Publ. Math. Debrecen
38 / 1–2 (1991), 33–38

A note on constructing E–rings

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1. Introduction

E-rings were introduced by PHILLIP SCHUTZ in 1973 as those rings R for which every endomorphism of their additive group R^+ can be achieved by left multiplication with some fixed ring element [S]. While appearing to be a rather specialized class of rings, E-rings turned up naturally in abelian group theory. For example, given a torsion-free abelian group G of finite rank, the center $R = Center(EndG)$ of its endomorphism ring $EndG$ is an E-ring in each of the following cases:

(1) G is strongly irreducible [R];

(2) G is strongly homogeneous [A, K, H3];

(3) G is reduced E-uniserial [H1].

In each of these cases, G is (or is very close to) a free module over an E-ring.

BOWSHELL and SCHULTZ characterized the finite-rank torsion-free E-rings as precisely those rings which are quasi-isomorphic to $R_1 \times R_2 \times \ldots \times R_n$ where each R_i is a strongly indecomposable subring of an algebraic number field and $Hom_Z(R_i, R_j) = 0$ for $i \neq j$ [BS]. In this context, E-rings had arisen earlier in the investigations of BEAUMONT and PIERCE on torsion-free rings [BP1, BP2, P]. Until recently, the only known E-rings of infinite rank were the pure subrings of the ring $\hat{Z}_p$ of p - adic integers. The question of whether E-rings of arbitrarily large cardinalities exist was settled in the affirmative by DUGAS, MADER and VINSONHALER in 1987 [DMV]. In [DH], a torsion-free E-ring is constructed which is a valuation domain but is not Noetherian. These results indicate that E-rings exist in abundance and that their structure may be rather complicated.

The purpose of this note is to outline a method for obtaining all E-rings. This method is then used to construct all torsion-free valuation domains which are E-rings. We will conclude with a number of problems and an example.

Our notation will be standard. All rings considered are unital. Maps are written to the right. Note that every E-ring is commutative [S,p. 65, Lemma 6]. Thus, an E-ring could also be defined as a ring R such that every endomorphism of R^+ is a *right* multiplication with some $r \in R$.

2. The E-ring core of a ring

For every ring R, BOWSHELL and SCHULTZ define a subring $T(R)$ of the center of R which they call the *T-core* of R [BS]. While $T(R)$ is always an E-ring, in general $T(R) \neq R$, even if R is an E-ring [BS, p. 202, 1.11]. The E-ring core of R defined below is an E-subring of the center of R which coincides with R if R is an E-ring.

For R a ring and $r \in R$, we let ρ_r be the right multiplication by r, i.e. $x\rho_r = xr$ for all $x \in R$. Define

$$\check{R} = \{r \in R | \rho_r \in Center(End_Z R^+)\}.$$

One verifies

Proposition 2.1. *(1) $\check{R}$ is a subring of the center of R, and $\check{R} \simeq Center(End_Z R^+)$, as rings. (2) R is an E-ring if and only if $\check{R} = R$. (3) If R is a domain and $0 \neq r, s \in R$, then $s \in \check{R}$ if both $rs \in \check{R}$ and $r \in \check{R}$.*

For any ring R, define subrings $C_\alpha(R)$ as follows: let $C_0(R) = R$. If $C_\mu(R)$ is defined for all $\mu < \alpha$, let $C_\alpha(R) = \bigcap_{\mu<\alpha} C_\mu(R)$ if α is a limit ordinal; if $\alpha = \mu + 1$, let $C_\alpha(R) = \check{C}_\mu(R)$. Then

$$R = C_0(R) \supseteq C_1(R) \supseteq \ldots \supseteq C_\alpha(R) \supseteq C_{\alpha+1}(R) \supseteq \ldots$$

is a descending series of subrings of R which we shall call the *descending C-chain of R*. For set theoretical reasons, there exist ordinals α such that $C_\alpha(R) = C_{\alpha+1}(R)$. The least α with $C_\alpha(R) = C_{\alpha+1}(R)$ is called the *E-ring length of R;* the ring $C_\alpha(R) = C_{\alpha+1}(R)$ is called the *E-ring core of R* and is denoted by $C_\infty(R)$.

We record the following result.

Theorem 2.2.. *For each ring R, the E-ring core $C_\infty(R)$ of R is an E-ring.*

3. E-Rings which are valuation domains

As an illustration, we shall construct all torsion-free valuation domains which are E-rings. By a valuation domain we mean an integral domain with linearly ordered ideal lattice.

Theorem 3.1. *Let R be a torsion-free valuation domain. Then* $C_\infty(R)$ *is an E-ring which is a torsion-free valuation domain.*

PROOF. If R^+ is divisible, $\check{R} \simeq Q$ is an E-ring with linearly ordered ideal lattice. Assume $R^+ = D \oplus H$ with D divisible and $H \neq 0$ reduced. We claim that $\check{R}$ is a reduced valuation domain. Let $\pi : D \oplus H \to H$ be the projection onto H along D. If $r \in \bigcap_{n\in N} n\check{R}$ and $0 \neq h \in H$,

$$hr = h\pi r = hr\pi \in (\bigcap_{n\in N} nR)\pi \subseteq D\pi = 0,$$

so that $r = 0$. Hence $\check{R}$ is reduced. Let $s, t \in \check{R}$. Since R is a valuation domain, there is no loss of generality in assuming $Rs \subseteq Rt$. Hence $s = rt$ for some $r \in R$. By 2.1, $r \in \check{R}$, which implies $\check{R}s \subseteq \check{R}t$ as desired. Thus, $C_1(R)$ is a valuation domain with reduced additive group. We claim that, for all $\alpha, C_\alpha(R)$ is a valuation domain. Assume, inductively, that λ is an ordinal less than or equal to the E-ring length of R such that, for all $\mu < \lambda, C_\mu(R)$ is a valuation domain. We need to distinguish cases. Suppose $\lambda = \mu + 1$. If $\mu = 0$ this has been established already. Assume $\mu \geq 1$. Then $C_\mu(R)^+$ is reduced. Note that the additive group of any ring with linearly ordered ideal lattice is *E-uniserial*, i.e. the lattice of fully invariant subgroups is a chain. Since $C_\lambda(R) = \check{C}_\mu(R) \simeq Center(EndC_\mu(R)^+)$, it follows from [H2, 3.5] that $C_\lambda(R)$ is a valuation domain. Suppose λ is a limit ordinal. Let $0 \neq a, b \in C_\lambda(R) = \bigcap_{\mu<\lambda} C_\mu(R)$. If, for all $\mu < \lambda, aC_\mu(R) = bC_\mu(R)$, then $a = bu$ with $u \in C_\mu(R)$ a unit. It follows that u is a unit in $C_\lambda(R)$ which implies $aC_\lambda(R) = bC_\lambda(R)$. Suppose there is $\mu < \lambda$ such that $aC_\mu(R) \neq bC_\mu(R)$. Without loss of generality, assume $aC_\mu(R) \subset bC_\mu(R)$. Note that this implies $aC_\nu(R) \subseteq bC_\nu(R)$ for all $\nu < \mu < \lambda$. For the same reason we must have $aC_\sigma(R) \subseteq bC_\sigma(R)$ for all $\sigma < \lambda$. Since R is a domain, $aC_\lambda(R) \subseteq bC_\lambda(R)$. It follows that $C_\lambda(R)$ is a valuation domain as claimed.

4. Problems and examples

Let G be a torsion-free abelian group, and let $R = Center(EndG)$ be the center of its endomorphism ring. If, given any nonzero fully invariant subgroup F of G, the quotient group G/F is bounded, G is said to be *strongly irreducible* [R]. For G strongly irreducible, R is an integral domain, $Q \otimes_Z R$ is a field, and G is a torsion-free R-module [H2, 2.2]. Moreover, R is a *strongly irreducible ring,* i.e. every nonzero ideal I of R contains a positive integer. In particular, $\check{R}$ is a strongly irreducible domain, too. Reid has shown that R is an E-ring if G has finite rank [R]. We pose

Problem 4.1. Must the E-ring core of a strongly irreducible domain be strongly irreducible ?

A similar situation occurs for *strongly homogeneous (E-transitive) groups: G is said to be strongly homogeneous (E-transitive) if AutG (EndG)* operates transitively on the pure rank-one subgroups of G [A, H3]. If G is E-transitive, R is a *strongly homogeneous ring,* i.e. R is a principal ideal domain with every ideal generated by a rational integer. Thus, R^+ is a torsion-free and strongly homogeneous group, too, so that $\check{R}$ is another strongly homogeneous domain. If G has finite rank, $R = \check{R}$ is an E-ring. We ask

Problem 4.2. Must the E-ring core of a strongly homogeneous integral domain be strongly homogeneous ?

Only recently, DUGAS and SHELAH have constructed strongly homogeneous and E-transitive groups with the property that the centers of their endomorphism rings are not E-rings: assuming Gödel's axiom of constructibility V=L , they show the existence of strongly homogeneous (and of E-transitive, but *not* strongly homogeneous) torsion-free abelian groups G with $R = Center(EndG)$ any predescribed cotorsion-free strongly homogeneous integral domain [DS]. Thus, the ring R of Example 4.7 below occurs as center of such an endomorphism ring without being an E-ring. However, $\check{R} = C_1(R)$ is an E-ring.

Problem 4.3. Is $C_\infty(R) = C_1(R)$ for any strongly homogeneous integral domain R ?

More general questions are:

Problem 4.4. If R is the center of the endomorphism ring of a torsion-free abelian group, under what conditions is $C_\infty(R) = C_1(R)$?

Problem 4.5. Given any ordinal λ, does there exist a ring R with E−ring length equal to λ ?

For a torsion-free abelian group G, if G is either strongly irreducible or reduced E-uniserial or E-transitive, the additive group of the center of its endomorphism ring has the same property. This allowed us to show that the E-ring core of a torsion-free valuation domain is a valuation domain.

Problem 4.6. What ring theoretical properties of R are inherited by its E-ring core?

We conclude with an example of a valuation domain R which is a strongly homogeneous principal ideal domain and has the property that $R \neq \check{R} = C_\infty(R)$.

Example 4.7. The p-height is a valuation on the field Q of rational numbers with value group the additive group Z of integers. Extend this valuation to the polynomial ring $Q[x]$ by defining $v(\sum_{i=0}^{n} a_i x^i) =$

$min\{h_p(a_i) + i \mid 0 \leq i \leq n\}$. By [G, p. 212, 18.4], v extends to a valuation on the quotient field K of $Q[x]$ with value group Z. Let $R = \{fg^{-1} \mid f, g \in Q[x], g \neq 0, v(f) \geq v(g)\}$ be the corresponding valuation ring. Since Z has a smallest strictly positive element, R is a principal ideal domain with maximal ideal pR [G, p. 193]. It follows that R is a strongly homogeneous domain. As usual, define the derivative of $f = \sum_{i=0}^{n} a_i x^i \in Q[x]$ by $f' = \sum_{i=1}^{n} i a_i x^{i-1}$, and let $(fg^{-1})\delta = \frac{f'g - fg'}{g^2}$. One verifies that $\delta \in End_Z R^+$. Let $h \in \check{R}$. Then $\rho_h \delta = \delta \rho_h$. It follows that $h\delta = 0$ and $h = a \in Z_p$ is a constant. Thus, $\check{R} = Z_p$, the ring of integers localized at p , and $\check{R} = C_1(R) = C_\infty(R)$ is an E-ring.

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(Received August 14, 1987)