Publ. Math. Debrecen
51 / 1-2 (1997), 175–189

$210 = 14 \times 15 = 5 \times 6 \times 7 = \binom{21}{2} = \binom{10}{4}$

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Abstract. It is given all the solutions of the diophantine equations

$$(y-1)y(y+1) = \binom{n}{4} \quad \text{and} \quad x(x+1) = \binom{n}{4}.$$

1. Introduction

The title of this paper illustrates the remarkable fact that the number 210 can be represented simultaneously as a product of two consecutive integers, a product of three consecutive integers, a triangular number, and as a binomial coefficient $\binom{n}{4}$ in a nontrivial way[1]. In other words, 210 is a common solution to the system of diophantine equations

$$x(x+1) = (y-1)y(y+1) = \binom{m}{2} = \binom{n}{4}, \tag{1}$$

where we take $x, y, m, n \in \mathbb{Z}$ without further restrictions, i.e. $\binom{m}{2} = \frac{1}{2}m(m-1)$ and $\binom{n}{4} = \frac{1}{24}n(n-1)(n-2)(n-3)$ are defined for all $m, n \in \mathbb{Z}$.

Mathematics Subject Classification: 11D25, 11G05.
Key words and phrases: combinatorial diophantine equations, Thue equations, elliptic curves.
*The research was supported in part by Grants T16975 and T19479 from the Hungarian National Foundation for Scientific Research.
†This author's research was supported by the Netherlands Mathematical Research Foundation SWON with financial aid from the Netherlands Organization for Scientific Research NWO.
[1]We prefer not to notice that 210 also is the product of the four smallest prime numbers.

The solution 210 occurs for $x = -15, 14$, $y = 6$, $m = -20, 21$, $n = -7, 10$. There is one other integer that can be represented in the above mentioned four ways: the number 0 occurs for $x = -1, 0$, $y = -1, 0, 1$, $m = 0, 1$, $n = 0, 1, 2, 3$.

In fact, the system (1) consists of six different diophantine equations. We will consider these equations in this paper.

The equation

$$x(x+1) = (y-1)y(y+1)$$

has been solved for the first time in 1963 by Mordell [M]. It has only the solutions $(x, y) = (-15, 6), (-3, 2), (-1, -1), (-1, 0), (-1, 1), (0, -1), (0, 0),$ $(0, 1), (2, 2), (14, 6)$.

The equation

$$x(x+1) = \binom{m}{2}$$

is essentially a Pell equation, and hence trivial. Its solutions are given by $(x, m) = (x_i, m_i)$ for $i = 0, 1, 2, \ldots$, where $x_{i+1} = 6x_i - x_{i-1} + 2$ and $m_{i+1} = 6m_i - m_{i-1} - 2$, with four different sets of initial values: $(x_0, m_0, x_1, m_1) = (0, 1, 2, 4), (0, 0, 2, -3), (-1, 1, -3, 4), (-1, 0, -3, -3)$.

The equation

$$(y-1)y(y+1) = \binom{m}{2}$$

has been solved for the first time in 1989 by Tzanakis and de Weger [TW]. It has only the solutions $(y, m) = (-1, 0)$, $(-1, 1)$, $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, $(2, -3)$, $(2, 4)$, $(5, -15)$, $(5, 16)$, $(6, -20)$, $(6, 21)$, $(10, -44)$, $(10, 45)$, $(57, -608)$, $(57, 609)$, $(637, -22736)$, $(637, 22737)$.

The equation

$$\binom{m}{2} = \binom{n}{4}$$

has been solved independently by the present two authors, [P] and [dW]. The only solutions are $(m, n) = (-20, -7)$, $(-20, 10)$, $(-5, -3)$, $(-5, 6)$, $(-1, -1)$, $(-1, 4)$, $(0, 0)$, $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, -1)$, $(2, 4)$, $(6, -3)$, $(6, 6)$, $(21, -7)$, $(21, 10)$.

It is the purpose of this note to solve the remaining two equations. We will prove the following two theorems.

Theorem 1. *The equation*

$$(y-1)y(y+1) = \binom{n}{4} \tag{2}$$

has only the solutions $(y, n) = (-1, 0)$, $(-1, 1)$, $(-1, 2)$, $(-1, 3)$, $(0, 0)$, $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(1, 3)$, $(6, -7)$, $(6, 10)$, $(22, -21)$, $(22, 24)$, $(26, -24)$, $(26, 27)$.

Theorem 2. *The equation*

$$x(x+1) = \binom{n}{4} \tag{3}$$

has only the solutions $(x, n) = (-15, -7)$, $(-15, 10)$, $(-1, 0)$, $(-1, 1)$, $(-1, 2)$, $(-1, 3)$, $(0, 0)$, $(0, 1)$, $(0, 2)$, $(0, 3)$, $(14, -7)$, $(14, 10)$.

2. Thue equations for Theorem 1

In equation (2) we put $X = 6y$ and $Y = \frac{3}{4}\left((2n-3)^2 - 5\right)$ (notice that $X, Y \in \mathbb{Z}$). Then equation (2) is seen to be equivalent to

$$Y^2 = X^3 - 36X + 9. \tag{4}$$

This equation defines an elliptic curve, that is of rank 2. We are interested in its integral points, but only in those with $6 \mid X$.

Let $\mathbb{K} = \mathbb{Q}(\theta)$, where θ is a root of $X^3 - 36X + 9$. Then an integral basis of $\mathbb{K}$ is $\{1, \theta, \frac{1}{3}\theta^2\}$, the class group is C_3, a system of fundamental units is

$$\epsilon = 1 - 4\theta - 2\frac{1}{3}\theta^2, \quad \eta = 1 - 4\theta + 2\frac{1}{3}\theta^2.$$

The ramifying primes are $3, 11$ and 23, and they ramify as follows:

$$\langle 3 \rangle = \mathfrak{p}_3^3, \quad \mathfrak{p}_3 = \left\langle -12 + \frac{1}{3}\theta^2 \right\rangle, \quad \langle 11 \rangle = \mathfrak{p}_{11}^2 \mathfrak{q}_{11}, \quad \langle 23 \rangle = \mathfrak{p}_{23}^2 \mathfrak{q}_{23},$$

where $\mathfrak{p}_{11}, \mathfrak{q}_{11}, \mathfrak{p}_{23}, \mathfrak{q}_{23}$ are non-principal prime ideals. Note that

$$X^3 - 36X + 9 = (X - \theta)\left(X^2 + \theta X + (\theta^2 - 36)\right),$$

and if a prime ideal $\mathfrak{p}$ divides both $\langle X - \theta \rangle$ and $\left\langle X^2 + \theta X + (\theta^2 - 36) \right\rangle$, then it divides $\left\langle (X + 2\theta)(X - \theta) - \left(X^2 + \theta X + (\theta^2 - 36)\right) \right\rangle = \langle 3^2\,(-4+$

$\frac{1}{3}\theta^2)\rangle = \mathfrak{p}_3^6\mathfrak{p}_{11}^2\mathfrak{p}_{23}^2$. Since $3 \mid X$ and $\mathrm{ord}_{\mathfrak{p}_3}(\theta) = 2$, we have $\mathrm{ord}_{\mathfrak{p}_3}(X-\theta) = 2$, and $\mathrm{ord}_{\mathfrak{p}_3}\left(X^2 + \theta X + (\theta^2 - 36)\right) = 4$. Thus from equation (4) we see that there are $a, b \in \{0, 1\}$ and an integral ideal $\mathfrak{a}$ such that

$$\langle X - \theta\rangle = \mathfrak{p}_3^2\mathfrak{p}_{11}^a\mathfrak{p}_{23}^b\mathfrak{a}^2.$$

On taking norms we find $Y^2 = 3^2 11^a 23^b (N\mathfrak{a})^2$, so that $a = b = 0$. Further it follows that $\mathfrak{a}^2$ is principal, hence so is $\mathfrak{a}$. There exist $m, n \in \{0, 1\}$ such that

$$X - \theta = \pm\epsilon^m\eta^n \left(-12 + \tfrac{1}{3}\theta^2\right)^2 \alpha^2,$$

where α is a generator of $\mathfrak{a}$.

Now we look at embeddings of $\mathbb{K}$ into $\mathbb{R}$. We write $\theta_1 = -6.12\ldots$, $\theta_2 = 0.25\ldots$, $\theta_3 = 5.87\ldots$, and then find that ϵ_2 and ϵ_3 are negative, whereas ϵ_1 and all conjugates of η are positive. Comparing norms, using that $N(X - \theta) = Y^2 > 0$ and $N\epsilon = N\eta = 1$, we see that the $\pm$-sign in (5) is $+$. Further, if $X \geq 6$ then $X - \theta_i > 0$ for $i = 1, 2, 3$, and it follows by studying the signs that $m = 0$. Notice that the solutions of (4) with $X < 6$ (and $6 \mid X$) are trivially found to be only $X = -6, 0$, leading to $Y = \pm 3$ in both cases, and further to $(y, n) = (-1, 0), (-1, 1), (-1, 2), (-1, 3), (0, 0), (0, 1), (0, 2), (0, 3)$.

2.1. The case $n = 0$

In (5) we now may put $\alpha = A + B\theta + C\frac{1}{3}\theta^2$, and if $n = 0$ we then find

$$X - \theta = \left(-12 + \frac{1}{3}\theta^2\right)^2 \left(A + B\theta + C\frac{1}{3}\theta^2\right)^2.$$

Expanding out and comparing coefficients, we obtain

$$X = 144A^2 + 72AB + 6AC + 9B^2, \tag{6}$$

$$1 = A^2 - 6BC, \tag{7}$$

$$0 = 4A^2 + 2AB - C^2. \tag{8}$$

Equation (7) implies that A is odd, and that A and B are coprime. Thus A and $2A+B$ are coprime, and equation (8), written as $C^2 = 2A(2A+B)$, is seen to imply the existence of $E, F \in \mathbb{Z}$ with

$$A = E^2, \quad B = 2F^2 - 2E^2, \quad C = 2EF.$$

Substituting these expressions into (7) we have

$$E^4 + 24E^3F - 24EF^3 = E(E^3 + 24E^2F - 24F^3) = 1.$$

Clearly $E = E^3 + 24E^2F - 24F^3 = \pm 1$, hence this is trivial: the only solutions are given by $(E, F) = \pm(1, -1), \pm(1, 0), \pm(1, 1)$, leading respectively to $(A, B, C) = (1, 0, -2)$, $(1, 0, 2)$, $(1, -2, 0)$, and further to $(X, Y) = (132, \pm 1515)$, $(36, \pm 213)$, $(156, \pm 1947)$, and finally to $(y, n) = (22, -21)$, $(22, 24)$, $(6, -7)$, $(6, 10)$, $(26, -24)$, $(26, 27)$.

2.2. The case $n = 1$

In (5) we again put $\alpha = A + B\theta + C\frac{1}{3}\theta^2$, and if $n = 1$ we then find by $1/\eta = 25 - 2\frac{1}{3}\theta^2$ that

$$\left(25 - 2\frac{1}{3}\theta^2\right)(X - \theta) = \left(-12 + \frac{1}{3}\theta^2\right)^2 \left(A + B\theta + C\frac{1}{3}\theta^2\right)^2.$$

Expanding out and comparing coefficients, we obtain

$$25X - 6 = 144A^2 + 72AB + 6AC + 9B^2, \tag{9}$$

$$1 = A^2 - 6BC, \tag{10}$$

$$\frac{2}{3}X = 4A^2 + 2AB - C^2. \tag{11}$$

Now $2 \times (9) + 12 \times (10) - 75 \times (11)$ gives

$$25C^2 + (4A - 24B)C + (-2AB + 6B^2) = 0.$$

We view this equation as a quadratic equation in C. If it is to have rational solutions, the discriminant must be a square, D^2 say. Hence

$$D^2 = (4A - 24B)^2 - 100(-2AB + 6B^2) = 8(A - B)(2A + 3B).$$

If p is a prime dividing both $A - B$ and $2A + 3B$, then it divides $5A$ and $5B$, and since A and B are coprime, it must be 5. It follows that we can write

$$A - B = eE^2, \quad 2A + 3B = fF^2$$

for unknown integers E, F, where for (e, f) we have four cases:

$$(e, f) = (1, 2), (2, 1), (5, 10), (10, 5).$$

So we get

$$A = \frac{3}{5}eE^2 + \frac{1}{5}fF^2, \quad B = -\frac{2}{5}eE^2 + \frac{1}{5}fF^2,$$
$$C = -\frac{6}{25}eE^2 \pm \frac{1}{25}\sqrt{2ef}EF + \frac{2}{25}fF^2, \quad D = 2\sqrt{2ef}EF.$$

Since F is defined up to sign, we can replace the $\pm$ sign by a $+$. Now we substitute the above expressions into equation (10), and find

$$-27e^2E^4 + 12e\sqrt{2ef}E^3F + 90efE^2F^2 - 6f\sqrt{2ef}EF^3 - 7f^2F^4 = 125.$$

On putting $U = 5\sqrt{2e/f}E$, $V = \sqrt{2e/f}E - F$, which are both integers, we get the Thue equation

$$U^4 - 8U^3V - 12U^2V^2 + 136UV^3 - 140V^4 = \frac{2500}{f^2}.$$

Notice that with $f = 1, 2, 5, 10$ we have $\frac{2500}{f^2} = 2500, 625, 100, 25$. The following Theorem treats these Thue equations. Its proof is postponed to a forthcoming section.

Theorem 3. *The Thue equations*

$$f_1(U, V) = U^4 - 8U^3V - 12U^2V^2 + 136UV^3 - 140V^4 = m, \quad m \in \{25, 100, 625, 2500\} \tag{12}$$

have only the solutions $(U, V) = \pm(3, 1)$ *at* $m = 25$, *and* $(U, V) = \pm(5, 0), \pm(5, 2)$ *at* $m = 625$.

The solutions $(U, V) = \pm(3, 1)$ lead to $(e, f) = (5, 10)$, and to non-integral E, F. The solutions $(U, V) = \pm(5, 0)$ lead to $(e, f) = (1, 2)$, $(E, F) = \pm(1, 1)$, $(A, B, C) = (1, 0, 0)$, $(X, Y) = (6, \pm 3)$, and finally to $(y, n) = (1, 0), (1, 1), (1, 2), (1, 3)$. The solutions $(U, V) = \pm(5, 2)$ lead to $(e, f) = (1, 2)$, $(E, F) = \pm(1, -1)$, and then to non-integral C.

This completes the proof of Theorem 1.

3. Thue equations for Theorem 2

In equation (3) we put $X = 2n - 3$ and $Y = 8x + 4$. Then equation (3) is seen to be equivalent to

$$6Y^2 = X^4 - 10X^2 + 105. \tag{13}$$

This equation defines an elliptic curve, that is of rank 2. We are interested in its integral points.

The right hand side of (13) can be written as

$$(X^2 - 5)^2 + 80 = \left(X^2 - 5 + 4\sqrt{-5}\right)\left(X^2 - 5 - 4\sqrt{-5}\right).$$

Let $\mathbb{K} = \mathbb{Q}\left(\sqrt{-5}\right)$. The class group is C_2, and we need to know the behaviour of the primes $2, 3$ and 5, which is as follows:

$$\langle 2\rangle = \mathfrak{p}_2^2, \quad \langle 3\rangle = \mathfrak{p}_3\overline{\mathfrak{p}}_3, \quad \langle 5\rangle = \mathfrak{p}_5^2, \quad \mathfrak{p}_5 = \left\langle\sqrt{-5}\right\rangle,$$

where $\mathfrak{p}_2, \mathfrak{p}_3$ are non-principal ideals, the bar denotes complex conjugation, and we have the relations

$$\overline{\mathfrak{p}}_2 = \mathfrak{p}_2, \quad \mathfrak{p}_2\mathfrak{p}_3 = \left\langle 1 + \sqrt{-5}\right\rangle, \quad \mathfrak{p}_3^2 = \left\langle 2 - \sqrt{-5}\right\rangle.$$

If $\mathfrak{p}$ is a prime ideal dividing both $\left\langle X^2 - 5 + \sqrt{-5}\right\rangle$ and $\left\langle X^2 - 5 - 4\sqrt{-5}\right\rangle$, then it divides $\left\langle\left(X^2 - 5 + 4\sqrt{-5}\right) - \left(X^2 - 5 - 4\sqrt{-5}\right)\right\rangle = \left\langle 8\sqrt{-5}\right\rangle = \mathfrak{p}_2^6\mathfrak{p}_5$. It follows by (13) that there exist $a, b, c, d \in \{0, 1\}$ and an integral ideal $\mathfrak{a}$ such that

$$\left\langle X^2 - 5 + 4\sqrt{-5}\right\rangle = \mathfrak{p}_2^a\mathfrak{p}_3^b\overline{\mathfrak{p}_3}^c\mathfrak{p}_5^d\mathfrak{a}^2.$$

Taking norms we have $6Y^2 = 2^a3^{b+c}5^d(N\mathfrak{a})^2$, hence $a = 1$, $(b, c) = (1, 0)$ or $(0, 1)$, $d = 0$. Notice that $\operatorname{ord}_{\mathfrak{p}_2}(X^2 - 1) \geq 6$, and $\operatorname{ord}_{\mathfrak{p}_2}\left(-4 + 4\sqrt{-5}\right) = 5$, so that we find $\operatorname{ord}_{\mathfrak{p}_2}(\mathfrak{a}) = 2$. Hence if $\mathfrak{a}$ is principal we may write $\mathfrak{a} = \left\langle 2A + 2B\sqrt{-5}\right\rangle$, and if $\mathfrak{a}$ is non-principal, then $\mathfrak{a}/\mathfrak{p}_2$ is principal, and we may write $\mathfrak{a} = \mathfrak{p}_2\left\langle A + B\sqrt{-5}\right\rangle$, where in both cases $A, B \in \mathbb{Z}$. We define $p = 0$ if $\mathfrak{a}$ is principal, and $p = 1$ if $\mathfrak{a}$ is non-principal. Then $\mathfrak{a}^2 = 2^{2-p}\left\langle A^2 - 5B^2 + 2AB\sqrt{-5}\right\rangle$.

3.1. The case $(b,c)=(1,0)$

In the case $(b,c)=(1,0)$, going from ideals to generators, we thus have

$$\pm 2^p\left(\frac{X^2-5}{4}+\sqrt{-5}\right)=\left(1+\sqrt{-5}\right)\left(A^2-5B^2+2AB\sqrt{-5}\right).$$

Comparing real and imaginary parts we get

$$\pm 2^p\frac{X^2-5}{4}=A^2-10AB-5B^2, \tag{14}$$

$$\pm 2^p=A^2+2AB-5B^2. \tag{15}$$

Then $4\times(14)+5\times(15)$ yields

$$2^pX^2=9A^2-30AB-45B^2=(3A-5B)^2-70B^2.$$

Thus the next field to study is $\mathbb{L}=\mathbb{Q}\left(\sqrt{70}\right)$. Its class group is C_2, a fundamental unit is $251+30\sqrt{70}$, and the primes $2,3,5$ and 7 behave as follows:

$$\langle 2\rangle=\mathfrak{p}_2^2,\quad \langle 3\rangle=\mathfrak{p}_3\mathfrak{q}_3,\quad \langle 5\rangle=\mathfrak{p}_5^2,\quad \mathfrak{p}_5=\left\langle 25+3\sqrt{70}\right\rangle,\quad \langle 7\rangle=\mathfrak{p}_7^2,$$

where $\mathfrak{p}_2,\mathfrak{p}_3,\mathfrak{q}_3,\mathfrak{p}_7$ are non-principal prime ideals. If $\mathfrak{p}$ is a prime ideal dividing both
$\left\langle 3A-5B+B\sqrt{70}\right\rangle$ and $\left\langle 3A-5B-B\sqrt{70}\right\rangle$, then it divides
$\left\langle\left(3A-5B+B\sqrt{70}\right)+\left(3A-5B-B\sqrt{70}\right)\right\rangle=\langle 2(3A-5B)\rangle$ and also
$\left\langle\left(3A-5B+B\sqrt{70}\right)-\left(3A-5B-B\sqrt{70}\right)\right\rangle=\left\langle 2B\sqrt{70}\right\rangle$.
Since A and B are relatively prime (by (15)) we find that $\mathfrak{p}$ divides $2,3,5$ or 7. It follows that there exist $a,b,c,d,e\in\{0,1\}$ and an integral ideal $\mathfrak{b}$ such that

$$\left\langle 3A-5B+B\sqrt{70}\right\rangle=\mathfrak{p}_2^a\mathfrak{p}_3^b\mathfrak{q}_3^c\mathfrak{p}_5^d\mathfrak{p}_7^e\mathfrak{b}^2.$$

Taking norms we find that $2^pX^2=2^a3^{b+c}5^d7^e(N\mathfrak{b})^2$, and thus that $a=p=0$ or 1, $b=c=0$ or 1, $d=e=0$. Since $\left\langle 3A-5B+B\sqrt{70}\right\rangle$, $\mathfrak{p}_3\mathfrak{q}_3$ and $\mathfrak{b}^2$ are principal ideals, it follows that $a=p=0$. Then it also follows that in (14) and (15) the $\pm$ sign is a $+$, because $A^2+2AB-5B^2=-1$ has no solutions.

If $\mathfrak{b}$ is principal, we may write $\mathfrak{b} = \left\langle E + F\sqrt{70}\right\rangle$, and if $\mathfrak{b}$ is non-principal, then $\mathfrak{b}\mathfrak{p}_2$ is principal, and we may write $\mathfrak{b}\mathfrak{p}_2 = \left\langle E + F\sqrt{70}\right\rangle$, where in both cases E, F are unknown integers. We let $q = 0$ if $\mathfrak{b}$ is principal, and $q = 1$ if $\mathfrak{b}$ is non-principal. Then, going from ideals to generators, we can write

$$\pm 2^q \left(3A - 5B + B\sqrt{70}\right) = \left(251 + 30\sqrt{70}\right)^n 3^b \left(E^2 + 70F^2 + 2EF\sqrt{70}\right),$$

where also n can be taken to be in $\{0, 1\}$. As A and B are defined up to sign, we may take the $\pm$ sign to be a $+$.

3.1.1. The case $n = 0$

In the case $n = 0$, writing $e = 2^{-q}3^b$ (thus $e \in \{1, 3, \frac{1}{2}, \frac{3}{2}\}$), and comparing coefficients, we obtain

$$\begin{aligned} 3A - 5B &= e(E^2 + 70F^2), \\ B &= 2eEF, \end{aligned}$$

hence

$$A = \frac{1}{3}e(E^2 + 10EF + 70F^2).$$

We substitute these expressions into (15), and thus get

$$\begin{aligned} E^4 + 32E^3F + 180E^2F^2 + 2240EF^3 \\ +4900F^4 = \frac{9}{e^2}. \end{aligned}$$

We prefer to substitute $E = U - 2V, F = V$, to get somewhat smaller coefficients. Notice that $U, V \in \mathbb{Z}$. This gives the Thue equations

$$U^4 + 24U^3V + 12U^2V^2 + 1872UV^3 + 900V^4 = m \tag{16}$$

for $m = \frac{9}{e^2} \in \{1, 4, 9, 36\}$. Below we will treat these Thue equations.

3.1.2. The case $n = 1$

In the case $n = 1$, again writing $e = 2^{-q}3^b$ (thus $e \in \{1, 3, \frac{1}{2}, \frac{3}{2}\}$), and comparing coefficients, we find

$$\begin{aligned} 3A - 5B &= e(251E^2 + 4200EF + 17570F^2), \\ B &= e(30E^2 + 502EF + 2100F^2), \end{aligned}$$

hence

$$A = \frac{1}{3}e(401E^2 + 6710EF + 28070F^2).$$

We substitute these expressions into (15), and thus get

$$192481E^4 + 6441632E^3F + 80841780E^2F^2 + 450914240EF^3 + 943156900F^4 = \frac{9}{e^2}.$$

We prefer to substitute $E = 3U - 31V, F = -\frac{5}{14}U + \frac{26}{7}V$, to get much smaller coefficients. Notice that $U, V \in \mathbb{Z}$. This gives in fact the Thue equations (16), but this time with $m = \frac{1764}{e^2} \in \{196, 784, 1764, 7056\}$.

In a forthcoming section we will prove the following result.

Theorem 4. *The Thue equations*

$$f_2(U,V) = U^4 + 24U^3V + 12U^2V^2 + 1872UV^3 + 900V^4 = m, \quad m \in \{1, 4, 9, 36, 196, 784, 1764, 7056\} \tag{17}$$

have only the solutions $(U,V) = \pm(1,0)$ *at* $m = 1$.

The solutions $(U,V) = \pm(1,0)$ lead to $m = 1$, $n = 0$, $e = 3$, $(E,F) = \pm(1,0)$, $(A,B) = (1,0)$, $(X,Y) = (\pm 3, \pm 4)$, and finally to $(x,n) = (-1,0), (-1,3), (0,0), (0,3)$.

3.2. The case $(b,c) = (0,1)$

In the case $(b,c) = (0,1)$, going from ideals to generators, we have

$$\pm 2^p \left(\frac{X^2 - 5}{4} + \sqrt{-5} \right) = \left(1 - \sqrt{-5}\right)\left(A^2 - 5B^2 + 2AB\sqrt{-5}\right).$$

Comparing real and imaginary parts we get

$$\pm 2^p \frac{X^2 - 5}{4} = A^2 + 10AB - 5B^2, \tag{18}$$

$$\mp 2^p = A^2 - 2AB - 5B^2. \tag{19}$$

Then $4 \times (18) - 5 \times (19)$ yields

$$\mp 2^p X^2 = A^2 - 50AB - 5B^2 = (A - 25B)^2 - 630B^2.$$

Again we work in $\mathbb{L} = \mathbb{Q}\left(\sqrt{70}\right)$. If $\mathfrak{p}$ is a prime ideal dividing both $\left\langle A - 25B + 3B\sqrt{70}\right\rangle$ and $\left\langle A - 25B - 3B\sqrt{70}\right\rangle$, then as above we see that $\mathfrak{p}$ divides $2, 3, 5$ or 7. It follows that there exist $a, b, c, d, e \in \{0, 1\}$ and an integral ideal $\mathfrak{b}$ such that

$$\left\langle A - 25B + 3B\sqrt{70}\right\rangle = \mathfrak{p}_2^a \mathfrak{p}_3^b \mathfrak{q}_3^c \mathfrak{p}_5^d \mathfrak{p}_7^e \mathfrak{b}^2.$$

Taking norms we find that $2^p X^2 = 2^a 3^{b+c} 5^d 7^e (N\mathfrak{b})^2$, and thus that $a = p = 0$ or 1, $b = c = 0$ or 1, $d = e = 0$. Since $\left\langle 3A - 5B + B\sqrt{70}\right\rangle$, $\mathfrak{p}_3\mathfrak{q}_3$ and $\mathfrak{b}^2$ are principal ideals, it follows that $a = p = 0$. Then it also follows that in (18) and (19) the $\pm$ and $\mp$ signs respectively are $-$ and $+$, because $A^2 - 2AB - 5B^2 = -1$ has no solutions.

If $\mathfrak{b}$ is principal, we may write $\mathfrak{b} = \left\langle E + F\sqrt{70}\right\rangle$, and if $\mathfrak{b}$ is non-principal, then $\mathfrak{b}\mathfrak{p}_2$ is principal, and we may write $\mathfrak{b}\mathfrak{p}_2 = \left\langle E + F\sqrt{70}\right\rangle$, where in both cases E, F are unknown integers. We let $q = 0$ if $\mathfrak{b}$ is principal, and $q = 1$ if $\mathfrak{b}$ is non-principal. Then, going from ideals to generators, we can write

$$\begin{aligned} &\pm 2^q \left(A - 25B + 3B\sqrt{70}\right) \\ &= \left(251 + 30\sqrt{70}\right)^n 3^b \left(E^2 + 70F^2 + 2EF\sqrt{70}\right), \end{aligned}$$

where also n can be taken to be in $\{0, 1\}$. As A and B are defined up to sign, we may take the $\pm$ sign to be a $+$.

3.2.1. The case $n = 0$

In the case $n = 0$, writing $e = 2^{-q}3^b$ (thus $e \in \{1, 3, \frac{1}{2}, \frac{3}{2}\}$), and comparing coefficients, we obtain

$$A - 25B = e(E^2 + 70F^2), \quad 3B = 2eEF,$$

hence

$$eA = \frac{1}{3}e(3E^2 + 50EF + 210F^2), \quad B = \frac{2}{3}eEF.$$

We substitute these expressions into (19), and thus get

$$E^4 + 32E^3F + \frac{1180}{3}E^2F^2 + 2240EF^3 + 4900F^4 = \frac{1}{e^2}.$$

We prefer to substitute $E = \frac{1}{3}U - \frac{19}{3}V, F = V$, to get somewhat smaller coefficients. Notice that $U, V \in \mathbb{Z}$. This gives the Thue equations

$$U^4 + 20U^3V + 234U^2V^2 + 2492UV^3 - 2423V^4 = m \tag{20}$$

for $m = \frac{81}{e^2} \in \{9, 36, 81, 324\}$. Below we will treat these Thue equations.

3.2.2. The case $n = 1$

In the case $n = 1$, again writing $e = 2^{-q}3^b$ (thus $e \in \{1, 3, \frac{1}{2}, \frac{3}{2}\}$), and comparing coefficients, we find

$$\begin{aligned} A - 25B &= e(251E^2 + 4200EF + 17570F^2), \\ 3B &= e(30E^2 + 502EF + 2100F^2), \end{aligned}$$

hence

$$\begin{aligned} A &= \frac{1}{3}e(1503E^2 + 25150EF + 105210F^2), \\ B &= \frac{1}{3}e(30E^2 + 502EF + 2100F^2). \end{aligned}$$

We substitute these expressions into (19), and thus get

$$\begin{aligned} 240481E^4 + 8048032E^3F + \frac{303005980}{3}E^2F^2 \\ +563362240EF^3 + 1178356900F^4 = \frac{1}{e^2}. \end{aligned}$$

We prefer to substitute $E = \frac{5}{3}U - \frac{221}{3}V, F = -\frac{1}{5}U + \frac{44}{5}V$, to get much smaller coefficients. Notice that $U, V \in \mathbb{Z}$. This gives in fact the Thue equations (20), but this time with $m = \frac{2025}{e^2} \in \{225, 900, 2025, 8100\}$.

In a forthcoming section we will prove the following result.

Theorem 5. *The Thue equations*

$$\begin{aligned} f_3(U,V) = U^4 + 20U^3V + 234U^2V^2 + 2492UV^3 - 2423V^4 = m, \\ m \in \{9, 36, 81, 225, 324, 900, 2025, 8100\} \end{aligned} \tag{21}$$

have only the solutions $(U, V) = \pm(3, 0)$ *at* $m = 81$, *and* $(U, V) = \pm(1, 1)$ *at* $m = 324$, *and* $(U, V) = \pm(17, -1)$ *at* $m = 8100$.

The solutions $(U, V) = \pm(3, 0)$ lead to $m = 81$, $e = 1$, $n = 0$, $(E, F) = \pm(1, 0)$, $(A, B) = (1, 0)$, $(X, Y) = (\pm 1, \pm 4)$, and finally to $(x, n) = (-1, 1), (-1, 2), (0, 1), (0, 2)$. The solutions $(U, V) = \pm(1, 1)$ lead to $m = 324$, $e = \frac{1}{2}$, $n = 0$, $(E, F) = \pm(-6, 1)$, $(A, B) = (3, -2)$, $(X, Y) = (\pm 17, \pm 116)$, and finally to $(x, n) = (-15, -7), (-15, 10), (14, -7), (14, 10)$. The solutions $(U, V) = \pm(17, -1)$ lead to $m = 8100$, $e = \frac{1}{2}$, $n = 1$, and then to non-integral F. This completes the proof of Theorem 2.

4. Solving the Thue equations

In this section we finally prove Theorems 3, 4 and 5, thus completing also the proofs of Theorems 1 and 2. Using the program package KANT (PC-DOS version) we obtain the following results:

Equation	*Solutions*	*486PC-CPU-time (sec)*
$f_1(x,y)=25$	$(-3,-1),(3,1)$	38
$f_1(x,y)=100$	–	33
$f_1(x,y)=625$	$(-5,-2),(-5,0),(5,0),(5,2)$	71
$f_1(x,y)=2500$	–	110
$f_2(x,y)=1$	$(-1,0),(1,0)$	15
$f_2(x,y)=4$	–	9
$f_2(x,y)=9$	–	9
$f_2(x,y)=36$	–	10
$f_2(x,y)=196$	–	10
$f_2(x,y)=784$	–	18
$f_2(x,y)=1764$	–	28
$f_2(x,y)=7056$	–	23
$f_3(x,y)=9$	–	15
$f_3(x,y)=36$	–	10
$f_3(x,y)=81$	$(-3,0),(3,0)$	23
$f_3(x,y)=225$	–	29
$f_3(x,y)=324$	$(-1,-1),(1,1)$	45
$f_3(x,y)=900$	–	36
$f_3(x,y)=2025$	–	60
$f_3(x,y)=8100$	$(-17,1),(17,-1)$	198

References

[M] L. J. Mordell, On the integer solutions of $y(y+1) = x(x+1)(x+2)$, *Pacific Journal of Mathematics* **13** (1963), 1347–1351.

[P] Á. Pintér, A note on the diophantine equation $\binom{x}{4} = \binom{y}{2}$, *Publ. Math. Debrecen* **47** (1995), 411–415.

[TW] N. Tzanakis and B. M. M. de Weger, On the practical solution of the Thue equation, *J. Number Theory* **31** (1989), 99–132.

[dW] B. M. M. de Weger, A binomial diophantine equation, *Quart. J. Math. Oxford 2nd Ser.* **47** (1996), 221–231.

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(Received October 7, 1996)