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Reflections of Riemannian manifolds

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1. Introduction

In the present paper we discuss mainly reflections of Riemannian manifolds. An isometry mapping of a connected Riemannian manifold onto itself is said to be a reflection if the fixed point set separates the manifold. The properties of such special involutive isometries can be applied in studies dealing with transformation groups of Riemannian manifolds.

As we shall see below, the notion of reflection can be introduced in the more general context of differentiable manifolds in similar form. At first the groups generated by such reflections were studied by J. L. KOSZUL (see [11; p. 45–50]). He proved that if a group acts properly on a simply connected manifold, then it is a Coxeter group. Later on M. W. DAVIS has generalized this theorem in Riemannian manifolds omitting the assumption that the manifold is simply connected (see [4]). The previously mentioned transformation groups have been discussed also by E. STRAUME (see [14]).

The purpose of this paper is to give a characterization of these reflections. In the second chapter we study topological aspects of reflections. We shall show that considering a reflection in a symmetric space, the set of fixed points, which is always a 1–codimensional submanifold, has at most two components, furthermore, if the fixed point set is not connected, then the components are homeomorphic to each other. Later on we shall prove that for any reflection the cardinal number of the components of the fixed point set is not greater than the order of the fundamental group of the Riemannian manifold. In the third chapter we shall characterize Riemannian manifolds admitting reflections by their Ricci tensor.

In this paper M (respectively N) denotes an n–dimensional connected Riemannian (respectively differentiable) manifold where $n > 0$. The tangent space of M at a point q will be denoted by T_qM. Furthermore, $<,>_q$ will denote the metric tensor of M at q. Let $\varrho : M \times M \to R$ be the distance function of M where R is the set of all real numbers.

If $\psi : N \to N$ is a differentiable map, let $N(\psi)$ denote the set of points which are left fixed by ψ, that is, $N(\psi) = \{p \in N \mid \psi(p) = p\}$. The tangent linear map of ψ will be denoted by $T\psi$.

At first let us consider the notion of reflection in differentiable manifolds.

Definition 1. Let N be a connected differentiable manifold. A diffeomorphism $\psi : N \to N$ is called a reflection if $N \setminus N(\psi)$ is disconnected and $\psi^2 = id_N$. (Clearly, id_N denotes the identical map of N.)

We shall essentially use the following result concerning reflections (see [11; p. 45]).

If $\psi : N \to N$ is a reflection, then $N(\psi)$ is a closed submanifold of codimension one, and $N \setminus N(\psi)$ has exactly two components which are carried onto each other by ψ.

We can easily show that if $\psi : N \to N$ is a reflection, then there exists a Riemannian metric on N such that ψ is an isometry with respect to this metric.

Let us consider a metric $(\, , \,)$ on N. We can define a new Riemannian metric $< \, , \, >_q : T_qN \times T_qN \to R$ where

$$< v, w >_q = (v, w)_q + (T_q\psi(v), \ T_q\psi(w))_{\psi(q)}$$

for any vectors $v, w \in T_qN$. Clearly, ψ is an isometry with respect to $< \, , \, >$.

Let $\varphi : M \to M$ be an isometry such that $M(\varphi)$ is not empty. It is known that each connected component of $M(\varphi)$ is a closed totally geodesic submanifold of M (see [10; p. 59]). Consider a point $p \in M(\varphi)$ and an open ball U_δ in T_pM of radius δ around the zero vector such that the restriction of the exponential map at p to this ball $exp_p|U_\delta$ is a diffeomorphism. The image of U_δ by exp_p is the open ball $B_\delta(p)$ of radius δ around p. It is well–known that the restriction of φ to $B_\delta(p)$ can be expressed in the following form (see [7; p. 61])

$$\varphi|B_\delta(p) = \exp_p \circ T_p\varphi \circ (\exp_p|U_\delta)^{-1}. \tag{1}$$

Definition 2. An isometry $\varphi : M \to M$ mapping a connected Riemannian manifold M onto itself is said to be a reflection if $M \setminus M(\varphi)$ is not connected.

It can be easily seen that a reflection φ is always involutive. Since $M \setminus M(\varphi)$ is not connected, $M(\varphi)$ has a 1–codimensional component F. Hence, regarding a point p of $F, T_p\varphi$ is equal to the orthogonal reflection on the hyperplane T_pF. Thus the equality (1) given above implies $\varphi^2 = id_M$. Threfore φ is also a reflection in the sense of diffeomorphisms. It follows from this that $M(\varphi)$ is a closed 1–codimensional submanifold of M (see [11; p. 45]). Obviously, for any point q left fixed by φ the tangent linear map $T_q\varphi$ is equal to the orthogonal reflection on the hyperplane $T_qM(\varphi)$.

2. Upper bounds for the number of the components of the fixed point set

Taking a submanifold F of a Riemannian manifold M, the normal vector bundle of F in M will be denoted by $\perp(F)$. Let $O : F \to \perp(F)$ be the zero cross section of $\perp(F)$. It is well–known that $O(F)$ is a submanifold of $\perp(F)$ diffeomorphic to F. Regarding the fixed point set of a reflection the following statement is true.

Proposition 1. *Let $\varphi : M \to M$ be a reflection of a Riemannian manifold M. Then the normal vector bundle of $M(\varphi)$ is trivial.*

PROOF. Let us consider a component F of $M(\varphi)$. As we have mentioned above F is a (connected) closed 1–codimensional submanifold of M. Hence there is a connected open neighborhood V of $O(F)$ in the normal bundle $\perp(F)$ such that the restriction of the exponential map to V $Exp|V$ is a diffeomorphism (see [5; p. 114–115]) and $Exp(V) \cap M(\varphi) = F$ holds.

Let us take a point p of F and an open ball $B_\delta(p)$ of radius δ around p such that $B_\delta(p)$ is included in $Exp(V)$ and the equality (1) holds. Clearly, F separates $B_\delta(p)$. It follows from (1) that the two components of $B_\delta(p) \setminus F$ are contained in distinct components of $M \setminus M(\varphi)$. Hence, it can be easily seen that $Exp(V) \setminus F$ has also exactly two components.

Since $Exp|V$ is a diffeomorphism, $O(F)$ separates V. Therefore $\perp(F) \setminus O(F)$ has exactly two components. Fixing one of them, a map $X : F \to \perp(F)$ can be defined by assigning to any point $q \in F$ the unit vector $X(q)$ included in this component. It can be shown that X is a differentiable normal vector field on F. This implies that $\perp(F)$ is a trivial vector bundle which completes the proof. □

Later on we shall apply the following simple statement.

Lemma 1. *Let $\varphi : M \to M$ be an isometry such that $M(\varphi)$ is not connected. If p_1, p_2 are two points belonging to distinct components F_1, F_2 of $M(\varphi)$ and $\gamma : [0,1] \to M$ is a minimizing geodesic between these points, then γ has no point in $M(\varphi)$ apart from p_1 and p_2.*

PROOF. It is clear that $\varphi \circ \gamma$ is another geodesic connecting p_1, p_2 the arc length of which is equal to $\varrho(p_1, p_2)$. This fact implies that p_2 is a cut point of p_1 along γ (see [10; p. 60]).

Assuming that there is a real number $s \in (0,1)$ having the property $\gamma(s) \in M(\varphi)$, we obtain immediately that $\gamma(s)$ is also a cut point of p_1 along γ, but this is impossible. □

It is known that if M is a symmetric space, then M has no geodesic intersecting itself (see [9; p. 144]). The following theorem characterizes reflections of Riemannian manifolds having the property given above.

Theorem 1. *Let M be a connected complete Riemannian manifold such that M has no geodesic intersecting itself and let $\varphi : M \to M$ be a reflection. Then $M(\varphi)$ has at most two components. Furthermore, if $M(\varphi)$ is disconnected, then the components are homeomorphic to each other.*

PROOF. Let us assume that $M(\varphi)$ is not connected. At first we shall prove that $M(\varphi)$ has exactly two components. Let F_1, F_2 be distinct components of $M(\varphi)$ and let p_1 be a point of F_1. Since M is complete and F_2 is a closed submanifold, applying the Hopf–Rinow theorem it can be easily seen that there is a geodesic $\gamma : [0 : 1] \to M$ joining F_2 and p_1 the arc length of which is equal to $\varrho(F_2, p_1)$. Let p_2 denote the initial point of γ and consider the geodesic $\widetilde{\gamma} : R \to M$ obtained by infinitely extending γ.

Since γ is a minimizing geodesic between F_2 and p_1, the tangent vector of γ at 0, denoted by $\dot{\gamma}(0)$, is normal to F_2 (see [1; p. 151]). $T_{p_2}\varphi$ is equal to the orthogonal reflection on $T_{p_2}F_2$, hence $T_{p_2}\varphi(\dot{\gamma}(0)) = -\dot{\gamma}(0)$ holds. Therefore $\varphi \circ \widetilde{\gamma}$ is a geodesic such that $\varphi \circ \widetilde{\gamma}(t) = \widetilde{\gamma}(-t)$ for any $t \in R$. Considering the point $p_1 = \widetilde{\gamma}(1)$ left fixed by φ, we obtain $\widetilde{\gamma}(1) = \widetilde{\gamma}(-1)$. M has no geodesic intersecting itself, thus $\widetilde{\gamma}$ is a periodic geodesic such that $\widetilde{\gamma}(t) = \widetilde{\gamma}(t+2)$ for any $t \in R$. Finally, notice that the tangent vector $\dot{\gamma}(1)$ is normal to F_1.

Suppose that there is a further component F_3 different from F_1, F_2. Let us regard a minimizing geodesic $\sigma : [0,1] \to M$ joining F_3 and $p_1(\sigma(0) \in F_3)$. As we have seen above, by infinitely extending σ we obtain a periodic geodesic $\widetilde{\sigma}$ and the tangent vector $\dot{\sigma}(1)$ is also orthogonal to $T_{p_1}F_1$. Therefore $\widetilde{\sigma}$ differs from $\widetilde{\gamma}$ only in parametrization, since $\dot{\widetilde{\sigma}}(1)$ is parallel to $\dot{\widetilde{\gamma}}(1)$. But $\widetilde{\sigma}$ includes a point of F_3, $\widetilde{\gamma}$ includes a point of F_2, thus by Lemma 1 our supposition is impossible.

At last we shall prove that the components F_1, F_2 of $M(\varphi)$ are homeomorphic to each other. Applying Proposition 1, let us take a differentiable normal unit vector field on F_2 denoted by X and the continuous function $\mu : F_2 \to R, \quad q \in F_2 \to \mu(q) = \varrho(q, F_1)$. Considering an arbitrary point $p_2 \in F_2$, take the geodesic $\gamma : [0, \mu(p_2)] \to M$ having the properties $\gamma(0) = p_2, \dot{\gamma}(0) = X(p_2)$. The endpoint of γ is included in F_1, therefore we can define a map $\psi : F_2 \to F_1$ by assigning the endpoint $\gamma(\mu(p_2))$ to p_2. It is clear that ψ is a bijective map. Since μX is a continuous normal vector field on F_2 and $\psi = Exp \circ \mu X$ holds, ψ is continuous. Similarly, we can see that the inverse map of ψ is also continuous which verifies our assertion. □

Since a symmetric space is always complete, our assertions given in Theorem 1 are valid in symmetric spaces.

Later on we shall apply the known result that a closed 1–codimensional connected submanifold separates a simply connected manifold. This fact can be proved as follows. Let N be a connected differentiable manifold and F be a closed 1–codimensional submanifold of N. If $\gamma : [0,1] \to N$

is a differentiable curve which is transversal to F, then we can see that the number of elements of the set $[0,1] \cap \gamma^{-1}(F)$ is finite. This number is called the intersection number of γ with respect to F. We need the following statement concerning homotopic curves.

Theorem. *(see [6; p. 78]). If $\gamma, \sigma : [0,1] \to N$ are homotopic differentiable curves which are transversal to F, then the parity of the intersection number of γ is equal to the parity of the intersection number of σ (with respect to F).*

Corollary 1. *Let N be a simply connected differentiable manifold and F be a connected closed 1–codimensional submanifold of N. Then F separates N and F is orientable.*

PROOF. Let us consider a local coordinate system in N which is adapted to the submanifold F. Using this coordinate system, we can define a differentiable curve $\gamma : [-1,1] \to N$ such that γ is transversal to F, the intersection number of γ with respect to F is equal to 1 and the points $p = \gamma(-1)$, $q = \gamma(1)$ are not contained in F.

Suppose that there exists a smooth curve $\sigma : [-1,1] \to N$ joining p and q such that σ has no point in F. It follows from this that σ is transversal to F. Since N is simply connected, γ and σ are homotopic curves. However, by the Theorem given above the intersection number of σ with respect to F is uneven, therefore our supposition is impossible.

Since p and q cannot be connected by a curve in $N \setminus F$, F separates N. (It is clear that $N \setminus F$ has exactly two components.) It follows from this that F is an orientiable submanifold (see [8; p. 107]). □

Proposition 2. *Let $\varphi : N \to N$ be a reflection of a simply connected manifold N. Then $N(\varphi)$ is a connected orientable submanifold.*

PROOF. The previous results imly that $N(\varphi)$ is orientable. It remained only to prove that $N(\varphi)$ is connected.

Let us suppose that $N(\varphi)$ has two distinct components F_1, F_2. In this case we can take a smooth curve $\gamma : [0,3] \to N$ such that γ is transversal to F_1 and F_2, $\gamma(1) \in F_1$, $\gamma(2) \in F_2$ and for any $t \in [0,1) \cup (1,2) \cup (2,3]$ the point $\gamma(t)$ is included in $N \setminus (F_1 \cup F_2)$. If we fix arbitrary two points from $\gamma(0)$, $\gamma\left(\frac{3}{2}\right)$, $\gamma(3)$, then as in the proof of Corollary 1 we can see that they cannot be joined by a curve in $N \setminus (F_1 \cup F_2)$. Hence, $N \setminus N(\varphi)$ has at least three components. Howeover, in the Introduction we have mentioned that $N \setminus N(\varphi)$ has exactly two components (see [11; p. 45]), therefore the supposition is impossible. □

Reflections in universal covering spaces of Riemannian manifolds

If M is a connected Riemannian manifold, then later on $\widetilde{M}$ will denote the universal Riemannian covering space of M, which is unique up to isometries. Furthermore, let $\omega : \widetilde{M} \to M$ denote an universal covering map of this simply connected Riemannian manifold $\widetilde{M}$ onto M.

Let us recall some notions which are important in the theory of covering spaces.

Definition 3. Let $\gamma : [0,1] \to M$, $\widetilde{\gamma} : [0,1] \to \widetilde{M}$ be continuous curves. $\widetilde{\gamma}$ is said to be a curve lying over γ if $\omega \circ \widetilde{\gamma} = \gamma$ is satisfied.

Definition 4. Let $\psi : M \to M$, $\widetilde{\psi} : \widetilde{M} \to \widetilde{M}$ be isometries of M and $\widetilde{M}$. $\widetilde{\psi}$ is called an isometry lying over ψ if $\omega \circ \widetilde{\psi} = \psi \circ \omega$ holds.

Later on we shall use the following basic results concerning the notions given above.

Lemma 2. *Let $\gamma : [0,1] \to M$ be a continuous curve in M ($\gamma(0) = p$). Fixing a point $\widetilde{p}$ of the set $\omega^{-1}(p)$, there exists exactly one curve $\widetilde{\gamma} : [0,1] \to \widetilde{M}$ lying over γ such that $\widetilde{\gamma}(0) = \widetilde{p}$ (see [12; p. 131]).*

Let $\gamma, \sigma : [0,1] \to M$ be homotopic continuous curves in M. If $\widetilde{\gamma}, \widetilde{\sigma} : [0,1] \to \widetilde{M}$ are curves lying over γ, respectively σ such that $\widetilde{\gamma}(0) = \widetilde{\sigma}(0)$, then $\widetilde{\gamma}$ and $\widetilde{\sigma}$ are also homotopic (see [12; p. 131–132]).

Lemma 3. *All the isometries of $\widetilde{M}$ lying over id_M form a group $\mathcal{F}$ which is isomorphic with the fundamental group of M (see [13; p. 197]).*

Furthermore, if $\mathcal{G}$ is a group of isometries in M, then all the isometries of $\widetilde{M}$ lying over the elements of $\mathcal{G}$ form a group $\widetilde{\mathcal{G}}$ such that $\mathcal{F}$ is a normal subgroup of $\widetilde{\mathcal{G}}$, and the quotient group $\widetilde{\mathcal{G}}|\mathcal{F}$ is isomorphic with $\mathcal{G}$ (see [2; p. 179–180]).

We shall prove that among the isometries lying over a reflection there exists at least one reflection.

Let $\varphi : M \to M$ be a reflection. Since ω is a local isometry, the components of $\omega^{-1}(M(\varphi))$ are closed 1–codimensional totally geodesic submanifolds in $\widetilde{M}$. Let us fix a point $\widetilde{p}$ of $\omega^{-1}(M(\varphi))$. We can define a map $\widetilde{\varphi} : \widetilde{M} \to \widetilde{M}$ attached to this point $\widetilde{p}$.

Take a point $\widetilde{q}$ of $\widetilde{M}$ and consider a continuous curve $\widetilde{\gamma} : [0,1] \to \widetilde{M}$ such that $\widetilde{\gamma}(0) = \widetilde{p}$ and $\widetilde{\gamma}(1) = \widetilde{q}$. Obviously, the initial point of $\varphi \circ \omega \circ \widetilde{\gamma}$ coincides with $\omega(\widetilde{p}) = p$. By Lemma 2 there exists exactly one curve $\widetilde{\sigma} : [0,1] \to \widetilde{M}$ lying over $\varphi \circ \omega \circ \widetilde{\gamma}$ such that $\widetilde{\sigma}(0) = \widetilde{p}$ is satisfied. We assign the endpoint of $\widetilde{\sigma}$ to $\widetilde{q}$ by $\widetilde{\varphi}$, that is, $\widetilde{\varphi}(\widetilde{q}) = \widetilde{\sigma}(1)$.

Applying the second assertion of Lemma 2, it can be easily seen that $\widetilde{\varphi}$ is well-defined, that is, the assignment given above does not depend on the choice of the curve $\widetilde{\gamma}$ joining $\widetilde{p}$ and $\widetilde{q}$.

Proposition 3. *The previously constructed map $\widetilde{\varphi}$ (attached to a fixed point $\widetilde{p}$ of $\omega^{-1}(M(\varphi))$) is a reflection of $\widetilde{M}$ lying over the reflection φ, and $\widetilde{M}(\widetilde{\varphi}) = \widetilde{F}$ where $\widetilde{F}$ denotes the component of $\omega^{-1}(M(\varphi))$ including $\widetilde{p}$.*

PROOF. a) At first we show that $\widetilde{\varphi}$ is a local isometry. Take an arbitrary point $\widetilde{q}$ of $\widetilde{M}$. Let us regard a positive number δ such that the restriction of $exp_{\tilde{q}}$ to the open ball in $T_{\tilde{q}}\widetilde{M}$ with radius δ around the zero vector $O_{\tilde{q}}$ (respectively the restriction of $exp_{\tilde{\varphi}(\tilde{q})}$ to the open ball in $T_{\tilde{\varphi}(\tilde{q})}\widetilde{M}$ with radius δ around $O_{\tilde{\varphi}(\tilde{q})}$) is a diffeomorphism. Obviously, the ranges of these restricted maps are open balls of radius δ around $\widetilde{q}$ and $\widetilde{\varphi}(\widetilde{q})$. Moreover, choosing δ, we require that $\omega|_{B_\delta(\tilde{q})}$ and $\omega|_{B_\delta(\tilde{\varphi}(\tilde{q}))}$ are isometries.

Let us consider the restriction of $\widetilde{\varphi}$ to $B_\delta(\widetilde{q})$. It is clear that a curve in $\widetilde{M}$ lying over a geodesic in M is always a geodesic. Hence, if in order to determine the map $\widetilde{\varphi}|_{B_\delta(\tilde{q})}$ we use minimizing geodesics starting at $\widetilde{q}$, then we obtain that

$$\widetilde{\varphi}|_{B_\delta(\tilde{q})} = (\omega|_{B_\delta(\tilde{\varphi}(\tilde{q}))})^{-1} \circ \varphi \circ \omega|_{B_\delta(\tilde{q})}$$

is satisfied. It follows from this that $\widetilde{\varphi}|_{B_\delta(\tilde{q})}$ is an isometry, thus our first statement is true.

b) We shall now prove that $\widetilde{\varphi}^2 = id_{\widetilde{M}}$. Take a point $\widetilde{q}$ of $\widetilde{M}$, and apply the notations given in the definition of $\widetilde{\varphi}$. In order to assign $\widetilde{\varphi}^2(\widetilde{q})$ let us consider $\widetilde{\sigma}$ from continuous curves joining $\widetilde{p}$ and $\widetilde{\varphi}(\widetilde{q})$. Since φ is a reflection, we have $\varphi^2 = id_M$. Therefore the equality $\omega \circ \widetilde{\sigma} = \varphi \circ \omega \circ \widetilde{\gamma}$ implies $\varphi \circ \omega \circ \widetilde{\sigma} = \omega \circ \widetilde{\gamma}$. According to this fact, the curve lying over $\varphi \circ \omega \circ \widetilde{\sigma}$ and starting at $\widetilde{p}$ coincides with $\widetilde{\gamma}$, thus $\widetilde{\varphi}^2(\widetilde{q}) = \widetilde{\gamma}(1) = \widetilde{q}$.

Since $\widetilde{\varphi}$ is involutive, it follows from the assertion a) that $\widetilde{\varphi}$ is an isometry of $\widetilde{M}$.

c) Finally, we show that $\widetilde{\varphi}$ is a reflection lying over φ and $\widetilde{M}(\widetilde{\varphi}) = \widetilde{F}$. Using again the equality $\varphi \circ \omega \circ \widetilde{\gamma} = \omega \circ \widetilde{\sigma}$, for the points $\widetilde{q} = \widetilde{\gamma}(1)$ and $\widetilde{\varphi}(\widetilde{q}) = \widetilde{\sigma}(1)$ we obtain that $\omega \circ \widetilde{\varphi}(\widetilde{q}) = \varphi \circ \omega(\widetilde{q})$ holds. Hence, $\widetilde{\varphi}$ is an isometry lying over φ.

It is clear that $\widetilde{M}(\widetilde{\varphi}) \subset \omega^{-1}(M(\varphi))$. If $\widetilde{q}$ is a point of $\widetilde{F}$, to assign $\widetilde{\varphi}(\widetilde{q})$ we can take a curve $\widetilde{\gamma}$ in $\widetilde{F}$ joining $\widetilde{p}$ and $\widetilde{q}$. Therefore $\widetilde{\varphi}(\widetilde{q}) = \widetilde{q}$ wich means that $\widetilde{F}$ is a component of $\widetilde{M}(\widetilde{\varphi})$. $\widetilde{M}$ is simply connected, thus by Corollary 1 $\widetilde{F}$ separates $\widetilde{M}$, that is, $\widetilde{\varphi}$ is a reflection. Applying Proposition 2, we obtain that $\widetilde{M}(\widetilde{\varphi}) = \widetilde{F}$ wich *completes the proof.* □

It is known that if $\psi : M \to M$ is an isometry of a connected Riemannian manifold M and p is a fixpoint of ψ, then $T_p\psi$ uniquely determines ψ (see [7; p. 62]).

Hence, considering Proposition 3, it can be easily seen that if $\widetilde{p}_1, \widetilde{p}_2$ are two points of the same component $\widetilde{F}$ of $\omega^{-1}(M(\varphi))$, and $\widetilde{\varphi}_1, \widetilde{\varphi}_2$ are the reflections lying over φ which are constructed to $\widetilde{p}_1, \widetilde{p}_2$ (respectively) as above, then $\widetilde{\varphi}_1 = \widetilde{\varphi}_2$.

The previous results immediately imply the following assertion.

Corollary 2. *The number of all reflections in $\widetilde{M}$ lying over a reflection φ of M is equal to the number of all components of $\omega^{-1}(M(\varphi))$.*

Applying the statements written above, we can give an upper bound for the number of the components of the fixed point set. The theorem given below is a genaralization of Proposition 2.

Theorem 2. *Let $\varphi : M \to M$ be a reflection. Then the cardinal number of the components of $M(\varphi)$ is not greater than the order of the fundamental group of M.*

PROOF. Let us regard a reflection $\widetilde{\varphi}$ of $\widetilde{M}$ lying over φ. It can be easily seen that if $\widetilde{\mu}$ is an element of $\mathcal{F}$, where $\mathcal{F}$ denotes the group of isometries in $\widetilde{M}$ lying over id_M, then $\widetilde{\varphi} \circ \widetilde{\mu}$ (respectively $\widetilde{\mu} \circ \widetilde{\varphi}$) is also an isometry in $\widetilde{M}$ lying over φ. By Lemma 3 the set of all isometries in $\widetilde{M}$ lying over φ coincides with the coset $\widetilde{\varphi} \circ \mathcal{F}$. Since the fundamental group of M is isomorphic with $\mathcal{F}$, the number of reflections included in $\widetilde{\varphi} \circ \mathcal{F}$ is not greater than the order of the fundamental group of M.

By Corollary 2 it follows from this that the number of all components of $\omega^{-1}(M(\varphi))$ is not greater than the order of $\mathcal{F}$. ω is a continuous map, therefore it is clear that the cardinal number of the components of $M(\varphi)$ is not greater than the cardinal number of the components of $\omega^{-1}(M(\varphi))$. This implies that our theorem is true. □

Concerning Theorem 2, it can be easily shown that this theorem is not true in general for involutive isometries. Furthermore, we can find a 2–dimensional Riemannian manifold M and a reflection $\varphi : M \to M$ such that the cardinality of the components of $M(\varphi)$ is equal to countable infinite.

3. Characterization of reflections by the Ricci tensor

In this chapter we shall study reflections of Riemannian manifolds by using their Ricci tensor. Considering a Riemannian manifold M, let $\mathbf{V}M$ denote the vector space of all differentiable vector fields on M and

∇ denote the Levi–Civita connection of M. Denoting the curvature tensor of M by R, for any elements $X, Y, Z \in \mathbf{V}M$ we have

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where $[X, Y]$ is the Lie bracket of X and Y.

The introduction of the following notion seems to be useful for us.

Definition 5. An isometry $\varphi : M \to M$ mapping a connected Riemannian manifold M onto itself is said to be a symmetry if φ is involutive and $M(\varphi)$ is not empty.

Obviously, reflections investigated in the second chapter are special symmetries. Later on we say that M is symmetric to a closed connected totally geodesic submanifold F, if there exists a symmetry φ in M such that F is a component of $M(\varphi)$. It is well–known that a Riemannian manifold is (locally) symmetric to every point if and only if the covariant derivative of the curvature tensor vanishes everywhere (see [7; p. 163]).

Let $\varphi : M \to M$ be a symmetry of a complete Riemannian manifold M. Let us consider a component F of $M(\varphi)$ and an arbitrary point p of F. Since $\varphi^2 = id_M$, by the chain rule of tangent linear maps $T_p\varphi$ is also involutive. Hence, $T_p\varphi$ is equal to the orthogonal reflection on the subspace T_pF.

Take a point q of $M \setminus F$. The image of q by φ can be obtained in the following way. Since F is closed and M is complete, using Hopf–Rinow theorem it can be seen that there exists a point p of F having the property $\varrho(p, q) = \varrho(F, q)$.

Let us regard a geodesic $\gamma : [0, 1] \to M$ joining a p and q the arc length of which is equal to $\varrho(p, q)$. γ is a minimizing geodesic between F and q, therefore the initial tangent vector $\dot{\gamma}(0)$ of γ is normal to T_pF (see [1; p. 151]). Infinitely extending γ, we obtain a geodesic $\widetilde{\gamma} : R \to M$. Applying the facts written above, we can see that $\varphi \circ \widetilde{\gamma}(t) = \widetilde{\gamma}(-t)$ holds for every $t \in R$, in particular $\varphi(q) = \bar{\gamma}(-1)$.

Let us consider the Ricci tensor $Ric : T_pM \times T_pM \to R$ of a Riemannian manifold M at a point p. Clearly, there exists exactly one linear map (called Ricci endomorphism) $A_p : T_pM \to T_pM$ such that

$$Ric(v, w) = < A_p(v), w > \tag{2}$$

holds for any vectors v, w of T_pM. Since the Ricci tensor is a symmetric bilinear form, A_p is a self–adjoint map. Hence there exists an orthonormal basis $e_1, \ldots, e_n$ of T_pM including characteristic vectors of the Ricci endomorphism A_p. Denoting the suitable characteristic values by $\lambda_1, \ldots, \lambda_n$, we have

$$A_p(e_i) = \lambda_i e_i \qquad (i = 1, \ldots, n). \tag{3}$$

Note that if A_p has n different characteristic values, then the orthonormal basis $e_1, \dots, e_n$ is unique apart from sign.

Let v be a vector of T_pM different from the zero vector. It is well–known that the value of the Ricci curvature in the direction of v by definition is

$$r(v) = \frac{Ric(v,v)}{< v, v >}.$$

Let α_i denote the angle between v and the i–th basis element e_i. Applying (2) and (3), $r(v)$ can be expressed in the following form

$$r(v) = \sum_{i=1}^{n} \lambda_i \cos^2 \alpha_i,$$

which is the Euler equation for the Ricci curvature.

According to the sentences written above, the following notions can be introduced.

Definition 6. The characteristic values of the Ricci endomorphism are called the principal values of the Ricci curvature.

Definition 7. Considering a principal value, the dimension of the subspace consisting of the characteristic vectors belonging to it is said to be the multiplicity of this value.

If all the principal values are of multiplicity one, the invariant 1–dimensional subspaces of the Ricci endomorphism are called the principal directions of the Ricci curvature.

Later on we shall mainly study those Riemannian manifolds where the principal values of the Ricci curvature at some point are of multiplicity one. Obviously, the manifolds mentioned above are not Einstein manifolds.

Proposition 4. *Let $\varphi : M \to M$ be an isometry of a Riemannian manifold M different from id_M. If $M(\varphi)$ has a point p such that the principal values of the Ricci curvature at p are of multiplicity one, then φ is a symmetry.*

PROOF. Suppose that $M(\varphi)$ is not empty and it has a point p where the Ricci endomorphism A_p has n different eigenvalues. Let $e_1, \dots, e_n$ be an orthonormal basis of T_pM determining the principal directions of the Ricci curvature. Since φ is an isometry, it is clear that $T_p\varphi$ maps any principal direction onto itself. Hence we have $T_p\varphi(e_i) = \pm e_i$ for any index $i (i = 1, \dots, n)$.

Let F denote the component of $M(\varphi)$ including p. Obviously, T_pF is spanned by some elements of the basis $e_1, \dots, e_n$ and $T_p\varphi$ is equal to the orthogonal reflection of T_pM on T_pF. Therefore we obtain that $T_p\varphi^2 = id_{T_pM}$. Applying (1), it follows from this that $\varphi^2 = id_M$ which verifies our assertion. $\square$

Corollary 3. *Let M be a Riemannian manifold such that at any point of M the principal values of the Ricci curvature are of multiplicity one and let $\varphi : M \to M$ be an isometry different from id_M. If $M(\varphi)$ is not empty, then φ is a symmetry.*

Let F be a submanifold of a Riemannian manifold M. Consider the real–valued function $h : F \to R$ where for any $p \in F$ $h(p)$ is equal to the mean curvature of F at p. The submanifold F is said to be of constant mean curvature if h is a constant function. It is known that totally umbilical submanifolds play an important role in the theory of submanifolds (see [3]). According to them we can state the following theorem.

Theorem 3. *Let F be a 1–codimensional totally umbilical submanifold with constant mean curvature, furthermore, let v be a vector at a point p of F which is normal to the hyperplane T_pF. Then v is a characteristic vector of the Ricci endomorphism A_p.*

PROOF. Let us consider a unit normal vector v at a point p of F. Let $e_1, \dots, e_n$ be an orthonormal basis of the tangent space T_pM such that $e_n = v$. It is clear that for any vectors w_1, w_2 the equation

$$Ric(w_1, w_2) = \sum_{i=1}^{n} < R(e_i, w_1)w_2, e_i >$$

holds. It follows from this that for every vector w of T_pM we have

$$A_p(w) = \sum_{i=1}^{n} R(w, e_i)e_i.$$

Hence, in order to prove the theorem we have to show

$$\sum_{i=1}^{n} < R(v, e_i)e_i, e_k >= 0 \tag{4}$$

where k runs the values $1, \dots, n-1$.
By a symmetric property of the curvature tensor (4) is equivalent to

$$\sum_{i=1}^{n} < R(e_i, e_k)v, e_i >= 0. \tag{5}$$

Regarding a local coordinate system (U, x) in M adapted to F such that p is included in U, we can consider differentiable vector fields $X_1, \dots, X_{n-1}, \xi$ on $U \cap F$ having the following properties:

a) $X_1, \dots, X_{n-1}$ are tangential local vector fields to F and $X_k(p) = e_k$ holds for any index k;
b) ξ is a normal unit vector field on $U \cap F$ such that $\xi(p) = v$ and $-h\xi(p)$ is the mean curvature vector at p.

It can easily be shown that for any $Y \in \mathbf{V}(U \cap F)$ the vector field $\nabla_Y \xi$ is tangential to F (see [5; p. 106]). Hence, since F is a totally umbilical submanifold in M with constant mean curvature h, the equation

$$\nabla_Y \xi = hY$$

holds for every tangential vector field Y on $U \cap F$. Therefore we obtain that

$$\nabla_{X_i} \nabla_{X_k} \xi - \nabla_{X_k} \nabla_{X_i} \xi - \nabla_{[X_i, X_k]} \xi = h(\nabla_{X_i} X_k - \nabla_{X_k} X_i - [X_i, X_k]).$$

Since the torsion tensor of ∇ vanishes, we have

$$R(X_i, X_k)\xi = 0. \tag{6}$$

Obviously, (6) immediately implies the equation (5) which verifies our theorem. □

Since a submanifold is totally geodesic if and only if it is totally umbilical and its mean curvature vector at any point is equal to the zero vector, Theorem 3 implies the following assertion.

Corollary 4. *Let $\varphi : M \to M$ be a reflection of a Riemannian manifold M and let F be a component of $M(\varphi)$. If v is a vector at a point p of F which is normal to T_pF, then v is a characteristic vector of the Ricci endomorphism A_p.*

Proposition 5. *Let $\varphi_1, \varphi_2 : M \to M$ be reflections such that $M(\varphi_1)$ and $M(\varphi_2)$ are different 1–codimensional submanifolds. If $M(\varphi_1) \cap M(\varphi_2)$ has a point p where the principal values of Ricci curvature are of multiplicity one, then for any point $q \in M(\varphi_1) \cap M(\varphi_2)$ the hyperplanes $T_qM(\varphi_1)$, $T_qM(\varphi_2)$ are orthogonal to each other.*

PROOF. Using the exponential map in M, it can easily be seen that if $M(\varphi_1)$ and $M(\varphi_2)$ have a point in common, then their intersection is a 2–codimensional totally geodesic submanifold in M. Let us suppose that $M(\varphi_1) \cap M(\varphi_2)$ has a point p such that the Ricci endomorphism A_p has n different eigenvalues.

Considering the proof of Proposition 4, it can easily be seen that the hyperplanes $T_pM(\varphi_1)$, $T_pM(\varphi_2)$ are orthogonal to each other, therefore $T_p(\varphi_1 \circ \varphi_2)^2 = id_{T_pM}$. This immediately implies that $(\varphi_1 \circ \varphi_2)^2 = id_M$.

Let us consider an arbitrary point q of $M(\varphi_1) \cap M(\varphi_2)$. Applying the chain rule of tangent linear maps, we obtain from the equality given above that the orthogonal reflections $T_q\varphi_1$, $T_q\varphi_2$ are commutable. It follows from this that our proposition is true. □

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