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Distributions of Finite Order in the Operational Calculus

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This paper generalizes on the basis of the Bittner operational calculus the algebraic – differential definition of a distribution of finite order introduced by SIKORSKI in [11].

1. Operational Calculus

The operational calculus [4] is referred to as the system

$$CO(L^0, L^1, S, T_q, s_q, q, Q),$$

where L^0 and L^1 are linear spaces over the same field Γ of real or complex numbers; the linear operation $S : L^1 \to L^0$ (which is denoted $S \in L(L^1, L^0)$), called derivative, is a surjection. The elements of the kernel of S, i.e. of the set $\mathrm{Ker}S := \{c \in L^1 : Sc = 0\}$ are called constants for the derivative S. Moreover, Q is an arbitrary set of indices q for the operations $T_q \in L(L^0, L^1)$ such that $ST_q y = y$, $y \in L^0$, called integrals, and for the operations $s_q \in L(L^1, L^1)$ such that $s_q x = x - T_q Sx$, $x \in L^1$, called limit conditions.
By induction a sequence of spaces L^k such that $L^k := \{x \in L^{k-1} : Sx \in L^{k-1}\}, k \in \mathbf{N}$ is defined. Then, there is $\ldots \subset L^k \subset L^{k-1} \subset \ldots \subset L^1 \subset L^0$ and $y = \underbrace{S(S(\ldots(S}_{k-\text{times}} x)\ldots)) =: S^k x$, where $y \in L^0, x \in L^k, S^k \in L(L^k, L^0)$, $S^k(L^k) = L^0$, $k \in \mathbf{N}$. It is also assumed that $S^0 := \mathrm{id}_{L^0}$ and $L^\infty := \bigcap_{k=0}^{\infty} L^k$. It can be shown that for the element $x \in L^k, k \in \mathbf{N}$, the following Taylor formula holds

$$(1) \qquad x = s_q x + T_q s_q Sx + \ldots + T_q^{k-1} s_q S^{k-1} x + T_q^k S^k x, \quad q \in Q.$$

If $s_q S^{k-1}x \neq 0$, then the expression

$$s_q x + T_q s_q Sx + \ldots + T_q^{k-1} s_q S^{k-1} x \tag{2}$$

is called the Taylor polynomial of $(k-1)$ –th degree for the element $x \in L^k$ (at the point $q \in Q$).
The Taylor polynomial (2) can be rewritten in the form

$$c_0 + T_q c_1 + \ldots + T_q^{k-1} c_{k-1},$$

where $c_i = s_q S^i x \in \text{Ker}S,\ i \in \mathcal{Z}_k := \{0, 1, \ldots, k-1\}$.
The element of the form

$$w_{q,\ell} := c_0 + T_q c_1 + \ldots + T_q^{\ell} c_\ell, \quad q \in Q,\ \ell \in \mathbf{N}_0 := \mathbf{N} \cup \{0\},$$

where $c_0, c_1, \ldots, c_\ell \in \text{Ker}\,S, c_\ell \neq 0$, is called a polynomial of the ℓ –th degree (at the point $q \in Q$).
It is evident that $w_{q,\ell} \in L^\infty$ and $S^k w_{q,\ell} = 0$ for $\ell \in \mathcal{Z}_k, \quad k \in \mathbf{N}$.

2. Distributions of Finite Order

In what follows, the set $\text{Ker}S^k := \{x \in L^k : S^k x = 0\}, k \in \mathbf{N}_0$ will be denoted by W_k. Hence we have in particular $W_0 = \{0\}, W_1 = \text{Ker}S$. It follows from the Taylor formula (1) that W_k is the set of all polynomials of degree lower than k, i.e. $W_k = \{w_{q,\ell} : q \in Q, \ell \in \mathcal{Z}_k\}, k \in \mathbf{N}$. Moreover, $W_0 \subset W_1 \subset \ldots \subset W_k \subset L^\infty$, $k \in \mathbf{N}_0$.
We will define, on the cartesian product $\mathbf{N}_0 \times L^0$, an equality relation

$$\left\{[m,x] = [n,y]\right\} \overset{\text{def}}{\Longleftrightarrow} \left\{(m \geq n \wedge x \in L^{m-n} \wedge S^{m-n}x - y \in W_n) \vee \right. \tag{3}$$
$$\left. \vee\, (m \leq n \wedge y \in L^{n-m} \wedge S^{n-m}y - x \in W_m)\right\}, m, n \in \mathbf{N}_0; x, y \in L^0$$

which is of equivalence type.
Reflexivity and symmetry of the relation (3) are evident. We will show that the relation (3) is transitive, i.e.

$$([m,x] = [n,y]) \wedge ([n,y] = [k,z]) \Rightarrow ([m,x] = [k,z]),$$
$$m, n, k \in \mathbf{N}_0; x, y, z \in L^0.$$

On the basis of the symmetry of the relation (3) we may assume $m \geq n \geq k$. Then it follows from the condition $[m, x] = [n, y]$ that $x \in L^{m-n}$ and $S^{m-n}x - y \in W_n$. By way of analogy from the equality $[n, y] = [k, z]$ we obtain $y \in L^{n-k}$ and $S^{n-k}y - z \in W_k$. Since $y \in L^{n-k}$ and $S^{m-n}x - y \in W_n \subset L^{\infty}$, so $S^{m-n}x \in L^{n-k}$. Hence $x \in L^{m-k}$. We also have $S^{n-k}(S^{m-n}x - y) \in W_k$, that is $S^{m-k}x - S^{n-k}y \in W_k$. From this and from the condition $S^{n-k}y - z \in W_k$ it follows that $S^{m-k}x - z \in W_k$. Finally $[m, x] = [k, z]$. Thus the set

$$\mathcal{D} := N_0 \times L^0/" = ",$$

where " = " is the equality relation (3), is composed of equivalence classes, which are called distributions of finite order (in short – distributions).

It follows from this that the representation of the distribution f in the form of an ordered pair $[m, x]$ is not unique. In fact, due to (3) we obtain

$$[m + k, y] = [m, x], \tag{4}$$

if

$$y \in L^k \text{ and } x = S^k y, \quad k \in \mathbf{N}. \tag{5}$$

Further on the distribution f will be identified with an arbitrary representative $[m, x]$ of the class which determines that distribution. Therefore

$$f = [m, x]. \tag{6}$$

The least number $m \in \mathbf{N}_0$ such that, for some element $x \in L^0$, the equality (6) holds is called an order of the distribution f and is denoted by the symbol $r(f)$.

It follows from (4) that the distribution $f = [m, x]$ of the m -th order can be written in the form

$$f = [n, y], \tag{7}$$

if $n \geq m$.
To this end it will be enough to assume

$$y = c_0 + T_q c_1 + \ldots + T_q^{n-m-1} c_{n-m-1} + T_q^{n-m} x, \tag{8}$$

where $q \in Q$ and $c_0, c_1, \ldots, c_{n-m-1} \in \mathrm{Ker} S$ are arbitrary, which follows from (5) and the Taylor formula (1).

Directly from the definition (3) we obtain the following properties of distributions :

$(a)\quad ([m,x]=[m,y]) \Longleftrightarrow (x-y\in W_m)$

$(a_1)\quad ([1,x]=[1,y]) \Longleftrightarrow (x-y=c\in \mathrm{Ker}S)$

$(a_2)\quad ([0,x]=[0,y]) \Longleftrightarrow (x=y)$

$(b)\quad ([m,x]=[n,y]) \Longrightarrow ([m+k,x]=[n+k,y])$

$(b_1)\quad ([m,x]=[n,y]) \Longrightarrow ([m+1,x]=[n+1,y])$

$(c)\quad ([m,x_1]=[n,y_1]\wedge[m,x_2]=[n,y_2]) \Longrightarrow ([m,\alpha x_1+\beta x_2]= = [n,\alpha y_1+\beta y_2]),\ \alpha,\beta\in\Gamma$

$(c_1)\quad ([m,x]=[n,y]) \Longrightarrow ([m,\alpha x]=[n,\alpha y]),\ \alpha\in\Gamma$

$(c_2)\quad ([m,x_1]=[n,y_1]\wedge[m,x_2]=[n,y_2]) \Longrightarrow ([m,x_1+x_2]=[n,y_1+y_2]).$

Let $f,g\in\mathcal{D}$. It follows from (7) that the distributions f and g can be written in the form

$$f=[k,x],\quad g=[k,y], \tag{9}$$

where $k\geq \max(r(f),r(g))$. This fact allows us to determine addition in the set $\mathcal{D}$:

$$f+g:=[k,x+y]. \tag{10}$$

The relation (3) of the equality of distributions is consistent with the operation (10). It follows from the property (c_2) that the sum (10) of the distributions f,g does not depend on the way they are exhibited in the form (9).
The product of the distribution (6) by the element $\alpha\in\Gamma$ is called a distribution defined by the form

$$\alpha f:=[m,\alpha x]. \tag{11}$$

It follows from the property (c_1) that the multiplication (11) of the distribution f by the number α does not depend on the way the distribution f is exhibited in the form (6). Thus the relation (3) of the equality of distributions is consistent with the operation (11).
The distribution $0:=[0,0]$ is called zero distribution.

It is easy to verify that $([m,x]=0)\Longleftrightarrow(x\in W_m)$.

Corollary 1. *The set of distributions $\mathcal{D}$ is a linear space over the field Γ.*

The elements of the space L^0 can be identified with the zero order distributions since the map $x\mapsto[0,x]$, where $x\in L^0,[0,x]\in\mathcal{D}$, is an isomorphism. In this sense L^0 is a linear subspace of the space $\mathcal{D}$ and the elements of the space L^0 are distributions, i.e.

$$x = [0, x]. \tag{12}$$

It should be noticed that the space $\mathcal{D}$ is significantly richer in elements than the space L^0.
The element

$$f = [1, x] \tag{13}$$

does not belong to L^0, for $x \in L^0 - L^1$, since for $f \in L^0$, there would be $[1, x] = [0, f]$, hence $x \in L^1$, what contradicts the assumption.
The operation $D : \mathcal{D} \to \mathcal{D}$ defined by the formula

$$D[m, x] := [m + 1, x] \tag{14}$$

is called a distributional derivative.

The distributional derivative D is an endomorphism of the space $\mathcal{D}$. It follows from the property (b_1) that the relation (3) of the equlity of distributions is consistent with the distributional differentation (14). Distributional derivatives of higher orders are determined by induction:

$$D^1 := D, \ D^{k+1} := DD^k, \qquad k \in \mathbf{N}.$$

In addition, we assume $D^0 := I$, where I is the identity operation determined over the space $\mathcal{D}$.

Corollary 2. *Each distribution has a distributional derivative of an arbitrarily high order.*

Corollary 3. *Each distribution $f = [m, x]$ is an m –th distributional derivative of the zero order distribution $[0, x]$, i.e. of the element $x \in L^0$.*

Corollary 4. *For each distribution $f \in L^k, k \in \mathbf{N}_0$ we have*

$$D^k f = S^k f. \tag{15}$$

PROOF. Since $f \in L^k$ and $S^k f - S^k f \in W_0, k \in \mathbf{N}_0$, thus $[k, f] = [0, S^k f]$, which means the equality (15).

Theorem 1. $KerD^k = W_k, \ k \in \mathbf{N}_0$.

PROOF. For $k = 0$ the theorem is evident. Let, then, $f = [m, x] \in \operatorname{Ker} D^k$ for the fixed $k \in \mathbf{N}$. Hence we obtain $[m + k, x] = [0, 0]$, thus $x \in L^{m+k}$ and $S^{m+k}x = 0$. Therefore $S^m x \in W_k$ which means that $S^m x$ can be an arbitrarily fixed polynomial of at most the $(k-1)$ –th degree. Let $S^m x = w_{q,\ell}$, where $q \in Q$ and $\ell \in \mathcal{Z}_k$ are fixed. Thus $S^m x - w_{q,\ell} \in W_0$. Hence, since $x \in L^m$, we obtain $f = [m, x] = [0, w_{q,\ell}] = w_{q,\ell}$, which, due to the arbitrariness of f, corresponds to $\operatorname{Ker} D^k \subset W_k$. Inclusion $W_k \subset \operatorname{Ker} D^k$ follows from the Corollary 4.

The distribution F is called a primitive distribution of the distribution f if $DF = f$.

Theorem 2. 1^0 *For each distribution f there exists a primitive distribution F.*
2^0 *The distribution*

$$\Phi = F + c, \qquad c \in KerS \tag{16}$$

is also a primitive distribution of f.
3^0 *Each primitive distribution Φ of f can be written in the form (16).*

PROOF. 1^0 Let $f = [m, x]$ and $y = c + T_q x$, where $c \in \mathrm{Ker} S$ and $q \in Q$ are arbitrarily fixed. It should be noticed that $[m+1, y] = [m, x]$, since $y \in L^1$ and $Sy - x \in W_0 \subset W_m$. Assuming $F := [m, y]$ we have then $DF = f$.
2^0 From the Theorem 1 we obtain $Dc = 0$, $c \in \mathrm{Ker} S$. Therefore $D\Phi = DF + Dc = f$.
3^0 Since $\Phi - F \in \mathrm{Ker} D$, we have $\Phi - F = c$, $c \in \mathrm{Ker} S$ on the basis of the Theorem 1.

The set of all primitive distributions of f is called an indefinite integral of the distribution f and is denoted by the symbol Tf.

It follows from the last theorem on primitive distributions that

$$Tf = \{F + c : c \in \mathrm{Ker} S\}, \tag{17}$$

where F is an arbitrary primitive distribution of f. We write

$$DTf = f \tag{18}$$

bearing in mind that the distributional derivative of each primitive distribution of f is equal to f.
The equality (17) can be written briefly as

$$Tf = F + c, \tag{19}$$

where F designates any primitive distribution of f, whereas c is an arbitrary constant called an integration constant.
Taking into account the formulas (18) and (19) we can write

$$TDF = F + c. \tag{20}$$

The map $T : \mathcal{D} \to \mathcal{D}$ defined by the formula

$$f \mapsto Tf$$

is called an indefinite integration operation.

It is a linear operation, i.e.

$$T(\alpha f + \beta g) = \alpha T f + \beta T g, \quad f, g \in \mathcal{D}; \quad \alpha, \beta \in \Gamma. \tag{21}$$

The equality (21), which has symbols of some primitive distribution sets on both sides, is considered true if and only if the distributional derivatives on its two sides are identical.
Due to formula (18), the equality (21) follows from the linearity of the operation D.
The element $X \in L^1$ is called a primitive element of the element $x \in L^0$ if $SX = x$.

It follows from the axiom $ST_q = \mathrm{id}_{L^0}, q \in Q$ of the operational calculus that the primitive elements corresponding to x have the form

$$T_q x, \qquad q \in Q.$$

It can easily be verified that, for arbitrary $q_1, q_2 \in Q$,

$$T_{q_1} x - T_{q_2} x \in \mathrm{Ker} S.$$

Hence, it follows that if we know at least one integral T_q corresponding to S then we know one primitive element of x. All other primitive elements corresponding to the element x are obtained by the addition of a constant. The set of all primitive elemets of x is called an indefinite integral of the element x and is denoted by the symbol Tx.

Therefore we have

$$Tx = \{T_q x + c : c \in \mathrm{Ker} S\}, \tag{22}$$

where T_q is an arbitrary integral corresponding to S.
We write here

$$STx = x. \tag{23}$$

The map $T : L^0 \to L^0$ given by the formula

$$x \mapsto Tx$$

is called an indefinite integration operation.

The identification (12) of the elements of L^0 with the zero order distributions preserves the indefinite integration operation. In fact, if Tx is an indefinite integral of the element x the distribution $[0, Tx]$ is then an indefinite integral of the distribution $[0, x]$, i.e. $T[0, x] = [0, Tx]$. Thus, on the basis of (23) we have

$$DT[0, x] = [1, Tx] = [0, x]. \tag{24}$$

It follows from (22) and (24) that each zero order distribution can be written in the form

$$[0, x] = [1, c + T_q x], \tag{25}$$

where $c \in \mathrm{Ker}S$ and $q \in Q$ are arbitrary.
The dependence (25) is a specific case of the equality (7) if we assume $m = 0, n = 1$ in (8).

Let $L \supset L^0$ be a linear space over the field Γ. Moreover, let T_q be an integral corresponding to S such that $T_q(L - L^0) \subset L^0 - L^1$.
The equality (25) suggests an extension of the notion of distribution over the elements of L. The element $f \in L$ will be identified with the distribution $[1, y]$:

$$(26) \qquad f = [1, y] \quad \text{if} \quad y = c + T_q f, \qquad c \in \mathrm{Ker}S.$$

It is not difficult to verify that this identification preserves algebraic operations determined on L.
In the case when $f \in L^0$ the identification (26) corresponds to (12). The element $f \in L$ is called a regular distribution. The element $f \in \mathcal{D} - L$ is called a singular distribution.

Let L^0 be a commutative algebra and $L^k, k \in \mathbf{N}$ its subalgebra. It is said that, for the derivative S, the Leibniz formula holds if

$$(27) \qquad S(x \cdot y) = Sx \cdot y + x \cdot Sy, \qquad x, y \in L^1.$$

Theorem 3. *If the derivative S satisfies the Leibniz condition (27), then the following Schwartz formula holds:*

$$(28) \quad x \cdot S^n y = S^n y \cdot x = \sum_{i=0}^{n} (-1)^i \binom{n}{i} S^{n-i}(S^i x \cdot y), \ x, y, \in L^n; \ n \in \mathbf{N}_0.$$

PROOF. For $n = 0$ the equality (28) is evident. Assuming the validity of the formula (28) for a fixed n we have

$$x \cdot S^{n+1} y = S(x \cdot S^n y) - Sx \cdot S^n y = \sum_{i=0}^{n} (-1)^i \binom{n}{i} S^{n+1-i}(S^i x \cdot y) +$$
$$+ \sum_{i=0}^{n} (-1)^{i+1} \binom{n}{i} S^{n-i}(S^{i+1} x \cdot y) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} S^{n+1-i}(S^i x \cdot y) +$$
$$+ \sum_{i=1}^{n+1} (-1)^i \binom{n}{i-1} S^{n+1-i}(S^i x \cdot y) = S^{n+1}(x \cdot y) +$$
$$+ \sum_{i=1}^{n} (-1)^i \left[\binom{n}{i} + \binom{n}{i-1} \right] S^{n+1-i}(S^i x \cdot y) + (-1)^{n+1} S^{n+1} x \cdot y =$$
$$= \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} S^{n+1-i}(S^i x \cdot y), \qquad x, y \in L^{n+1}.$$

An application of the induction principle finishes the proof for $n \geq 0$.

If S satisfies the Leibniz condition, the equality (28) then suggests the following definition of the product of the element $x \in L^\infty$ by the distribution $f = [n, y] \in \mathcal{D}$:

$$x \cdot f = f \cdot x := \sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{n-i}(S^i x \cdot y). \tag{29}$$

Using the definition of the distributional derivative D equality (29) can be rewritten in the form

$$x \cdot f = f \cdot x = \sum_{i=0}^{n} (-1)^i \binom{n}{i} [n - i, S^i x \cdot y]. \tag{30}$$

Theorem 4. *The relation (3) of the equality of distributions is consistent with the multiplication (29).*

PROOF. It must be shown that the distribution (29) does not depend on the way the distribution f is presented in the form of an ordered pair. In other words, it must be proved that if

$$f = [n, y] = [n, z], \tag{31}$$

then

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{n-i}(S^i x \cdot y) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{n-i}(S^i x \cdot z). \tag{32}$$

Moreover, if

$$u \in L^1, \quad Su = y \tag{33}$$

that is

$$f = [n + 1, u] = [n, y],$$

then

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{n-i}(S^i x \cdot y) = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} D^{n+1-i}(S^i x \cdot u). \tag{34}$$

It follows from (31) that $y - z \in W_n$, i.e. the difference $y - z$ can be an arbitrarily fixed polynomial of at most $(n-1)$ -th degree. Let $y - z = w_{q,\ell}$, where $q \in Q$ and $\ell \in \mathcal{Z}_n$ are fixed. Using the linearity of the operation D we infer that the difference R between the left and the right sides of the formula (32) is equal to

$$R = \sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{n-i}(S^i x \cdot w_{q,\ell}).$$

Since $x, w_{q,\ell} \in L^\infty$, thus $S^i x \cdot w_{q,\ell} \in L^{n-i}, i = 0, 1, \dots, n$. From this and from the Corollary 4 we obtain

$$D^{n-i}(S^i x \cdot w_{q,\ell}) = S^{n-i}(S^i x \cdot w_{q,\ell}), \qquad i = 0, 1, \dots, n,$$

which means that

$$R = \sum_{i=0}^{n} (-1)^i \binom{n}{i} S^{n-i}(S^i x \cdot w_{q,\ell}).$$

Thus $R = x \cdot S^n w_{q,\ell}$ on the basis of the Schwartz formula (28). Hence $R = 0$. If $u \in L^1$, then $S^i x \cdot u \in L^1, i = 0, 1, \dots, n$. Thus

$$D(S^i x \cdot u) = S(S^i x \cdot u), \qquad i = 0, 1, \dots, n$$

on the basis of Corollary 4. Hence, from the Leibniz formula (27) and from (33) we obtain

$$\begin{aligned}
&\sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{n-i}(S^i x \cdot y) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{n-i}[S(S^i x \cdot u) - S^{i+1} x \cdot u] \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{n+1-i}(S^i x \cdot u) + \sum_{i=0}^{n} (-1)^{i+1} \binom{n}{i} D^{n-i}(S^{i+1} x \cdot u) = \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{n+1-i}(S^i x \cdot u) + \sum_{i=1}^{n+1} (-1)^i \binom{n}{i-1} D^{n+1-i}(S^i x \cdot u) = \\
&= D^{n+1}(x \cdot u) + \sum_{i=1}^{n} (-1)^i \left[\binom{n}{i} + \binom{n}{i-1} \right] D^{n+1-i}(S^i x \cdot u) + \\
&\qquad + (-1)^{n+1} S^{n+1} x \cdot u = \\
&\qquad = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} D^{n+1-i}(S^i x \cdot u)
\end{aligned}$$

that is the equality (34).

For $n = 0$ the equalities (28) and (29) are identical and consistent with the multiplication in the algebra L^0.
It should be noticed, that for $x \in L^\infty, y \in L^n, n \in \mathbf{N}$ we have

$$D^n y = S^n y,\ D^{n-i}(S^i x \cdot y) = S^{n-i}(S^i x \cdot y), \qquad i = 0, 1, \ldots, n,$$

which follows from Corollary 4.
In this case the product (28) also represents the multiplication (29) of the regular distributions x and $S^n y$.
It is not difficult to prove that

$$(x + y)f = x \cdot f + y \cdot f,\ x(f + g) = x \cdot f + x \cdot g,$$

$$(x \cdot y)f = x(y \cdot f),$$

where $x, y \in L^\infty$ and $f, g \in \mathcal{D}$.
From (30) we obtain the formula for the product of the constant $c \in \mathrm{Ker}S$ by the distribution $f = [n, y] \in \mathcal{D}$:

$$c \cdot f = [n, c \cdot y]. \tag{35}$$

If L^0 is an algebra with the unity e belonging to $\mathrm{Ker}S$ and

$$\bigwedge_{c \in \mathrm{Ker}S} \bigvee_{\alpha \in \Gamma} c = \alpha e,$$

i.e. $\dim \mathrm{Ker}S = 1$, then the product (35) is identical with the previously defined product (11) of the element $\alpha \in \Gamma$ by the distribution f.

Theorem 5. *For the distributional derivative D the Leibniz formula*

$$D(x \cdot f) = Dx \cdot f + x \cdot Df \tag{36}$$

holds, where $x \in L^\infty, f \in \mathcal{D}$.

PROOF. Since $x \in L^\infty$, on the basis of Corollary 4 we have $Dx = Sx$. Using formula (29) we see that the following must be proved

$$\sum_{i=0}^{n}(-1)^i \binom{n}{i} D^{n+1-i}(S^i x \cdot y) = \sum_{i=0}^{n}(-1)^i \binom{n}{i} D^{n-i}(S^{i+1} x \cdot y) +$$

$$+ \sum_{i=0}^{n+1}(-1)^i \binom{n+1}{i} D^{n+1-i}(S^i x \cdot y).$$

The proof of this equality runs as follows:

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}D^{n-i}(S^{i+1}x\cdot y)+\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}D^{n+1-i}(S^ix\cdot y)=$$
$$=\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i-1}D^{n+1-i}(S^ix\cdot y)+D^{n+1}(x\cdot y)-$$
$$-\sum_{i=1}^{n}(-1)^{i-1}\binom{n+1}{i}D^{n+1-i}(S^ix\cdot y)=D^{n+1}(x\cdot y)+$$
$$+\sum_{i=1}^{n}(-1)^{i-1}\left[\binom{n}{i-1}-\binom{n+1}{i}\right]D^{n+1-i}(S^ix\cdot y)=$$
$$=\sum_{i=0}^{n}(-1)^i\binom{n}{i}D^{n+1-i}(S^ix\cdot y).$$

Theorem 6. *If $x\in L^\infty$ and $f\in\mathcal{D}$, then the following formula holds for partial integration:*

$$T(x\cdot Df)=x\cdot f-T(Dx\cdot f). \tag{37}$$

PROOF. Formula (37) should be understood just as the equality (21). Since

$$D[x\cdot f-T(Dx\cdot f)]=Dx\cdot f+x\cdot Df-Dx\cdot f=x\cdot Df=DT(x\cdot Df),$$

which follows from the Leibniz formula (36), therefore the equality (37) is true.

3. Convergence in a Distribution Space

Let us assume the following definition of a partially ordered linear space, which BITTNER in [4] calls Mikusiński space.
By a Mikusiński space we mean a real linear space Z, where we distinguish a cone K of the following properties:

1. $0\in K$
2. if $z_1,z_2\in K$ then $z_1+z_2\in K$
3. if $z\in K,\alpha\geq 0$ then $\alpha z\in K$
4. if $z_1\in K$ and there exists $z_2\in K$ such that for each $n\in\mathbf{N}$ $z_2-nz_1\in K$ then $z_1=0$
5. for each $z\in Z$ there exist $z_1,z_2\in K$ such that $z=z_1-z_2$.

If $z_1,z_2\in Z$ then we assume by definition

$$z_1\leq z_2 \quad\text{if and only if}\quad z_2-z_1\in K.$$

The notation $z_2 \geq z_1$ is also used.
In addition, for each element $z \in Z$ we define a modulus $|z| \in K$ satisfying the following conditions:

6. $|z| = z$ for $z \geq 0$
7. if $|z| = 0$ then $z = 0$
8. $|z_1 + z_2| \leq |z_1| + |z_2|$
9. $|\alpha z| = |\alpha||z|, \quad z, z_1, z_2 \in Z; \alpha \in R^1$.

The notion of regular convergence (convergence with a regulator f) [4] can be introduced in a Mikusiński space Z (cf [7, 8]).
It is said that a sequence of elements $z_k \in Z, k \in \mathbf{N}_0$ converges to the element $z \in Z$, written $(z_k) \Longrightarrow z$, if and only if

$$\bigvee_{f \in K} \bigwedge_{\varepsilon > 0} \bigvee_{n \in \mathbf{N}_0} \bigwedge_{k \geq n} |z_k - z| \leq \varepsilon \cdot f. \tag{38}$$

In [4], the convergence (38) has also been called Mikusiński convergence.
The convergence in L^0 can be introduced as the convergence in a space normed by elements belonging to the cone K in the Mikusiński space Z.
It is said that in the space L^0 a norm $\|x\| \in Z, x \in L^0$ is determined, i.e. a norm of values from the Mikusiński space Z, if

1. $\|x\| \geq 0$; if $\|x\| = 0$ then $x = 0$
2. $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$
3. $\|\gamma x\| = |\gamma|\|x\|, \quad x, x_1, x_2 \in L^0; \gamma \in \Gamma$ [4].

It is said that the sequence $(x_k)_{k \in \mathbf{N}_0}$ of elements from L^0 converges to the element $x \in L^0$, written $(x_k) \Longrightarrow x$, if $\|x_k - x\| \Longrightarrow 0$ in the space Z [4].

The space L^0 together with the norm $\|\cdot\|$, i.e. the pair $(L^0, \|\cdot\|)$, is called a normed space.

The (commutative), algebra L^0, which is the normed space, is called a (commutative) quasi–normed algebra.

Henceforth we will be assuming that L^0 is a quasi–normed commutative algebra, in which multiplication is sequentially–continuous, i.e.

$$\text{if } x_k, y_k, x, y \in L^0, k \in \mathbf{N}_0 \text{ and } (x_k) \Longrightarrow x, (y_k) \Longrightarrow y \quad \text{then } (x_k \cdot y_k) \Longrightarrow x \cdot y.$$

We also require that the integrals $T_q, q \in Q$ be continuous operations and the sets

$$W_{q,\ell} := \{w_{q,\ell} : w_{q,\ell} = c_0 + T_q c_1 + \ldots + T_q^\ell c_\ell, c_i \in \operatorname{Ker} S, i = 0, 1, \ldots, \ell\}, \quad q \in Q; \ell \in \mathbf{N}_0$$

be closed in L^0.
It is said that the sequence of distributions $f_k \in \mathcal{D}, k \in \mathbf{N}_0$ converges to the distribution $f \in \mathcal{D}$, written

$$(f_k) \to f, \tag{39}$$

if and only if there exists a number $m \in \mathbf{N}_0$, a sequence $x_k \in L^0, k \in \mathbf{N}_0$ and an element $x \in L^0$ such that

$$f_k = [m, x_k], \quad f = [m, x], \quad k \in \mathbf{N}_0$$

and $(x_k) \Longrightarrow x$.

If (39) holds we also say that the sequence $(f_k)_{k \in \mathbf{N}_0}$ is distributionally convergent to f.
Each sequence $(x_k)_{k \in \mathbf{N}_0}$ of elements from L^0 convergent according to the norm $\|\cdot\|$ is distributionally convergent, since

$$\Big\{(x_k) \Longrightarrow x\Big\} \equiv \Big\{([0, x_k]) \to [0, x]\Big\}.$$

Theorem 7. 1^0 *The convergence in the space $\mathcal{D}$ is determined uniquely.*
2^0 *The operations of addition of distributions and of multiplication of a distribution by a number from the field Γ are distributionally sequentially–continuous.*

PROOF. 1^0 Let us assume that $(f_k) \to f$ and $(f_k) \to g$. Then there exist $m, n \in \mathbf{N}_0; x_k, y_k, x, y \in L^0, k \in \mathbf{N}_0$ such that

$$\begin{aligned} &f_k = [m, x_k], \quad f = [m, x], \quad (x_k) \Longrightarrow x, \\ &f_k = [n, y_k], \quad g = [n, y], \quad (y_k) \Longrightarrow y, \quad k \in \mathbf{N}_0. \end{aligned}$$

We put, say $m \le n$. Assuming $\tilde{x}_k := T_q^{n-m} x_k, k \in \mathbf{N}_0$, where $q \in Q$ is fixed, we have $(\tilde{x}_k) \Longrightarrow T_q^{n-m} x =: \tilde{x}$ and

$$f_k = [n, \tilde{x}_k] = [m, x_k], \; f = [n, \tilde{x}] = [m, x], \; k \in \mathbf{N}_0.$$

Now $f_k = [n, \tilde{x}_k] = [n, y_k], k \in \mathbf{N}_0$ so $\tilde{x}_k - y_k = w^k_{q,\ell}, k \in \mathbf{N}_0$, where $w^k_{q,\ell}, k \in \mathbf{N}_0$ is an arbitrarily fixed polynomial of degree $\ell \in \mathcal{Z}_n$. In addition, $W_{q,\ell} = \overline{W}_{q,\ell}$. Thus $(\tilde{x}_k - y_k) \Longrightarrow \tilde{x} - y = w_{q,\ell}$, where $(w^k_{q,\ell}) \Longrightarrow w_{q,\ell}$. Hence

$$f = [n, \tilde{x}] = [n, y] = g.$$

2^0 Let us assume that $(f_k) \to f$ and $(g_k) \to g$. Then there exist $m, n \in \mathbf{N}_0; x_k, y_k, x, y \in L^0, k \in \mathbf{N}_0$ such that

$$f_k = [m, x_k],\ f = [m, x],\ (x_k) \Longrightarrow x,$$
$$g_k = [n, y_k],\ g = [n, y],\ (y_k) \Longrightarrow y,\ k \in \mathbf{N}_0.$$

For each $k \in \mathbf{N}_0$ the distributions f_k and g_k can be written in the form $f_k = [\nu, \tilde{x}_k], g_k = [\nu, \tilde{y}_k]$, where $\nu \geq \max(r(f_k), r(g_k))$. Let $m \leq n$. Then $\nu = n$ can be assumed, as well as $\tilde{x}_k := T_q^{n-m} x_k, \tilde{y}_k := y_k, k \in \mathbf{N}_0$, where $q \in Q$ is fixed. Thus we have $f_k = [n, \tilde{x}_k] = [m, x_k], k \in \mathbf{N}_0$ and $f = [n, \tilde{x}] = [m, x]$, where $\tilde{x} := T_q^{n-m} x, (\tilde{x}_k) \Longrightarrow \tilde{x}$. Therefore

$$(f_k) + (g_k) = ([n, \tilde{x}_k + y_k]) \to [n, \tilde{x} + y] = f + g,$$

since $(\tilde{x}_k + y_k) \Longrightarrow \tilde{x} + y$.
If $\lim_{k\to\infty} \alpha_k = \alpha, \quad \alpha_k, \alpha \in \Gamma, k \in \mathbf{N}_0$, then the condition $(x_k) \Longrightarrow x$ implies $(\alpha_k x_k) \Longrightarrow \alpha x$. Thus from the condition $(f_k) = ([m, x_k]) \to [m, x] = f$ we obtain

$$(\alpha_k f_k) = ([m, \alpha_k x_k]) \to [m, \alpha x] = \alpha f.$$

Theorem 8. *If $x_k, x \in L^\infty; f_k, f \in \mathcal{D}, k \in \mathbf{N}_0$ and $(S^i x_k) \Longrightarrow S^i x$, $i \in \mathbf{N}_0, (f_k) \to f$, then $(x_k \cdot f_k) \to x \cdot f$.*

PROOF. Let $(f_k) \to f$. Then there exists a number $n \in \mathbf{N}_0$, a sequence $y_k \in L^0, k \in \mathbf{N}_0$ and an element $y \in L^0$ such that

$$f_k = [n, y_k],\ f = [n, y],\ (y_k) \Longrightarrow y,\ k \in \mathbf{N}_0.$$

Since, for each $i \in \mathbf{N}_0$, we have $(S^i x_k \cdot y_k) \Longrightarrow S^i x \cdot y$, we get

$$([n - i, S^i x_k \cdot y_k]) \to [n - i, S^i x \cdot y], \qquad i \leq n.$$

From this and from definition (30) we obtain the proposition.

Corollary 5. *If $x \in L^\infty; f_k, f \in \mathcal{D}, k \in \mathbf{N}_0$ and $(f_k) \to f$, then $(x \cdot f_k) \to x \cdot f$.*

Theorem 9. *The derivative D is distributionally sequentially–continuous, i.e.*

$$(f_k) \to f \quad \text{implies} \quad (Df_k) \to Df, \quad f_k, f \in \mathcal{D}, k \in \mathbf{N}_0.$$

PROOF. If $(f_k) \to f$, then there exists a number $m \in \mathbf{N}_0$, a sequence $x_k \in L^0, k \in \mathbf{N}_0$ and an element $x \in L^0$ such that

$$f_k = [m, x_k],\ f = [m, x],\ (x_k) \Longrightarrow x,\ k \in \mathbf{N}_0.$$

Since, for each $k \in \mathbf{N}_0$, we have $Df_k = [m + 1, x_k], Df = [m + 1, x]$, by substituting the number $m + 1$ for m in the distributional convergence $(f_k) \to f$ we obtain the proposition.

Let us consider the sequence of distributions $f_k \in \mathcal{D}, k \in \mathbf{N}_0$. The sequence $(\sigma_n)_{n\in\mathbf{N}_0}$ of partial sums

$$\sigma_n := \sum_{k=0}^{n} f_k$$

is called a distributional series and is denoted by the symbol $\sum\limits_{k=0}^{\infty} f_k$. We write $f = \sum\limits_{k=0}^{\infty} f_k$ if and only if $(\sigma_n) \to f$.

Theorem 10. *If* $f = \sum\limits_{k=0}^{\infty} f_k$, *then* $Df = \sum\limits_{k=0}^{\infty} Df_k$.

PROOF. It follows from the last theorem that the condition $(\sigma_n) \to f$ implies $(D\sigma_n) \to Df$, which means the proposition.

Theorem 11. *If* $y_n := \sum\limits_{k=0}^{n} x_k;\ x_k, y \in L^\infty, k, n \in \mathbf{N}_0; f \in \mathcal{D}$ *and* $(S^i y_n) \Longrightarrow S^i y,\ i \in \mathbf{N}_0$, *then* $\left(\sum\limits_{k=0}^{\infty} x_k\right) f = \sum\limits_{k=0}^{\infty} x_k \cdot f$.

PROOF. The proposition of the theorem follows from the Theorem 8.

Theorem 12. *If* $f = \sum\limits_{k=0}^{\infty} f_k,\ x \in L^\infty$, *then* $x \cdot f = \sum\limits_{k=0}^{\infty} x \cdot f_k$.

PROOF. The proposition of the theorem follows from the Corollary 5.

The normed space $(L^0, \|\cdot\|)$ is called a Weierstrass space if, for each element $x \in L^0$, there exists a sequence of polynomials $(w_{q,k})_{k\in\mathbf{N}_0}$ such that $(w_{q,k}) \Longrightarrow x$.

Theorem 13. *If L^0 is a Weierstrass space then, for each distribution $f \in \mathcal{D}$, there exists a sequence of polynomials $(w_{q,k})_{k\in\mathbf{N}_0}$ such that $(w_{q,k}) \to f$.*

PROOF. Let $f = [m, x] \in \mathcal{D}$. Since L^0 is a Weierstrass space, for $x \in L^0$ there exists a sequence of polynomials $(v_{q,k})_{k\in\mathbf{N}_0}$ such that

$$(v_{q,k}) \Longrightarrow x. \tag{40}$$

Let $w_{q,k} := S^m v_{q,k},\ k \in \mathbf{N}_0$. Thus $[m, v_{q,k}] = [0, w_{q,k}] = w_{q,k}, k \in \mathbf{N}_0$. From this and from (40) it follows that $(w_{q,k}) \to f$.

4. Examples

A. Let $Q \subset R^1$ be a fixed bounded or unbounded interval. Moreover, let there be given the classical model of operational calculus, in which

$$L^0 := C^0(Q, R^1), \quad L^1 := C^1(Q, R^1)$$

and

$$Sx := \left\{ \frac{dx(t)}{dt} \right\}, \; T_q y := \left\{ \int_q^t y(\tau) d\tau \right\}, \; s_q x := \{x(q)\},$$

where $q \in Q, x = \{x(t)\} \in L^1, y = \{y(t)\} \in L^0$.
In this case the definition of distributions of finite order presented in the paper agrees with the algebraico–differential definition formulated by SIKORSKI in [11]. It follows from the assumed operational calculus and from (12) that all the continuous functions on Q are distributions. If we denote by L the space of functions locally integrable on Q (in the Lebesgue sense) then we obtain $L \subset \mathcal{D}$ from (26). In addition, the element (13), i.e. the pair $[1, \{x(t)\}]$, where $\{x(t)\}$ is a continuous function on Q which has no derivative at least at one point, is a distribution which is not a function.
Let us assign to each element $x = \{x(t)\} \in L^0$ a sequence of real quasi–norms

$$\|x\|_k := \sup\{ \, |x(t)| : t \in \overline{Q}_k \, \}, \qquad k \in \mathbf{N}_0,$$

where $(\overline{Q}_k)_{k \in \mathbf{N}_0}$ is a sequence of closed, bounded intervals such that $\bigcup_{k \in \mathbf{N}_0} \overline{Q}_k = Q$.
The set $C(N_0)$ of real sequences $z = (z_k)_{k \in \mathbf{N}_0}$ is a real linear space with the usual addition of sequences and the multiplication of a sequence by a real number. It is a Mikusiński space for the ordering:

$$z \geq 0 \quad \text{if and only if} \quad z_k \geq 0 \quad \text{for each} \quad k \in \mathbf{N}_0$$

and for the modulus

$$|z| := (|z_k|)_{k \in \mathbf{N}_0}.$$

L^0 is a normed space if

$$\|x\| := (\|x\|_k)_{k \in \mathbf{N}_0} \in C(N_0), \qquad x \in L^0. \tag{41}$$

The convergence induced by the norm (41) is an almost uniform convergence on the interval Q. With the help of this convergence the convergence (39) of distributions in the Sikorski sense is determined.

The development of the theory of distributions based on the Sikorski definition is presented in [12].

It can be shown (see [1]) that the notion of distributions in the Sikorski sense is equivalent to the sequential approach to distributions in the Mikusiński sense. This equivalence is established by identification of the distribution determined by the pair $[m, \{x(t)\}]$ with the distribution $\{x^{(m)}(t)\}$ (in sequential approach), i.e. with the distribution which is an m -th order distributional derivative of the continuous function $\{x(t)\}$.

B. In the Euler model for operational calculus, in which $q \in Q := (0, +\infty)$, L^0 and L^1 are determined as previously, the norm (41) is given in L^0 as well as

$$Sx := \left\{ t\frac{dx(t)}{dt} \right\}, \; T_q y := \left\{ \int\limits_q^t \frac{y(\tau)}{\tau} d\tau \right\}, \; s_q x := \{x(q)\},$$

where $x = \{x(t)\} \in L^1, y = \{y(t)\} \in L^0$, the sequence of regular distributions

$$f_k = \{(k+1)t\cos(k+1)^2 t\}, \qquad k \in \mathbf{N}_0$$

is divergent in the space L^0 and convergent in the space $\mathcal{D}$. Hence we have

$$(f_k) = \left(\left[1, \left\{ \frac{\sin(k+1)^2 t}{k+1} \right\} \right] \right) \to [1, 0] = 0.$$

C. Let $x = (x_k),\ y = (y_k) \in C(\mathbf{N}_0)$. We write $x = y$ if and only if

(42) $$x_k = y_k \quad \text{for each} \quad k \in \mathbf{N}_0.$$

In the discrete model for operational calculus, in which $L^0 = L^1 := C(\mathbf{N}_0)$ and

$$Sx := (x_{k+1} - x_k), \qquad s_{k_0} x := (x_{k_0}),$$

$$T_{k_0} x := \begin{cases} -\sum\limits_{i=k}^{k_0 - 1} x_i & \text{for } k < k_0 \\ 0 & \text{for } k = k_0, \ q = k_0 \in Q := \mathbf{N}_0 \\ \sum\limits_{i=k_0}^{k-1} x_i & \text{for } k > k_0 \end{cases}$$

each distribution of the form (13) is a sequence from $C(\mathbf{N}_0)$. In this case the relation (3) different from (42) must be used in comparing the sequences.

D. In the models for operational calculus, in which derivatives have, for example, the form

$$S := a(t)\frac{d}{dt},\ S := \frac{\partial}{\partial \xi} + b\frac{\partial}{\partial \eta} \qquad \text{(see [6])},$$

where $\{a(t)\}$ is a continuous function determined on the interval $[\alpha, \beta]$ such that $a(t) \neq 0$, for each $t \in [\alpha, \beta]$ and $b \in R^1 - \{0\}$, the multiplication of the distributions defined by the formula (29) can be introduced since these operations satisfy the Leibniz condition (27) for the usual function multiplication.

E. The abstract definition of distributions discussed in the paper permits forming a space of random distributions (so called generalized stochastic processes, cf [14]). To this end the space of the second order stochastic processes (Hilbert processes), continuous in the quadratic mean sense, should be assumed as L^0 while as operations

$$S := \frac{d}{dt}, \qquad T_q := \int_q^t$$

the quadratic mean derivative and the Riemann quadratic mean integral of the second order stochastic process respectively (see [13]).

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