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Direct arithmetics of relational systems

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The aim of this paper is to define and study direct operations of addition, multiplication and exponentiation for relational systems. By a relational system we understand a set G together with a set of mappings of another set into G. More precisely, let G and I be non–empty sets. Then a set of mappings $R \subseteq G^I$ is called a *relation* on G and the ordered pair $\mathbf{G} = (G, R)$ is said to be a *relational system*. The set G is called the *carrier* of $\mathbf{G}$ and the set I is called the *domain* of $\mathbf{G}$. The relation R of $\mathbf{G}$ (i.e. on G) will sometimes be denoted by $\Re(\mathbf{G})$.

Birkhoff's arithmetics of ordered sets discussed in [1] and [2] has been generalized by several mathematicians – see e.g. [3], [4]. Especially, in [4] V. Novák deals with direct operations of addition, multiplication and mainly exponentiation for relational structures, i.e. for sets endowed with n–ary relations (where n is a positive integer). As n–ary relations coincide with relations of finite domains, the presented results can be considered as a generalization and a completion of those of [4].

Definition 1. Let $\mathbf{G} = (G, R)$, $\mathbf{H} = (H, S)$ be two relational systems of the same domain. Let $\varphi : G \to S$ be a mapping for which the implication $f \in R \Rightarrow \varphi \circ f \in S$ holds. Then φ is called a *homomorphism* of $\mathbf{G}$ into $\mathbf{H}$. The set of all homomorphisms of $\mathbf{G}$ into $\mathbf{H}$ is denoted by $\mathrm{Hom}(\mathbf{G}, \mathbf{H})$. A bijective homomorphism φ of $\mathbf{G}$ onto $\mathbf{H}$ such that φ^{-1} is a homomorphism of $\mathbf{H}$ onto $\mathbf{G}$ is called an *isomorphism* of $\mathbf{G}$ onto $\mathbf{H}$. We write $\mathbf{G} \sim \mathbf{H}$ and say that the relational systems $\mathbf{G}$ and $\mathbf{H}$ are *isomorphic* iff there exists an isomorphism of $\mathbf{G}$ onto $\mathbf{H}$.

Clearly, the class of all relational systems of the same domain together with homomorphisms as morphisms form a category.

Definition 2. Let $\{\mathbf{G}_j \ / \ j \in J\} = \{(G_j, R_j) \ / \ j \in J\}$ be a family of relational systems of the same domain I and let $G_{j_1} \cap G_{j_2} = \emptyset$ whenever $j_1, j_2 \in J$, $j_1 \neq j_2$. The *direct sum* $\sum\limits_{j \in J} \mathbf{G}_j$ of the family $\{\mathbf{G}_j \ / \ j \in J\}$

is the relational system $\mathbf{G} = (G, R)$ of domain I where $G = \bigcup_{j \in J} G_j$ and $R = \bigcup_{j \in J} R_j$.

If $J = \{j_1, \ldots, j_n\}$, then we write $\sum_{j \in J} \mathbf{G}_j = \mathbf{G}_{j_1} + \cdots + \mathbf{G}_{j_n}$.

Having a family of sets $\{G_j \ / \ j \in J\}$, we denote by κ_j $(j \in J)$ the j–th canonical insertion, i.e. the mapping $\kappa_j : G_j \to \bigcup_{j \in J} G_j$ defined by $\kappa_j(x) = x$ whenever $x \in G_j$.

Let $\mathbf{G} = (G, R)$, $\mathbf{H} = (G, S)$ be two relational systems of the same domain and with the same carrier. Put $\mathbf{G} \leq \mathbf{H}$ iff $R \subseteq S$. Clearly, $\leq$ is an order on the class of all relational systems of the same domain and with the same carrier.

Proposition 1. *Let $\{\mathbf{G}_j \ / \ j \in J\} = \{(G_j, R_j) \ / \ j \in J\}$ be a family of relational systems of the same domain I. Let $G_{j_1} \cap G_{j_2} = \emptyset$ whenever $j_1, j_2 \in J$, $j_1 \neq j_2$ and let $\mathbf{G} = (G, R) = \sum_{j \in J} \mathbf{G}_j$. Then $\mathbf{G}$ is the least element (with respect to $\leq$) in the class of all such relational systems $\mathbf{H}$ of the same domain I and with the same carrier G for which every canonical insertion $\kappa_j (j \in J)$ is a homomorphism of $\mathbf{G}_j$ into $\mathbf{H}$.*

PROOF. Clearly, every canonical insertion κ_j $(j \in J)$ is a homomorphism of $\mathbf{G}_j$ into $\mathbf{G}$. Let $\mathbf{H} = (G, S)$ be such a relational system of domain I and with carrier G for which all canonical insertions κ_j $(j \in J)$ are homomorphisms of $\mathbf{G}_j$ into $\mathbf{H}$. Let $f \in R$. Then there exists $j \in J$ such that $f \in R_j$. Consequently, $\kappa_j \circ f \in S$. Since $\kappa_j \circ f = f$, we have $f \in S$. Therefore $\mathbf{G} \leq \mathbf{H}$ and the assertion is proved.

Evidently, there holds:

Theorem 1. *Let $\{\mathbf{G}_j \ / \ j \in J\} = \{(G_j, R_j) \ / \ j \in J\}$ be a family of relational systems of the same domain and let $G_{j_1} \cap G_{j_2} = \emptyset$ whenever $j_1, j_2 \in J$, $j_1 \neq j_2$. Let $\{J_k \ / \ k \in K\}$ be a decomposition of the set J. Then*

$$\sum_{k \in K} \sum_{j \in J_k} \mathbf{G}_j = \sum_{j \in J} \mathbf{G}_j.$$

Definition 3. Let $\{\mathbf{G}_j \ / \ j \in J\} = \{(G_j, R_j) \ / \ j \in J\}$ be a family of relational systems of the same domain I. The *direct product* $\prod_{j \in J} \mathbf{G}_j$ of the family $\{\mathbf{G}_j \ / \ j \in J\}$ is the relational system $\mathbf{G} = (G, R)$ of domain I where $G = \bigtimes_{j \in J} G_j$ and R is defined as follows: $f \in (\bigtimes_{j \in J} G_j)^I$, $f \in R \Leftrightarrow pr_j \circ f \in R_j$ for all $j \in J$.

Of course, $pr_j\,(j \in J)$ means here the j–th projection, i.e. $pr_j : \bigtimes_{j\in J} G_j \to G_j$ is the mapping defined by $pr_j(x) = x(j)$ whenever $x \in \bigtimes_{j\in J} G_j$.

If $J = \{j_1, \ldots, j_n\}$, then we write $\prod_{j\in J} \mathbf{G}_j = \mathbf{G}_{j_1} \cdot \ \ldots\ \cdot \mathbf{G}_{j_n}$.

Proposition 2. *Let $\{G_j \,/\, j \in J\} = \{(G_j, R_j) \,/\, j \in J\}$ be a family of relational systems of the same domain I. Let $\mathbf{G} = (G, R) = \prod_{j\in J} \mathbf{G}_j$. Then $\mathbf{G}$ is the greatest element (with respect to $\leq$) in the class of all such relational systems $\mathbf{H}$ of the same domain I and with the same carrier G for which every projection $pr_j\,(j \in J)$ is a homomorphism of $\mathbf{H}$ onto $\mathbf{G}_j$.*

PROOF. It can easily be seen that every projection $pr_j(j \in J)$ is a homomorphism of $\mathbf{G}$ onto $\mathbf{G}_j$. Let $\mathbf{H} = (G, S)$ be such a relational system of domain I and with carrier G for which all projections $pr_j\ (j \in J)$ are homomorphisms of $\mathbf{H}$ onto $\mathbf{G}_j$. Let $f \in S$. Then $pr_j \circ f \in R_j$ for all $j \in J$. Consequently, $f \in R$. Hence $\mathbf{H} \leq \mathbf{G}$ and the proof is complete.

Theorem 2. *Let $\{\mathbf{G}_j \,/\, j \in J\}$ be a family of relational systems of the same domain and let $\{J_k \,/\, k \in K\}$ be a decomposition of the set J. Then*

$$\prod_{k\in K} \prod_{j\in J_k} \mathbf{G}_j \sim \prod_{j\in J} \mathbf{G}_j.$$

PROOF. If $j \in J$, let $\mathbf{G}_j = (G_j, R_j)$, and let the domain of $\mathbf{G}_j$ be I. Let $\varphi : \bigtimes_{k\in K} \bigtimes_{j\in J_k} G_j \to \bigtimes_{j\in J} G_j$ be the mapping defined as follows: for any $x \in \bigtimes_{k\in K} \bigtimes_{j\in J_k} G_j$ we put $\varphi(x) = y$ where $y : J \to \bigcup_{j\in J} G_j$ is the mapping fulfilling $y(j) = x(k)(j)$ for all $j \in J$ where $k \in K$ is the index with $j \in J_k$. Obviously, φ is a bijection. Clearly, there holds $pr_j(\varphi(x)) = \varphi(x)\,(j) = x(k)\,(j) = pr_j(pr_k(x))$ for all $x \in \bigtimes_{k\in K} \bigtimes_{j\in J_k} G_j$, $k \in K$ and $j \in J_k$. Let $f \in \Re(\prod_{k\in K} \prod_{j\in J_k} \mathbf{G}_j)$. Then $pr_j \circ f \in \Re(\prod_{j\in J_k} \mathbf{G}_j)$ and so $pr_j \circ pr_k \circ f \in R_j$ for all $k \in K$, $j \in J_k$. Let $i \in I$, $k \in K$ and $j \in J_k$. Then we have $pr_j(\varphi(f(i))) = pr_j(pr_k(f(i)))$ and so $pr_j \circ \varphi \circ f = pr_j \circ pr_k \circ f \in R_j$. Therefore $\varphi \circ f \in \Re(\prod_{j\in J} \mathbf{G}_j)$ and φ is a homomorphism. Conversely, let $f \in \Re(\prod_{j\in J} \mathbf{G}_j)$. Then $pr_j \circ f \in R_j$ for all $j \in J$. Let $i \in I$, $k \in K$ and $j \in J_k$. Then $pr_j(pr_k(\varphi^{-1}(f(i)))) = pr_j(f(i))$ and so $pr_j \circ pr_k \circ \varphi^{-1} \circ f = pr_j \circ f \in R_j$. Hence $\varphi^{-1} \circ f \in \Re(\prod_{k\in K} \prod_{j\in J_k} \mathbf{G}_j)$. Thus φ^{-1} is a homomorphism.

Consequently, $\prod_{k\in K}\prod_{j\in J_k}\mathbf{G}_j$ and $\prod_{j\in J}\mathbf{G}_j$ are isomorphic. The statement is proved.

Let $\mathbf{G}=(G,R)$ be a relational system of domain I. The system $\mathbf{G}$ is called

(1) *discrete* iff for every mapping $f\in G^I$ there holds $f\in R \Leftrightarrow \exists x\in G:\ f(i)=x$ for all $i\in I$,

(2) *reflexive* iff for the discrete relational system $\mathbf{H}$ of domain I and with carrier G there holds $\mathbf{H}\leq\mathbf{G}$,

(3) *complete* iff $R=G^I$.

Theorem 3. *Let $\mathbf{G}=(G,R)$ be a relational system of domain I and let $\{\mathbf{G}_j\ /\ j\in J\}=\{(G_j,R_j)\ /\ j\in J\}$ be a family of relational systems of the same domain I. Let $G_{j_1}\cap G_{j_2}=\emptyset$ whenever $j_1,j_2\in J$, $j_1\neq j_2$. Let $\mathbf{G}_j\sim\mathbf{G}$ for every $j\in J$ and let $\mathbf{J}$ be a reflexive relational system of domain I and with carrier J. Then there exists a bijective homomorphism of $\sum_{j\in J}\mathbf{G}_j$ onto $\mathbf{J}\cdot\mathbf{G}$. If $\mathbf{J}$ is even discrete, then*

$$\sum_{j\in J}\mathbf{G}_j\sim\mathbf{J}\cdot\mathbf{G}.$$

PROOF. Let φ_j be an isomorphism of $\mathbf{G}_j$ onto $\mathbf{G}$ for every $j\in J$. For any element $x\in\bigcup_{j\in J}G_j$ put $\varphi(x)=(j,\varphi_j(x))$ where $j\in J$ is the index with $x\in G_j$. Clearly, $\varphi:\bigcup_{j\in J}G_j\to J\times G$ is a bijection. Let $f\in\Re(\sum_{j\in J}\mathbf{G}_j)$. Then there exists $j\in J$ such that $f\in R_j$. Let $i\in I$. Then $\varphi(f(i))=(j,\varphi_j(f(i)))$. Since $\mathbf{J}$ is reflexive and since $\varphi_j\circ f\in R$, we have $\varphi\circ f\in\Re(\mathbf{J}\cdot\mathbf{G})$. So φ is a homomorphism of $\sum_{j\in J}\mathbf{G}_j$ onto $\mathbf{J}\cdot\mathbf{G}$. Let even $\mathbf{J}$ be discrete and let $f\in\Re(\mathbf{J}\cdot\mathbf{G})$. Then there exist $j\in J$ and $g\in R$ such that $f(i)=(j,g(i))$ for all $i\in I$. Consequently, for any $i\in I$ we have $\varphi^{-1}(f(i))=\varphi^{-1}((j,g(i)))=\varphi^{-1}((j,\varphi_j(\varphi_j(g(i)))))=\varphi_j^{-1}(g(i))$ because $\varphi_j^{-1}(g(i))\in G_j$. Thus $\varphi^{-1}\circ f=\varphi_j^{-1}\circ g$. As φ_j^{-1} is a homomorphism of $\mathbf{G}$ onto $\mathbf{G}_j$, there is $\varphi_j^{-1}\circ g\in R_j$. Hence $\varphi^{-1}\circ f\in R_j\subseteq\Re(\sum_{j\in J}\mathbf{G}_j)$. Therefore φ^{-1} is a homomorphism of $\mathbf{J}\cdot\mathbf{G}$ onto $\sum_{j\in J}\mathbf{G}_j$. This proves the statement.

Theorem 4. *Let $\{\mathbf{G}_j\ /\ j\in J\}=\{(G_j,R_j)\ /\ j\in J\}$ be a family of relational systems of the same domain and let $\{J_k\ /\ k\in K\}$ be a decomposition of the set J. Let $G_{j_1}\cap G_{j_2}=\emptyset$ whenever there exists $k\in K$*

such that $j_1, j_2 \in J_k$ *and* $j_1 \neq j_2$. *Denote* $P = \bigtimes_{k \in K} J_k$. *Then*

$$\prod_{k \in K} \sum_{j \in J_k} \mathbf{G}_j = \sum_{p \in P} \prod_{k \in K} \mathbf{G}_p(k).$$

PROOF. At first, it can easily be seen that the carriers of the systems $\prod_{k \in K} \mathbf{G}_{p_1(k)}$ and $\prod_{k \in K} \mathbf{G}_{p_2(k)}$ are disjoint whenever $p_1, p_2 \in P$, $p_1 \neq p_2$. Hence, the direct sum $\sum_{p \in P} \prod_{k \in K} \mathbf{G}_{p(k)}$ is defined. Next, the carriers of the systems $\prod_{k \in K} \sum_{j \in J_k} \mathbf{G}_j$ and $\sum_{p \in P} \prod_{k \in K} \mathbf{G}_{p(k)}$ are clearly equal. We shall prove that also the relations of these systems are equal. On that account, let $f \in \Re(\prod_{k \in K} \sum_{j \in J_k} \mathbf{G}_j)$. Then for every $k \in K$ there exists $j \in J_k$ such that $pr_k \circ f \in R_j$. In other words, there exists $p \in P$ such that for every $k \in K$ we have $pr_k \circ f \in \Re(\mathbf{G}_{p(k)})$. Therefore $f \in \Re(\prod_{k \in K} \mathbf{G}_{p(k)}) \subseteq \Re(\sum_{p \in P} \prod_{k \in K} \mathbf{G}_{p(k)})$. We have proved the inclusion $\prod_{k \in K} \sum_{j \in J_k} \mathbf{G}_j \subseteq \sum_{p \in P} \prod_{k \in K} \mathbf{G}_{p(k)}$. Reversing the arguments we can easily prove the inverse inclusion.

Definition 4. Let $\mathbf{G} = (G, R)$, $\mathbf{H} = (H, S)$ be a relational systems of the same domain I. The direct power $\mathbf{G}^{\mathbf{H}}$ is the relational system $\mathbf{K} = (K, T)$ of domain I where $K = \mathrm{Hom}(\mathbf{H}, \mathbf{G})$ and T is defined as follows: $f \in K^I$, $f \in T \Leftrightarrow {}^x f \in R$ for all $x \in H$. By ${}^x f$ we understand the mapping ${}^x f : I \to G$ defined by ${}^x f(i) = f(i)(x)$ whenever $i \in I$.

Theorem 5. *Let* $\mathbf{G}$ *be a relational system of domain* I *and let* $\{\mathbf{G}_j \mid j \in J\}$ *be a family of relational systems of the same domain* I. *Let* $\mathbf{G} \sim \mathbf{G}_j$ *for every* $j \in J$ *and let* $\mathbf{J}$ *be a relational system of domain* I *and with carrier* J. *Then there exists an isomorphic embedding of* $\mathbf{G}^{\mathbf{J}}$ *into* $\prod_{j \in J} \mathbf{G}_j$. *If* $\mathbf{G}$ *is complete or if* $\mathbf{G}$ *is reflexive and* $\mathbf{J}$ *is discrete, then*

$$\mathbf{G}^{\mathbf{J}} \sim \prod_{j \in J} \mathbf{G}_j.$$

PROOF. Let $\mathbf{G} = (G, R)$ and $\mathbf{G}_j = (G_j, R_j)$ for all $j \in J$. Let φ_j be an isomorphism of $\mathbf{G}$ onto $\mathbf{G}_j$ for every $j \in J$. For any $x \in \mathrm{Hom}(\mathbf{J}, \mathbf{G})$ put $\varphi(x) = y$ where $y : J \to \bigcup_{j \in J} G_j$ is the mapping defined by $y(j) = \varphi_j(x(j))$

whenever $j \in J$. Clearly, $\varphi : \mathrm{Hom}(\mathbf{J}, \mathbf{G}) \to \bigtimes_{j \in J} G_j$ is an injection. Let $f \in \Re(\mathbf{G}^{\mathbf{J}})$. Then we have $pr_j(\varphi(f(i))) = \varphi(f(i))\,(j) = \varphi_j(f(i)(j)) = \varphi_j({}^j f(i))$ for any $i \in I$ and $j \in J$. Hence $pr_j \circ \varphi \circ f = \varphi_j \circ {}^j f$ for every $j \in J$. As ${}^j f \in R$ for all $j \in J$, there holds $\varphi_j \circ {}^j f \in R$, i.e. $pr_j \circ \varphi \circ f \in R_j$ for all $j \in J$. From this $\varphi \circ f \in \Re(\prod_{j \in J} \mathbf{G}_j)$ and therefore φ is a homomorphism of $\mathbf{G}^{\mathbf{J}}$ into $\prod_{j \in J} \mathbf{G}_j$. Conversely, let $f \in (\mathrm{Hom}(\mathbf{J}, \mathbf{G}))^I$ be a mapping with $\varphi \circ f \in \Re(\prod_{j \in J} \mathbf{G}_j)$, i.e. with $pr_j \circ \varphi \circ f \in R_j$ for all $j \in J$. Then ${}^j f(i) = f(i)(j) = \varphi_j^{-1}(\varphi_j(f(i)(j))) = \varphi_j^{-1}(\varphi(f(i))(j)) = \varphi_j^{-1}(pr_j(\varphi(f(i))))$ for any $i \in I$ and $j \in J$. Hence ${}^j f = \varphi_j^{-1} \circ pr_j \circ \varphi \circ f \in R$ for all $j \in J$ so that $f \in \Re(\mathbf{G}^{\mathbf{J}})$. Thus φ is an isomorphic embedding of $\mathbf{G}^{\mathbf{J}}$ into $\prod_{j \in J} \mathbf{G}_j$. Now suppose that $\mathbf{G}$ is complete or that $\mathbf{G}$ is reflexive and $\mathbf{J}$ is discrete. Then clearly $\mathrm{Hom}(\mathbf{J}, \mathbf{G}) = \mathbf{G}^{\mathbf{J}}$ and hence φ is a bijection. Therefore φ is an isomorphism of $\mathbf{G}^{\mathbf{J}}$ onto $\prod_{j \in J} \mathbf{G}_j$. The assertion is proved.

Corollary. *Let $\mathbf{G}, \mathbf{J}, \mathbf{K}$ be relational systems of the same domain I. Let $\mathbf{G}$ be complete or let $\mathbf{G}$ be reflexive and both $\mathbf{J}$ and $\mathbf{K}$ be discrete. Then*

$$(\mathbf{G}^{\mathbf{J}})^{\mathbf{K}} \sim \mathbf{G}^{\mathbf{J} \cdot \mathbf{K}}.$$

PROOF. If $\mathbf{G}$ is complete, then clearly $\mathbf{G}^{\mathbf{J}}$ is complete, too. If $\mathbf{G}$ is reflexive and both $\mathbf{J}$ and $\mathbf{K}$ are discrete, then it can easily be seen that $\mathbf{G}^{\mathbf{J}}$ is reflexive and $\mathbf{J} \cdot \mathbf{K}$ is discrete. Therefore, putting $\mathbf{G}_{jk} = \mathbf{G}$ for all $j \in J$ and all $k \in K$, in consequence of Theorems 2 and 5 we obtain $(\mathbf{G}^{\mathbf{J}})^{\mathbf{K}} \sim \prod_{k \in K} \prod_{j \in J} \mathbf{G}_{jk} \sim \prod_{(j,k) \in J \times K} \mathbf{G}_{jk} \sim \mathbf{G}^{\mathbf{J} \cdot \mathbf{K}}$ whenever $\mathbf{G}$ is complete or $\mathbf{G}$ is reflexive and both $\mathbf{J}$ and $\mathbf{K}$ are discrete.

Obviously, the law $(\mathbf{G}^{\mathbf{J}})^{\mathbf{K}} \sim \mathbf{G}^{\mathbf{J} \cdot \mathbf{K}}$ does not hold generally, i.e. for arbitrary relational systems $\mathbf{G}, \mathbf{J}, \mathbf{K}$ of the same domain. At the end of the article we give other sufficient conditions for the validity of this law.

The following statement is evident:

Theorem 6. *Let $\mathbf{G}, \mathbf{J}$ be relational systems of the same domain and let $\mathbf{J}$ be reflexive. Then there exists an isomorphic embedding of $\mathbf{J}$ into $\mathbf{J}^{\mathbf{G}}$. If moreover $\mathbf{J}$ is a singleton, then*

$$\mathbf{J} \sim \mathbf{J}^{\mathbf{G}}.$$

Theorem 7. *Let $\{\mathbf{G}_j \mid j \in J\} = \{(G_j, R_j) \mid j \in J\}$ be a family of relational systems of the same domain I. Let $\mathbf{H} = (H, S)$ be a relational* *system of domain I. Then*

$$(\prod_{j\in J}\mathbf{G}_j)^{\mathbf{H}} \sim \prod_{j\in J}\mathbf{G}_j^{\mathbf{H}}.$$

PROOF. For any $f \in \mathrm{Hom}(\mathbf{H}, \prod_{j\in J}\mathbf{G}_j)$ and any $j \in J$ denote $f_j = pr_j \circ f$. Then $f_j \in \mathrm{Hom}(\mathbf{H}, \mathbf{G}_j)$ for any $j \in J$. Let $\varphi : \mathrm{Hom}(\mathbf{H}, \prod_{j\in J}\mathbf{G}_j) \to \mathop{\times}_{j\in J} \mathrm{Hom}(\mathbf{H}, \mathbf{G}_j)$ be the mapping defined as follows: $f \in \mathrm{Hom}(\mathbf{H}, \prod_{j\in J}\mathbf{G}_j), \varphi(f) = g$ where $g : J \to \bigcup_{j\in J}\mathrm{Hom}(\mathbf{H}, \mathbf{G}_j)$ is the mapping with $g(j) = f_j$ whenever $j \in J$. It can easily be seen that φ is a bijection. Let $h \in \Re((\prod_{j\in J}\mathbf{G}_j)^{\mathbf{H}}$. Then ${}^x h \in \Re(\prod_{j\in J}\mathbf{G}_j)$ for all $x \in H$. Thus $pr_j \circ {}^x h \in R_j$ for all $j \in J$ and $x \in H$. We have $\varphi \circ h \in \Re(\prod_{j\in J}\mathbf{G}_j^{\mathbf{H}}) \Leftrightarrow pr_j \circ \varphi \circ h \in \Re(\mathbf{G}_j^{\mathbf{H}})$ for all $j \in J \Leftrightarrow {}^x(pr_j \circ \varphi \circ h) \in R_j$ for all $x \in H,\ j \in J$. There holds ${}^x(pr_j \circ \varphi \circ h)(i) = (pr_j \circ \varphi \circ h)(i)(x) = pr_j(\varphi(h(i)))(x) = \varphi(h(i))(j)(x) = (h(i))_j(x) = (pr_j \circ h(i))(x) = (pr_j \circ {}^x h)(i)$ for every $i \in I,\ j \in J,\ x \in H$. Thus ${}^x(pr_j \circ \varphi \circ h) = pr_j \circ {}^x h \in R_j$ for all $x \in H,\ j \in J$. Therefore $\varphi \circ h \in \Re(\prod_{j\in J}\mathbf{G}_j^{\mathbf{H}})$. Consequently, φ is a homomorphism of $(\prod_{j\in J}\mathbf{G}_j)^{\mathbf{H}}$ into $\prod_{j\in j}\mathbf{G}_j^{\mathbf{H}}$. By the reverse considerations we can easily show that φ^{-1} is a homomorphism of $\prod_{j\in J}\mathbf{G}_j^{\mathbf{H}}$ into $(\prod_{j\in J}\mathbf{G}_j)^{\mathbf{H}}$ and the proof is complete.

Theorem 8. *Let $\mathbf{G} = (G, R)$ be a relational system of domain I. Let $\{\mathbf{H}_j \mid j \in J\} = \{(H_j, S_j) \mid j \in J\}$ be a family of relational systems of the same domain I and let $H_{j_1} \cap H_{j_2} = \emptyset$ whenever $j_1, j_2 \in J,\ j_1 \neq j_2$. Then*

$$\mathbf{G}^{\sum_{j\in J}\mathbf{H}_j} \sim \prod_{j\in J}\mathbf{G}^{\mathbf{H}_j}.$$

PROOF. For any $f \in \mathrm{Hom}(\sum_{j\in J}\mathbf{H}_j, \mathbf{G})$ and any $j \in J$ denote by f_j the restriction f/H_j. Then $f_j \in \mathrm{Hom}(\mathbf{H}_j, \mathbf{G})$ for any $j \in J$. Let $\varphi : \mathrm{Hom}(\sum_{j\in J}\mathbf{H}_j, \mathbf{G}) \to \mathop{\times}_{j\in J}\mathrm{Hom}(\mathbf{H}_j, \mathbf{G})$ be the mapping defined as follows: $f \in \mathrm{Hom}(\sum_{j\in J}\mathbf{H}_j, \mathbf{G}),\ \varphi(f) = g$ where $g : J \to \bigcup_{j\in J}\mathrm{Hom}(\mathbf{H}_j, \mathbf{G})$ is the mapping fulfilling $g(j) = f_j$ whenever $j \in J$. It can easily be seen that φ is a bijection. Let $h \in \Re(\mathbf{G}^{\sum_{j\in J}\mathbf{H}_j})$. Then ${}^x h \in R$ for all $x \in \bigcup_{j\in J} H_j$. We have $\varphi\circ h \in \Re(\prod_{j\in J} \mathbf{G}^{\mathbf{H}}) \Leftrightarrow pr_j\circ\varphi\circ h \in \Re(\mathbf{G}^{\mathbf{H}_j})$ for all $j \in J \Leftrightarrow {}^x(pr_j\circ\varphi\circ h) \in R$ for all $j \in J, x \in H_j$. There holds ${}^x(pr_j \circ \varphi \circ h)(i) = (pr_j \circ \varphi \circ h)(i)(x) = pr_j(\varphi(h(i)))(x) = \varphi(h(i))(j)(x) = (h(i))_j(x) = (h(i)/H_j)(x) = h(i)(x) = {}^x h(i)$ for every $i \in I,\ j \in J,\ x \in H_j$. Thus ${}^x(pr_j \circ \varphi \circ h) = {}^x h \in R$ for all $j \in J,\ x \in H_j$. Therefore $\varphi \circ h \in \Re(\prod_{j\in J} \mathbf{G}^{\mathbf{H}_j})$. Consequently, φ is a homomorphism of $\mathbf{G}^{\sum_{j\in J}\mathbf{H}_j}$ into $\prod_{j\in J} \mathbf{G}^{\mathbf{H}_j}$. Reversing the arguments we can easily show that φ^{-1} is a homomorphism of $\prod_{j\in J} \mathbf{G}^{\mathbf{H}_j}$ into $\mathbf{G}^{\prod_{j\in J}\mathbf{H}_j}$. The theorem is proved.

Let $\mathbf{G} = (G, R)$ be a relational system of domain I. The system $\mathbf{G}$ is called *diagonal* iff the following is valid:
Let $\{f_i \ /\ i \in I\}$ be a family of elements of G^I such that $f_i \in R$ for all $i \in I$. Let $\{g_j \ /\ j \in I\}$ be the family of elements of G^I where $g_j(i) = f_i(j)$ for all $i, j \in I$. If $g_j \in R$ for all $j \in I$, then putting $h(i) = f_i(i)$ for every $i \in I$ we get $h \in R$.

Let us note that for a relational system $\mathbf{G} = (G, R)$ of finite domain the diagonality of $\mathbf{G}$ coincides with the diagonal property of R defined in [4]. If R is even a binary relation on G, then $\mathbf{G}$ is diagonal iff R is transitive.

Theorem 9. *Let $\mathbf{G}, \mathbf{H}, \mathbf{K}$ be relational systems of the same domain and let $\mathbf{H}$ and $\mathbf{K}$ be reflexive. Then there exists an isomorphic embedding of $\mathbf{G}^{\mathbf{H}\cdot\mathbf{K}}$ into $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$. If moreover $\mathbf{G}$ is diagonal, then*

$$\mathbf{G}^{\mathbf{H}\cdot\mathbf{K}} \sim (\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}.$$

PROOF. Let $\mathbf{G} = (G, R)$, $\mathbf{H} = (H, S)$, $\mathbf{K} = (K, T)$ and let I be their domain. Let $f \in \mathrm{Hom}(\mathbf{H} \cdot \mathbf{K}, \mathbf{G})$ and $y \in K$. By f_y we denote the mapping $f_y : H \to G$ defined by $f_y(x) = f(x, y)$ whenever $x \in H$. Let $g \in S$. Putting $g^*(i) = (g(i), y)$ for all $i \in I$, we have $g^* \in \Re(\mathbf{H} \cdot \mathbf{K})$. Hence $f \circ g^* \in R$. But $f \circ g^* = f_y \circ g$ and thus $f_y \circ g \in R$. Therefore $f_y \in \mathrm{Hom}(\mathbf{H}, \mathbf{G})$. Let $x \in H$ and $h \in T$. Put $\bar{h}(i) = (x, h(i))$ for every $i \in I$. Then $\bar{h} \in \Re(\mathbf{H} \cdot \mathbf{K})$. Consequently, $f \circ \bar{h} \in R$. Let $f' : K \to \mathrm{Hom}(\mathbf{H}, \mathbf{G})$ be the mapping defined by $f'(y) = f_y$ for any $y \in K$. There holds ${}^x(f' \circ h)(i) = f'(h(i))(x) = f_{h(i)}(x) = f(x, h(i)) = f(\bar{h}(i))$ for every

$x \in H$ and $i \in I$. This yields ${}^x(f' \circ h) = f \circ \bar{h}$ so that ${}^x(f' \circ h) \in R$ for all $x \in H$. Therefore $f' \circ h \in \Re(\mathbf{G}^{\mathbf{H}})$ and thus $f' \in \mathrm{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$. Let $\varphi : \mathrm{Hom}(\mathbf{H} \cdot \mathbf{K}, \mathbf{G}) \to \mathrm{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ be the mapping defined by $\varphi(f) = f'$ for every $f \in \mathrm{Hom}(\mathbf{H} \cdot \mathbf{K}, \mathbf{G})$. Let $f, g \in \mathrm{Hom}(\mathbf{H} \cdot \mathbf{K}, \mathbf{G})$, $f \neq g$. Then there exists $(x, y) \in H \times K$ with $f(x, y) \neq g(x, y)$. Hence $f_y(x) \neq g_y(x)$ for some $x \in H$ and $y \in K$. Thus $f'(y) = f_y \neq g_y = g'(y)$ for some $y \in K$. We have $\varphi(f) = f' \neq g' = \varphi(g)$ which implies that φ is an injection. Let $p \in \Re(\mathbf{G}^{\mathbf{H}\cdot\mathbf{K}})$. Then ${}^{(x,y)}p \in R$ for all $(x, y) \in H \times K$. For any $y \in K$, by $\hat{y}$ we denote the mapping $\hat{y} : \mathrm{Hom}(\mathbf{H} \cdot \mathbf{K}, \mathbf{G}) \to \mathrm{Hom}(\mathbf{H}, \mathbf{G})$ defined by $\hat{y}(f) = f_y$ whenever $f \in \mathrm{Hom}(\mathbf{H} \cdot \mathbf{K}, \mathbf{G})$. There holds ${}^x(\hat{y} \circ p)(i) = \hat{y}(p(i))(x) = (p(i))_y(x) = p(i)(x, y) = {}^{(x,y)}p(i)$ for every $i \in I$, $x \in H$, $y \in K$. Consequently, ${}^x(\hat{y} \circ p) = {}^{(x,y)}p$ for all $x \in H, y \in K$, so that ${}^x(\hat{y} \circ p) \in R$ for all $x \in H, y \in K$. Therefore $\hat{y} \circ p \in \Re(\mathbf{G}^{\mathbf{H}})$ for all $y \in K$. There holds ${}^y(\varphi \circ p)(i) = \varphi(p(i))(y) = (p(i))'(y) = (p(i))_y = \hat{y}(p(i))$ for all $i \in I$, $y \in K$. This yields ${}^y(\varphi \circ p) = \hat{y} \circ p$ for all $y \in K$, so that ${}^y(\varphi \circ p) \in \Re(\mathbf{G}^{\mathbf{H}})$ for all $y \in K$. From this $\varphi \circ p \in \Re((\mathbf{G}^{\mathbf{H}})^{\mathbf{K}})$. Hence φ is a homomorphism of $\mathbf{G}^{\mathbf{H}\cdot\mathbf{K}}$ into $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$. Conversely, let $p \in (\mathrm{Hom}(\mathbf{H} \cdot \mathbf{K}, \mathbf{G}))^I$ and suppose that $\varphi \circ p \in \Re((\mathbf{G}^{\mathbf{H}})^{\mathbf{K}})$. Reversing the previous considerations we can easily show that $p \in \Re(\mathbf{G}^{\mathbf{H}\cdot\mathbf{K}})$. Thus φ is an isomorphic embedding of $\mathbf{G}^{\mathbf{H}\cdot\mathbf{K}}$ into $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$.

Suppose moreover that $\mathbf{G}$ is diagonal. Let $p \in \mathrm{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ and put $f(x, y) = p(y)(x)$ for any $x \in H$, $y \in K$. Let $h \in \Re(\mathbf{H} \cdot \mathbf{K})$. Then there exist $h_1 \in S$ and $h_2 \in T$ such that $h(i) = (h_1(i), h_2(i))$ for all $i \in I$. From $p \circ h_2 \in \Re(\mathbf{G}^{\mathbf{H}})$ it follows that ${}^x(p \circ h_2) \in R$ for all $x \in H$. Thus, putting $f_i = {}^{h_1(i)}(p \circ h_2)$ we have $f_i \in R$ for all $i \in I$. Next, as $p(y) \in \mathrm{Hom}(\mathbf{H}, \mathbf{G})$ for all $y \in K$, there is $p(h_2(j)) \in \mathrm{Hom}(\mathbf{H}, \mathbf{G})$ for all $j \in I$. Therefore, putting $g_j = p(h_2(j)) \circ h_1$ we obtain $g_j \in R$ for all $j \in I$. There holds $f_i(j) = {}^{h_1(i)}(p \circ h_2)(j) = p(h_2(j))(h_1(i)) = f(h_1(i), h_2(j))$ and $g_j(i) = (p(h_2(j)) \circ h_1)(i) = p(h_2(j))(h_1(i)) = f(h_1(i), h_2(j))$ for all $i, j \in I$. So $f_i(j) = g_j(i)$ for every pair $i, j \in I$. Since $f_i(i) = f(h(i))$ for all $i \in I$ and since $\mathbf{G}$ is diagonal, we have $f \circ h \in R$. Consequently, $f \in \mathrm{Hom}(\mathbf{H} \cdot \mathbf{K}, \mathbf{G})$. Now, according to the first part of the proof, there holds $p(y)(x) = f(x, y) = f_y(x) = f'(y)(x) = \varphi(f)(y)(x)$ for all $x \in H$, $y \in K$. Hence $p = \varphi(f)$. Therefore φ is a surjection and the proof is complete.

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