Existence of maximal elements and equilibria

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Abstract. The objective of this note is to extend the results available in the literature by proving a general theorem on the existence of maximal elements and giving applications.

1. Introduction

Suppose that K is a subset of a Hausdorff topological vector space E. Then each binary relation P on K gives rise to a multivalued map $T: K \to 2^K$ as follows: if $x \in K$, then $T(x)=\{y \in K: (x, y) \in P\}$. Conversely, if $T: K \to 2^K$ is a multivalued map, then a binary relation P on K is defined as follows: $(x, y) \in P$ if and only if $y \in T(x)$. A point x_0 of K is said to be a maximal element of the map $T: K \to 2^K$, with respect to the binary relation defined above, if $T(x_0)=\emptyset$.

Theorems on the existence of maximal elements have important applications in mathematical economics. For example, in recent work in general equilibrium theory without ordered preferences, the existence of an equilibrium in an abstract economy or qualitative game is often proved by constructing a map P, which may bc construed as a 'preference map', on a compact convex subset K of a Hausdorff topological vector space and then by showing that there exists a point x_0 such that $P(x_0)=\emptyset$. Sueh a point may, therefore, also be regarded as an 'equilibrium point' of the map P.

Following the path-breaking work of GALE and MAS-COLELL [1975] several theorems on the existence of maximal elements have been proved by BORGLIN and KEIDING [1976], TOUSSAINT [1984], YANNELIS and PRABHAKAR [1983] and others. These theorems have been used to generalize the work of Gale and Mas-Colell. The object of this note is to extend the results available in the literature by proving a general theorem on the existence of maximal elements and giving applications.

The note is organized as follows. In section 2, the main theorem is proved. In section 3, this theorem is used to prove some general theorems on the existence of equilibria in games and economies.

2. Maximal Elements

In what follows we assume that E is a real Hausdorff linear topological space and K a compact convex subset of E.

Let L be a class of maps. Then by a L-map we mean a map belonging to the class L. A map $F: K \to 2^K$ is said to be L-majorized if for each $x \in K$ such that $F(x) \neq \emptyset$, there is an open neighbourhood U_x of x and a L-map T_x such that for every z in U_x we have $F(z)\subseteqq T_x(z)$. The map T_x is said to be a L-majorant of F at x.

Definition 1. A map $F: K\to 2^K$ is said to be *weakly B-majorized* if for each $x\in K$ with $F(x)\neq\emptyset$ there exists an open neighbourhood U_x of x and a convexvalued map $T_x: K\to 2^K$ with $z\notin T_x(z)$ for all $z\in K$ and $F(z)\subseteqq T_x(z)$ for $z\in U_x$ and the family $\{T_x, U_x: x\in K$ with $F(x)\neq\emptyset\}$ satisfies the condition that for each $x\in K$ with $F(x)\neq\emptyset$ there is $y\in K$ such that $x\in\bigcap\{\text{Int } T_z^{-1}(y): x\in U_z\}$ where Int A denotes the interior of the set A.

Although the concept of weak B-majorization is cumbersome, it will enable us to obtain neat results of some generality as we shall see in the sequel.

Our work in this paper is based on the following fixed point theorem of TARAFDAR [1977] (and also by BEN-EL-MECHAIEKH et. al. [1982] in equivalent form) which is a generalization of a fixed point theorem of BROWDER [1968, Theorem 1].

Theorem 1. *Let K be a compact convex subset of a Hausdorff linear topological space E. Suppose that $F: K\to 2^K$ is a map such that the following conditions are satisfied:*

a) *for each $x\in K$, $F(x)$ is nonempty and convex;*

b) *for each $x\in K$, there is $y\in K$ such that $x\in\text{Int } F^{-1}(y)$. Then there exists $x_0\in K$ such that $x_0\in F(x_0)$.*

We now prove the following theorem.

Theorem 2. *Let $F: K\to 2^K$ be weakly B-majorized. Then there exists x_0 in K such that $F(x_0)=\emptyset$.*

PROOF. Suppose that the theorem is false. Then $F(x)\neq\emptyset$ for all $x\in K$. Since F is weakly B-majorized, for each x, there exist an open neighbourhood U_x and a convex-valued map T_x with $z\notin T_x(z)$ for every z, such that for every $u\in U_x$ we have $F(u)\subseteqq T_x(u)$. As K is compact, there exists a finite set of points $\{x_1, \dots, x_n\}$ such that $K=\bigcup_{i=1}^{n} U_{x_i}$. Let $\{V_{x_1}, \dots, V_{x_n}\}$ be a closed refinement of $\{U_{x_1}, \dots, U_{x_n}\}$. Define $G: K\to 2^K$ by

$$G(x) = \bigcap_{j\in i(x)} T_{x_j}(x) \quad \text{where} \quad i(x) = \{j\in(1, 2, \dots, n): x\in V_{x_j}\}.$$

Clearly, for each $x\in K$, $G(x)$ is convex and nonempty. We prove now that for each $x\in K$ there is $y\in K$ such that $x\in\text{Int } G^{-1}(y)$.

By virtue of our definition of a weakly B-majorized map, we have that for each $x\in K$ there is $y\in K$ such that $x\in\bigcap\{\text{Int } T_z^{-1}(y): x\in U_z\}$. Hence, for every $j\in i(x)$, $x\in\text{Int } T_{x_j}^{-1}(y)$ as $j\in i(x)$ implies that $x\in V_{x_j}\subseteqq U_{x_j}$. Therefore $x\in\bigcap_{j\in i(x)}\text{Int } T_{x_j}^{-1}(y)= =\text{Int}\bigcap_{j\in i(x)} T_{x_j}^{-1}(y)$ which is an open set. Now let $A(x)=O_x\cap\{\text{Int}\bigcap_{j\in i(x)} T_{x_j}^{-1}(y)\}$ where $O_x=K\setminus\bigcup_{j\notin i(x)} V_{x_j}$. Thus $A(x)$ is an open set. Now, $A(x)\subseteqq G^{-1}(y)$. Indeed if $u\in A(x)$ then $i(u)\subseteqq i(x)$ and $u\in\text{Int}\bigcap_{j\in i(x)} T_{x_j}^{-1}(y)\subseteqq\text{Int}\bigcap_{j\in i(u)} T_{x_j}^{-1}(y)$. Thus $u\in \in G^{-1}(y)$ and hence $x\in A(x)\subseteqq\text{Int } G^{-1}(y)$.

Hence, all conditions of TARAFDAR's [1977] fixed-point theorem are satisfied and we conclude that G has a fixed-point x_0. This implies that $x_0 \in G(x_0) = \bigcap_{j \in i(x_0)} T_{x_j}(x_0)$ a contradiction since for each $j \in i(x_0)$, $x_0 \notin T_{x_j}(x_0)$ by the definition of the map T_{x_j}. The contradiction proves the theorem. q.e.d.

Remark 1. Suppose that the maps T_x in the above proof are not convex-valued but satisfy the condition that z does not belong to the convex hull of $T_x(z)$ for each $z \in K$. Then the proof proceeds along the same lines except that we now define $G(x) = \bigcap_{j \in i(x)} \text{co } T_{x_j}(x)$ where co $T_{x_j}(x)$ is the convex hull of $T_{x_j}(x)$.

In the definition of weak B-majorization we now require $x \in \bigcap \{\text{Int co } T_z^{-1}(y)\colon x \in U_z$ and $F(z) \neq \emptyset\}$.

Corollary 1. *Let $F\colon K \to 2^K$ be B-majorized. Then there exists a point $x \in K$ such that $F(x) = \emptyset$, where $T\colon K \to 2^K$ is said to be a B-map if*

a) *$T(x)$ is convex for each $x \in K$;*

b) *$x \notin T(x)$ for each $x \in K$;*

c) *$T^{-1}(y) = \{x \in K\colon y \in T(x)\}$ is open in K for every $y \in K$.*

PROOF. It is easy to verify that a B-majorized map is weakly B-majorized. q.e.d.

The above corollary is due to TOUSSAINT [1984, Theorem 2.2]. A similar result has also been obtained by YANNELIS and PRABHAKAR [1983, Corollary 5.1].

3. On Games: Applications of Weak B-Majorization

We proceed now to the problem of the existence of equilibrium points of games and generalized games or abstract economies.

We consider a (possibly infinite) set I of agents. With each $i \in I$ we associate a nonempty choice set or strategy set X_i contained in an arbitrary topological vector space. Let $X = \prod_{i \in I} X_i$. The preferences of agent i are given by a multivalued map $P_i\colon X \to 2^{X_i}$ such that $x_i \notin P_i(x)$ where $x \in X$, and x_i is the ith coordinate of x. The collection $(X_i, P_i)_{i \in I}$ is said to be a qualitative game.

Definition 2. Let $(X_i, P_i)_{i \in I}$ be a qualitative game. A point $\bar{x} \in X$ said to be an *equilibrium point* of the game if $P_i(\bar{x}) = \emptyset$ for all $i \in I$.

The following theorem on the existence of equilibrium points for games has been proved by TOUSSAINT [1984].

Theorem 3. *Let $(X_i, P_i)_{i \in I}$ be a qualitative game such that for each $i \in I$ the following conditions are satisfied:*

a) *X_i is compact and convex;*

b) *P_i is B-majorized;*

c) *$\{x \in X\colon P_i(x) \neq \emptyset\}$ is open in X.*

The the game (X_i, P_i) $i \in I$ has an equilibrium point.

In the above theorem, the preferences of agent i are assumed to be B-majorized. It is interesting to ask if the requirement in the definition of a B-majorized map that the point inverses are open can be relaxed. To show that this can be done we need the following generalization of a B-map.

Definition 3. A map $F: K\to 2^K$ is said to be an *I-map* if the following conditions are satisfied:

a) $F(x)$ is convex for each $x\in K$;

b) $x\notin F(x)$ for each $x\in K$;

c) for each $x\in K$ there exists a point $y\in K$ such that $x\in \operatorname{Int} F^{-1}(y)$.

The following theorem is a consequence of Theorem 1.

Theorem 4. *Let K be a compact convex subset of a Hausdorff topological vector space E. Suppose that $F: K\to 2^K$ is an I-map. Then there exists a point x_0 such that $F(x_0)=\emptyset$.*

We now have the following generalization of Theorem 3.

Theorem 5. *Let $(X_i, P_i)_{i\in I}$ be a qualitative game such that for each $i\in I$ the following conditions are satisfied:*

a) *X_i is compact and convex;*

b) *P_i is I-majorized;*

c) *$\{x\in X: P_i(x)\neq\emptyset\}$ is open in X.*

Then the game $(X_i, P_i)_{i\in I}$ has an equilibrium point.

PROOF. Let $I(x)=\{i: P_i(x)\neq\emptyset\}$.
For each i, define $\hat{P}_i: X\to 2^X$ by $\hat{P}_i(x)=\{y\in X: \Pi_i(y)\in P_i(x)\}$ where $\Pi_i: X\to X_i$ is the natural projection. We now define a map $P: X\to 2^X$ by

$$P(x)=\begin{cases}\bigcap_{i\in I(x)} \hat{P}_i(x) & \text{if } I(x)\neq\emptyset\\ \emptyset & \text{if } I(x)=\emptyset\end{cases}$$

We note that $P(x)\neq\emptyset$ whenever $I(x)\neq\emptyset$.

We claim that P is weakly B-majorized. Suppose that $P(x)\neq\emptyset$.

Then $\hat{P}_i(x)\neq\emptyset$ for some i (indeed $\hat{P}_i(x)\neq\emptyset$ for all $i\in I(x)$) and hence $P_i(x)\neq\emptyset$ for the same i. Hence, there exists an I-map $T_{i,x}: X\to 2^{X_i}$ and an open neighbourhood $U_{i,x}$ of x such that $T_{i,x}(z)\supseteq P_i(z)$ for every $z\in U_{i,x}$. We now define a map

$$\hat{T}_{i,x}: X\to 2^X \quad\text{by}\quad \hat{T}_{i,x}(u)=\{y\in X: \Pi_i(y)\in T_{i,x}(u)\},\quad u\in X.$$

Then, clearly $\hat{T}_{i,x}(u)$ is convex for each $u\in X$ and $u\notin\hat{T}_{i,x}(u)$ since each $T_{i,x}$ is an I-map.

Moreover, for $y\in X$,

$$\begin{aligned}\hat{T}_{i,x}^{-1}(y) &= \{u\in X: y\in\hat{T}_{i,x}(u)\}\\ &= \{u\in X: \Pi_i(y)\in T_{i,x}(u)\}\\ &= \{u\in X: u\in T_{i,x}^{-1}(\Pi_i(y))\}\\ &= T_{i,x}^{-1}(\Pi_i(y))\ldots\ldots\ldots(*).\end{aligned}$$

Set $T_x=\hat{T}_{i,x}$.

Let $W_x = W_i \cap U_{i,x}$ where $W_i = \{x: P_i(x) \neq \emptyset\}$. Then for each $z \in W_x$, we have

$$P(z) = \bigcap_{j \in I(z)} \hat{P}_j(z) \subseteqq \hat{P}_i(z) \subseteqq \hat{T}_{i,x}(z) = T_x(z).$$

Thus we have shown that for each x with $P(x) \neq \emptyset$ we have a map $T_x: X \to 2^X$ and an open neighbourhood W_x of x such that for each $u \in X$, $T_x(u)$ is convex, $u \notin T_x(u)$ and $P(z) \subseteqq T_x(z)$ for each $z \in W_x$. We need to show that for each $x \in X$, there exists a point $y \in X$ such that $x \in \cap \{\text{Int}\, T_z^{-1}(y): x \in W_z\}$. Let $A(x) = \{z \in X: x \in W_z\}$. By construction, each W_z and T_z are of the forms $W_z = W_k \cap U_{k,z}$ and $T_z = \hat{T}_{k,z}$ for some $k \in I$.

Let $J = \{i: W_z \subseteqq U_{i,z}$ and $z \in A(x)\}$. Then as each $T_{i,z}$ is an I-map, for each $x \in X$, there exists $y_i \in X_i$ such that $x \in \text{Int}\, T_{i,z}^{-1}(y_i)$. Thus for each $i \in J$ there exists y_i such that $x \in \bigcap_{i \in J} \text{Int}\, T_{i,z}^{-1}(y_i)$. Let y be any point in X such that $\Pi_i(y) = y_i$ for every $i \in J$. Then by (*)

$$x \in \bigcap_{i \in J} \text{Int}\, T_{i,z}^{-1}(y_i) = \bigcap_{i \in J} \text{Int}\, \hat{T}_{i,z}^{-1}(y) = \cap \{\text{Int}\, T_z^{-1}(y):\ x \in W_z\}.$$

This proves that P is weakly B-majorized. Thus Theorem 2 implies that there exists x_0 such that $P(x_0) \neq \emptyset$. Hence, $P_i(x) = \emptyset$ for all i and x_0 is an equilibrium point of the game. q.e.d.

We now turn our attention to generalized qualitative games. Let $(X_i, P_i)_{i \in I}$ be a qualitative game. Let $A_i: X \to X_i$ be a multivalued mapping such that $A_i(x) \neq \emptyset$ for all $x \in X$. A_i is called the ith agent's constraint correspondence. The family $(X_i, P_i, A_i)_{i \in I}$ is called a generalized qualitative game or abstract economy.

Definition 4. A point $\bar{x} \in X$ is said to be an *equilibrium point of the qualitative game* $(X_i, P_i, A_i)_{i \in I}$ if for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and $P_i(\bar{x}) \cap A_i(\bar{x}) = \emptyset$.

For the existence of equilibrium points of generalized games Toussaint [1984] has proved the following theorem, the statement of which is included for convenience of presentation.

Theorem 6. *Let $(X_i, P_i, A_i)_{i \in I}$ be a generalized qualitative game satisfying for each $i \in I$:*

i) *X_i is compact and convex;*
ii) *$x_i \notin \text{co}\, P_i(x)$ for $x \in X$;*
iii) *$P_i(x)$ is open in X_i for each $x \in X$;*
iv) *$P_i^{-1}(y_i)$ is open in X for each $y_i \in X_i$;*
v) *A_i has a closed graph in $X \times X_i$;*
vi) *There exists $B_i: X \to 2^{X_i}$ such that*
  a) *$B_i(x)$ is non-empty and convex for each $x \in X$;*
  b) *$B_i^{-1}(y_i)$ is open in X for each $y_i \in X_i$;*
  c) *$\text{cl}\, B_i(x) = A_i(x)$.*

Then the game $(X_i, P_i, A_i)_{i \in I}$ has an equilibrium point.

We now generalize the above theorem by replacing the condition that both P_i and B_i have open inverse images by a weaker condition.

Theorem 7. *Assume that all the conditions of Theorem* 6, *except* iv) *and* vi b) *hold. Further, assume that* iv) *and* vi b) *are replaced by a weaker condition:* (**) *for* *each x, such that* $(P_i \cap B_i)\ (x) = P_i(x) \cap B_i(x) \neq \emptyset$, *there exists* y_i *such that* $x \in \operatorname{Int}[(P_i \cap B_i)^{-1}(y_i)]$ *and for each x such that* $P_i(x) \cap B_i(x) = \emptyset$, *there exists* y_i *such that* $x \in \operatorname{Int} B^{-1}(y_i)$.

Then the game $(X_i, P_i, A_i)_{i \in I}$ *has an equilibrium point.*

PROOF. The general line of argument is the same as in TOUSSAINT [1984]. For each $i \in I$, define:

$$Q_i = \begin{cases} P_i(x) \cap B_i(x) & \text{if} \quad x_i \in A_i(x) \\ B_i(x) & \text{if} \quad x_i \notin A_i(x) \end{cases}$$

and

$$T_i = \begin{cases} \operatorname{co} P_i(x) \cap B_i(x) & \text{if} \quad x_i \in A_i(x) \\ B_i(x) & \text{if} \quad x_i \notin A_i(x) \end{cases}$$

where $Q_i, T_i \colon X \to 2^{X_i}$.

We claim that the game $(X_i, Q_i)_{i \in I}$ satisfies the conditions of Theorem 5.

Condition a) is trivially satisfied. Next we observe that $W_i = \{x \in X \colon Q_i(x) \neq \emptyset\} = \{x \in X \colon x_i \notin A_i(x)\} U \{x \in X \colon P_i(x) \cap B_i(x) \neq \emptyset\}$. The set $\{x \in X \colon x_i \notin A_i(x)\}$ is open because A_i has a closed graph. Now let $x_0 \in \{x \in X \colon P_i(x) \cap B_i(x) \neq \emptyset\}$. By condition (**) there exists y_i such that $x_0 \in \operatorname{Int}[(P_i \cap B_i)^{-1}(y_i)] = V$. Hence, $y_i \in (P_i \cap B_i)(z)$ for every z in the open neighbourhood V of x_0. This proves that the set $\{x \in X \colon P_i(x) \cap B_i(x) \neq \emptyset\}$ is open. Hence, W_i is an open set and the condition c) of Theorem 5 is satisfied.

It remains to prove that condition b) is satisfied. We prove this by considering two cases.

Case 1. Let $x \in X$ and suppose that $x_i \notin A_i(x)$. This implies that $Q_i(x) \neq \emptyset$. Since A_i has a closed graph, there exists a neighbourhood U_x of x such that if $z \in U_x$, then $z_i \notin A_i(z)$.

First, we suppose that $(P_i \cap B_i)(x) = \emptyset$. Then condition (**) implies that there exists y_i such that $x \in \operatorname{Int} B_i^{-1}(y_i) = V_x$, say. It is clear that $y_i \in B_i(z)$ for every $z \in V_x$. Let $W_x = U_x \cap V_x$. Then if $z \in W_x$, $y_i \in B_i(z) = T_i(z)$ by the definition of T_i as $z_i \notin A_i(z)$. Hence $z \in T_i^{-1}(y_i)$, i.e. $W_x \subseteq T_i^{-1}(y_i)$. Now it follows that $x \in \operatorname{Int} T_i^{-1}(y_i)$.

Next we suppose that $P_i(x) \cap B_i(x) \neq \emptyset$. Then the condition (**) implies that there is a point y_i such that $x \in \operatorname{Int} P_i^{-1}(y_i) \cap \operatorname{Int} B_i^{-1}(y_i) \subseteq \operatorname{Int} B_i^{-1}(y_i) = \hat{V}_x$, say. Now arguing exactly as we did above we can show that $x \in W_x = U_x \cap \hat{V}_x \subseteq T_i^{-1}(y_i)$ which implies that $x \in \operatorname{Int} T_i^{-1}(y_i)$.

Case 2. Let $x \in X$ and suppose that $x_i \in A_i(x)$, with $Q_i(x) \neq \emptyset$. Then $(P_i \cap B_i)(x) \neq \emptyset$ and condition (**) implies that there exists y_i such that $x \in \operatorname{Int}(P_i \cap B_i)^{-1}(y_i) \subseteq \operatorname{Int}(\operatorname{co} P_i \cap B_i)^{-1}(y_i) \subseteq \operatorname{Int} T_i^{-1}(y_i)$, the last inequality being a consequence of the fact that $(\operatorname{co} P_i \cap B_i)^{-1}(y) \subseteq T_i^{-1}(y_i)$ for all $y_i \in X_i$.

Hence we have proved that for each x, there exists y_i such that $x \in \operatorname{Int} T_i^{-1}(y_i)$.

Since $Q_i(x) \subseteq T_i(x)$ for all $x \in X$, and T_i is clearly a convex-valued map such that $x_i \notin T_i(x)$ for all $x \in X$, we have proved that T_i is an I-map and that each Q_i is I-majorized. Hence, condition b) of Theorem 5 holds.

By Theorem 5 we conclude that the game $(X_i, Q_i)_{i \in I}$ has anli equibrium point. This implies that there exists a point $\bar{x} \in X$ such that $\bar{x}_i \in A_i(\bar{x})$ and $P_i(\bar{x}) \cap B_i(\bar{x}) = \emptyset$. Conditions iii) and vi) c) now imply that $P_i(\bar{x}) \cap A_i(\bar{x}) = \emptyset$. q.e.d.

Remark 2. If $P_i^{-1}(y_i)$ and $B_i^{-1}(y_i)$ are open for all y_i, then clearly condition (**) is satisfied so that Theorem 7 is a generalization of Theorem 6.

Remark 3. If it is assumed that $B_i^{-1}(y_i)$ is open for all y_i then condition (**) may be replaced by the following condition:

(***) for each x, such that $(P_i \cap B_i)(x) \neq \emptyset$ there exists y_i such that

$$x \in \operatorname{Int}(P_i \cap B_i)^{-1}(y_i).$$

The preceding material on games can be used to prove the existence of Walrasian general equilibrium. In TARAFDAR and MEHTA [1985] it is shown that Theorem 7 can be used to generalize TOUSSAINT's theorem [1984] on the existence of equilibria in economies with infinitely many commodities and without ordered preferences.

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