Products of cyclic permutations

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Abstract. If a cyclic permutation π of length n is a product of cyclic permutations $\pi_1, \dots, \pi_r$ of lengths $n_1, \dots, n_r$ respectively then $n \leqq 1-r+\sum_{i=1}^{r} n_i$. We consider the case where equality holds. Using graph theoretic means we show in how many different ways a cyclic permutation π can be written as a product $\pi=\pi_1 \dots \pi_r$ of cyclic permutations as above.

1. Introduction and preliminaries

It is well known (see, for example, [1], [2], [6]) that there are n^{n-2} different ways in which an n-cycle can be written as a product of $n-1$ transpositions. This may be proved either by recursion, using a formula of Abel, or by showing that the number one is looking for equals the number of labelled trees on n vertices, which by a result of Cayley equals n^{n-2}. We shall give a generalization of this result.

Let $P=\{\pi_1, \dots, \pi_r\}$ be a subset of S_n, the symmetric group on $N(n):=\{1, 2, \dots, n\}$. Define $G(P)$ to be the graph with vertex set $N(n)$ where distinct vertices i, j are connected by an edge if and only if there exists $k \in \{1, 2, \dots, r\}$ and $s \in \mathbf{N}$ such that $j=i\pi_k^s$. It is clear that $G(P)$ is connected if and only if $\langle P \rangle$ is transitive on $N(n)$.

Now let $P=\{\pi_1, \dots, \pi_r\}$ be a set of cyclic permutations of lengths $n_1, \dots, n_r$ respectively, and suppose $\langle P \rangle$ is transitive on $N(n)$. Then $G(P)$ is connected, and for every $i \in N(n)$ there exists $\pi \in P$ with $i \in \operatorname{supp}(\pi)=\{j \in N(n) \mid j\pi \neq j\}$. It follows that without loss of generality we can assume that the elements of P are numbered in such a way that whenever $j>1$ then there exists $a \in N(n)$ and $i_j<j$ such that $a \in \operatorname{supp}(\pi_j) \cap \operatorname{supp}(\pi_{i_j})$. Thus we get

$$n \leqq n_1+\sum_{i=2}^{r}(n_i-1)=1-r+\sum_{i=1}^{r} n_i.$$

In the following we shall consider the extremal case where $n_1, \dots, n_r$ are such that $n=1-r+\sum_{i=1}^{r} n_i$. The graph theoretic concepts we use shall be defined as in [3].

Theorem 1. *Let $P=\{\pi_1, \dots, \pi_r\}$ be a set of cyclic permutations in S_n with $n>1$, where π_i has length $n_i>1$, and such that $n=1-r+\sum_{i=1}^{r} n_i$. Then the following are equivalent.*

(1) $A_n \leqq \langle P \rangle$.

(2) *$\langle P\rangle$ is transitive on $N(n)$.*
(3) *$\pi_1\pi_2\ldots\pi_r$ is an n-cycle.*
(4) *$G(P)$ is a connected graph.*
(5) *$G(P)$ is a connected graph whose blocks are cliques.*
(6) *$G(P)$ is a connected graph which has r blocks which are complete graphs on $n_1, \ldots, n_r$ vertices respectively.*

Proof. First assume (2). Clearly $G(P)$ is connected. Let $a\in N(n)$ be such that a lies in two of the cycles in P. Then a is a cutpoint of $G(P)$. For, otherwise we could replace a by $n+1$ in one of the cycles, and the altered set P' would generate a transitive group on $N(n+1)$, hence $n+1\leqq 1-r+\sum_{i=1}^{r} n_i$, which is a contradiction. Thus if B is a block of $G(P)$ then there exists $\pi\in P$ such that the vertex set of B is contained in supp (π). But the subgraph of $G(P)$ induced on supp (π) is a block, and hence we have (6). Clearly, whenever a block is a complete graph then it is a clique, therefore (6) implies (5). Trivially, (5) implies (4).

Consider the permutations $\varrho=(1, 2, \ldots, k)$ and $\sigma=(k, k+1, \ldots, n)$. Note that $\sigma\varrho=(1, 2, \ldots, k, k+1, \ldots, n)$ and $\varrho\sigma=(1, 2, \ldots, k-1, k+1, \ldots, n, k)$, furthermore $\varrho\sigma^{-1}\varrho^{-1}\sigma=(k-1, k, k+1)$, hence also $(\sigma\varrho)^{-z}\varrho\sigma^{-1}\varrho^{-1}\sigma(\sigma\varrho)^z=(k+z-1, k+z, k+z+1)$ for $2-k\leqq z\leqq n-1-k$. Let $G(P)$ be connected. Then, using this remark together with induction and the fact that A_{n-1} and S_{n-1} are maximal subgroups of A_n and S_n respectively, we get that $\langle P\rangle=A_n$ or $\langle P\rangle=S_n$. Thus (4) implies (1). Trivially, (1) implies (2) (note that if $n=2$ then we have $P=S_2$). Similarly it follows that (6) implies (3), and again (3) implies (2) trivially. ∎

2. The number of decompositions into products

We now consider the number of different ways in which an n-cycle can be factorized as in (3) of Theorem 1 into a product of shorter cycles. If $n=1+\sum_{k\geqq 2} m_k(k-1)$ and $\pi\in S_n$ then we define $f(n; m_2, m_3, \ldots)$ to be the number of m-tuples $(\pi_1, \ldots, \pi_m)$ with $m=\sum_{k\geqq 2} m_k$ of cyclic permutations π_i where there are m_k permutations of cycle length k such that $\pi=\pi_1\ldots\pi_m$. This is clearly independent of π. If furthermore σ is a permutation of $\{n\in\mathbf{N}|n\geqq 2\}$ then denote by $F^*(n; m_2, m_3, \ldots)$ the number of m-tuples $(\pi_1, \ldots, \pi_m)$ as above with the further property that $\pi_1, \ldots, \pi_{m_{\sigma(2)}}$ have length $\sigma(2)$, that $\pi_{m_{\sigma(2)}+1}, \ldots, \pi_{m_{\sigma(2)}+m_{\sigma(3)}}$ have length $\sigma(3)$, and so forth, that is, we first have all $\sigma(2)$-cycles, then all $\sigma(3)$-cycles, and so forth. It is clear that this is independent of π, and we shall see that it is also independent of σ. In the following, note that Σ and Π shall mean $\sum_{k\geqq 2}$ and $\prod_{k\geqq 2}$ respectively.

Theorem 2. *Let $m=\Sigma m_k$. Then*

$$F(n;\ m_2, m_3, \ldots) = n^{m-1}\cdot m!\cdot\left(\Pi(m_k!)\right)^{-1}.$$

Proof. Let C be the set of all m-tuples $(\pi_1, \ldots, \pi_m)$ of cyclic permutations in S_n of which there are m_k of length k and such that $\pi_1\ldots\pi_m$ is an n-cycle. As the number of n-cycles in S_n is $(n-1)!$, we get $|C|=(n-1)!.F(n; m_2, m_3, \ldots)$. By Theorem 1, we have that $G(P)$ with $P=\{\pi_1, \ldots, \pi_m\}$ is a connected graph which for $k\geqq 2$

has m_k blocks which are complete graphs on k vertices. KODI HUSIMI [4] (see also [5, Cor. 4.2.2.]) proved that the number of such (labelled) graphs is

$$(n-1)!\cdot n^{m-1}\cdot(\Pi(((k-1)!)^{m_k}m_k!)^{-1}.$$

Furthermore it is easy to see that the number of elements of C which give rise to the same labelled $G(P)$ is equal to $(\Pi((k-1)!)^{m_k})\cdot m!$, as the vertices of each block on k vertices can be formed into a k-cycle in $(k-1)!$ ways, and there are $m!$ ways of ordering the cycles. Thus

$$|C| = (n-1)!\cdot n^{m-1}\cdot(\Pi(m_k!))^{-1}\cdot m!,$$

which gives the result. ▮

Theorem 3. *Let* $m=\Sigma m_k$. *Then*

$$F^*(n;\ m_2, m_3, \dots) = n^{m-1}.$$

PROOF. Let D be the set of all m-tuples $(\pi_1, \dots, \pi_m)$ of cyclic permutations in S_n where $\pi_1, \dots, \pi_{m_{\sigma(2)}}$ are $\sigma(2)$-cycles, $\pi_{m_{\sigma(2)}+1}, \dots, \pi_{m_{\sigma(2)}+m_{\sigma(3)}}$ are $\sigma(3)$-cycles, and so forth. As above, we get $|D|=(n-1)!\cdot F^*(n;\, m_2, m_3, \dots)$. Again, consider the labelled graph $G(P)$ where $P=\{\pi_1, \dots, \pi_m\}$. This time, the number of elements of D which give rise to the same labelled $G(P)$ is equal to $\Pi((k-1)!)^{m_k}\cdot\Pi(m_k!)$, as we now only can order the cycles of the same length. We thus get $|D|=(n-1)!\cdot n^{m-1}$ which proves the theorem. ▮

In [1], JÓZSEF DÉNES poses as a problem to find a natural bijection between the $(n-1)$-tuples of transpositions whose product is a given n-cycle π and the set of labelled trees on n vertices. This generalization of his theorem seems to indicate that there may not be such a bijection, as the cardinalities of the corresponding sets will differ in general. The numbers $F(n;\, m_2, m_3, \dots)$ and in particular $F^*(n;\, m_2, m_3, \dots)$ can even be expressed more simply than the corresponding numbers of labelled graphs. This suggests to ask the following question. Is there a natural bijection between the set of m-tuples of cyclic permutations counted by $F^*(n;\, m_2, m_3, \dots)$ and the set of mappings from an $(m-1)$-element set to an n-element set?

References

[1] JÓZSEF DÉNES, The representation of a permutation as the product of a minimal number of transpositions, and its connection with the theory of graphs. *Publ. Math. Inst. Hung. Acad. Sci.* **4** (1959) 63—71.

[2] O. DZIOBEK, Eine Formel der Substitutionstheorie. Sitzungsber. *Berliner Math. Ges.* **16** (1917), 64—67.

[3] FRANK HARARY, Graph theory. Addison-Wesley, Reading, Menlo Park, *London*. Don Mills (3rd printing, 1972).

[4] KODI HUSIMI, Note on Mayer's theory of cluster integrals. *J. Chem. Phys.* **18** (1950) 682—684.

[5] J. W. MOON, Counting labelled trees, *Canadian Math. Monographs* **1**, Can. Math. Congress (1970).

[6] PROBLEM E. 3058. *Amer. Math. Monthly* **91** (1984) 516 and **93** (1986) 820—821.

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