}/k)^{\vee})^2(D)\psi\big)_0 \geqq 0 \quad \text{for} \quad \psi \in C_\pi^\infty.$$

The inequalities (5.1), (5.19) and (5.20) imply by Theorem 5.3 that $R(L_{2,k}^{\sim}+aI) = \mathscr{H}_k^\pi$ for $a>c$. Similarly the inequalities (5.32), (5.33) and (5.34) imply by Theorem

5.3 that $R(L'^{\sim}_{2,1/k^{\sim}}+aI)=\mathscr{H}^{\pi}_{1/k^{\sim}}$ for $a>c$. Hence the identity (5.31) is showed as in Corollary 4.2. □

The Corollary 5.4 can be also formulated as follows (cf. the computation performed in (5.33))

Corollary 5.5. *Suppose that* $L\in L^{t\pi}$ *such that with some constants* $C>0$ *and* $c>0$ *the estimates*

(5.1) $$\|L\varphi\|_k \leqq C\|\varphi\|_{kk^{\sim}}$$

(5.29) $$\|L\varphi\|_{k/k^{\sim}} \leqq C\|\varphi\|_k$$

(5.35) $$\mathrm{Re}\left((L'+cI)(k^{\check{}})^2(D)\varphi, \varphi\right)_0 \geqq 0$$

(5.36) $$\mathrm{Re}\left((L'+cI)((kk^{\check{}})^{\check{}})^2(D)\varphi, \varphi\right)_0 \geqq 0$$

and

(5.30) $$\mathrm{Re}\left(k^2(D)(L+cI)(k^{\check{}})^2(D)\varphi, \varphi\right)_0 \geqq 0 \quad \textit{for} \quad \varphi\in C^{\infty}_{\pi}$$

hold, where $k^{\sim}\geqq 1$.

Then the equality

(5.31) $$L^{\sim}_k := L^{\sim}_{2,k} = L'^{\#}_{2,k} =: L'^{\#}_k$$

is valid.

The assumptions (5.35), (5.36) and (5.30) mean a semiboundedness (from below) condition for $(L'+cI)(k^{\check{}})^2(D)$, $(L'+cI)\left((kk^{\check{}})^{\check{}}\right)^2(D)$ and $k^2(D)(L+cI)(k^{\check{}})^2(D)$. The assumptions (5.1) and (5.29) mean a boundedness condition for L.

The formulation of Theorem 5.3 can be similarly modified (that is, (5.19) and (5.20) can be replaced (5.35) and (5.36)).

Corollary 5.6. *Suppose that* $L\in L^{t\pi}$ *such that with some constants* $C>0$ *and* $c>0$ *the estimates*

(5.37) $$\|L\varphi\|_0 \leqq C\,\|\varphi\|_k$$

(5.35) $$\|L\varphi\|_{1/k^{\sim}} \leqq C\|\varphi\|_0$$

(5.39) $$\mathrm{Re}\left((L+cI)\varphi, \varphi\right)_0 \geqq 0$$

(5.40) $$\mathrm{Re}\left((L'+cI)((k^{\check{}})^{\check{}})^2(D)\varphi, \varphi\right)_0 \geqq 0$$

and

(5.41) $$\mathrm{Re}\left((L+cI)(k^{\check{}})^2(D)\varphi, \varphi\right)_0 \geqq 0$$

are valid, where $k^{\sim}\geqq 1$.

Then the equality

(5.42) $$L^{\sim} := L^{\sim}_{2,1} = L'^{\#}_{2,l} =: L'^{\#}$$

holds.

Remark 5.7. The inequalities (5.35), (5.37) and (5.38) imply that $N(L^{\sim}+aI)=\{0\}$ and that $R(L^{\sim}+aI)=\mathscr{H}^{\pi}$ for $a>c$. $\mathscr{H}^{\pi}$ is the space of all periodic $L^2(W)$-(Lebesgue) integrable functions.

5.3. Suppose that L is an elliptic partial differential operator with C_π^∞-coefficients, that is,

$$L(x, D) = \sum_{|\sigma|\leqq 2m} a_\sigma D^\sigma, \quad m\in \mathbf{N},\ a_\sigma \in \mathbf{C}.$$

Then for each $t\in \mathbf{N}_0$ there exists a constant $c>0$ and $\gamma>0$ such that

$$\operatorname{Re}\big((L(x,D)+c)\varphi, \varphi\big)_0 \geqq \gamma \|\varphi\|_{k_{2m}} \quad \text{for all} \quad \varphi\in C_\pi^\infty \tag{5.43}$$

(cf. [3], p. 117). Let t be in $\mathbf{N}_0$. Choosing $\bar{k}_t(l)=(\sum_{|\alpha|\leqq t} l^{2\alpha})^{1/2}$ and $k^\sim(l)=\sum_{|\alpha|\leqq m} l^{2\alpha}$ one sees that $L'(x,D)\bar{k}_t^2(D)$, $L'(x,D)(\bar{k}_t k^\sim)^2(D)$ and $\bar{k}_t^2(D)L(x,D)(k^\sim)^2(D)$ are also elliptic operators (of order $2(m+t)$ and $2(2m+t)$, resp.). Hence by (5.43) there exists a constant $c>0$ such that (note that $k^\sim \geqq 1$)

$$\operatorname{Re}\big(L'(x,D)\bar{k}_t^2(D)\varphi, \varphi\big)_0 + c\big(\bar{k}_t^2(D)\varphi, \varphi\big)_0 \geqq 0,$$

$$\operatorname{Re}\big(L'(x,D)(\bar{k}_t k^\sim)^2(D)\varphi, \varphi\big)_0 + c\big((\bar{k}_t k^\sim)^2(D)\varphi, \varphi\big)_0 \geqq 0$$

and

$$\operatorname{Re}\big(\bar{k}_t^2(D)L(x,D)(k^\sim)^2(D)\varphi, \varphi\big)_0 + c\big((\bar{k}_t^2 k^\sim)^2(D)\varphi, \varphi\big)_0 \geqq 0.$$

Hence the inequalities (5.35), (5.36) and (5.30) hold. The inequalities (5.1) and (5.29) are also valid (cf. [3], p. 111).

Using the evaluation such as in (5.33)—(5.34) one sees that the inequalities (5.35), (5.36) and (5.30) hold also in the case when $k=(\bar{k}_t)^{-1}$. For example, (5.30) is proved as follows: The operator $\bar{k}_t^2(D)k^{\sim 2}(D)L'(x,D)$ is elliptic. Hence there exists $c\geqq 0$ such that

$$\operatorname{Re}\big(\bar{k}_t^2(D)k^{\sim 2}(D)L'(x,D)\varphi, \varphi\big)_0 + c\big(\bar{k}_t^2(D)k^{\sim 2}(D)\varphi, \varphi\big)_0 \geqq 0$$

and then

$$\operatorname{Re}\big(k^{\sim 2}(D)(L'(x,D)+c)\varphi, \bar{k}_t^2(D)\varphi\big)_0 \geqq 0.$$

Choosing $\varphi=\bar{k}_t^{-2}(D)\psi$ one has

$$\operatorname{Re}\big(k^{\sim 2}(D)(L'(x,D)+c)\bar{k}_t^2(D)\psi, \psi\big)_0 \geqq 0.$$

Thus one has (cf. (5.33) and note that $k^{\sim\vee}=k^\sim$ and $\bar{k}_t^\vee=\bar{k}_t$)

$$\operatorname{Re}\big(\bar{k}_t^{-2}(D)\bar{\psi}, (L(x,D)+c)k^{\sim 2}(D)\bar{\psi}\big)_0 \geqq 0,$$

which implies (5.30).

Since $\bar{k}_t$ and k_t are equivalent weight functions we obtain

Corollary 5.8. *Let $L(x,D)$ be an elliptic partial differential operator with C_π^∞-coefficients and let t be in $\mathbf{Z}$. Then one has*

$$N(L_{k_t}'^{\#}+aI) = \{0\} \quad \text{and} \quad R(L_{k_t}^\sim + aI) = \mathscr{H}_{k_t}^\pi \tag{5.44}$$

for a large enough and

$$L_{k_t}^\sim = L_{k_t}'^{\#}. \tag{5.45}$$

For elliptic operators one has

$$\|L(X,D)\varphi\|_{k_t} + c\|\varphi\|_{k_t} \geqq \gamma\|\varphi\|_{k_t+2m} \quad \text{for} \quad \varphi\in C_\pi^\infty.$$

Hence we obtain from (5.45)

$$D(L'^{\#}_{k_\varepsilon}) = D(L^{\sim}_{k_\varepsilon}) \subset \mathscr{H}^{\pi}_{k_\varepsilon+2m}.$$

We remark that in the proof of Corollary 5.7 we need only the validity of

$$\operatorname{Re}\big(L(x, D)\varphi, \varphi\big)_0 + c(\varphi, \varphi)_0 \geqq 0 \quad \text{for} \quad \varphi \in C^{\infty}_{\pi}$$

for elliptic operators; we not need the strong estimate (5.43).

6. References

[1] R. BEALS, Advanced Mathematical Analysis. Springer-Verlag, *New York—Heidelberg—Berlin* 1973.

[2] L. BERS and M. SCHECHTER, Elliptic equations. Partial differential equations (by L. Bers, F. John and M. Schechter). Interscience Publishers, a division of John Wiley Sons, Inc., *New York—London—Sydney*, 1964.

[3] J. EELLS, Elliptic operators on manifolds. Complex analysis and its applications. International Centre for Theoretical Physics, Trieste, Internat. Atomic Energy Agency, *Vienna*, 1976, pp. 95—152.

4] A. FRIEDMAN, Partial differential operators. Halt Rinehart and Wiston, Inc., *New York—Chicago—San Francisco—Atlanta—Dallas—Montreal—Toronto—London—Sydney*, 1969.

[5] W. GREUB, Linear Algebra. Springer-Verlag, *New York—Heidelberg—Berlin*, 1981.

[6] T. KATO, Perturbation theory for linear operators. Springer-Verlag, *Berlin—Heidelberg—New York*, 1966.

[7] A. A. KULESHOV, Linear equations in space of periodic generalized functions. *Differents. Uravn.*, **20**, No. 2, 1984.

[8] I. S. LOUHIVAARA and C. G. SIMADER, Über periodische Lösungen koerzitiver linearer partieller Differentialgleichungen. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **4**, 1979, pp. 87—108.

[9] M. SCHECHTER, Spectral of partial differential operators. North-Holland series in applied mathematics and mechanics 14. North-Holland Publishing Company, *Amsterdam—London*, 1971.

[10] J. TERVO, On one-sided invertibility of linear coercive Fourier series operators. *Publ. Math. Debrecen* **34** (1987), 43—60.

[11] J. TERVO, On compactness of Fourier series operators. *Ber Univ. Jyväskylä Math. Inst. Ber.* **33**, 1987.

[12] K. YOSIDA, Functional analysis. Springer-Verlag, *Berlin—Heidelberg—New York*, 1974.

(Received May 26, 1987)