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Central automorphisms of standard wreath products

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1. Introduction

Let $W = AW_rB$ be the standard wreath product of the groups A and B, where if $|B| = 2$ then $|A| \neq 2$ or A is not the dihedral group of order $4m+2$. It is known that the group of central automorphisms $\mathrm{Aut}_c(W)$ of the group W is the direct product of the groups K_c and I_c where K_c is the group of all central automorphisms which leave the group B elementwise fixed and I_c is the group of all inner central automorphisms which are induced by the elements of the base group A^B. (cf. J. PANAGOPOULOS [3]). In this paper we study the groups K_c and I_c. It is proved that the group K_c contains a normal subgroup H_c and the factor group K_c/H_c is isomorphic to a subgroup of $\mathrm{Aut}_c(A)$. (Proposition 3.1). If the group B is finite then there is an one–one correspondence of the group H_c onto the group $\mathrm{Hom}(A, C)$, where C is a suitable subgroup of the center $Z(A)$ of A (Proposition 3.3). As far as the group I_c is concerned, it is proved that I_c is trivial if and only if $\mathrm{Hom}(B, Z(A))$ is trivial, where A is not of exponent 2 when $|B| = 2$. If in $W = AW_rB$ the group A is of exponent $2, A \neq C_2$ and $|B| = 2$, then $I_c = I_1$, where I_1 is the group of inner automorphisms which are induced by the elements of the base group A^B. In the case of $W = C_2W_rC_2$ we have that $\mathrm{Aut}_c(W) = I(W)$. (cf. J. PANAGOPOULOS [3]).

2. Preliminaries

We know that an automorphism α of A induces an automorphism $\alpha^\star$ of $W = AW_rB$ by

$$(bf)^{\alpha^\star} = bf^{\alpha^\star} \quad \text{for all} \quad b \in B,\ f \in A^B \quad \text{where}$$

$f^{\alpha^\star}(x) = (f(x))^\alpha$ for all $x \in B$.
The group $A^\star$ of all such automorphisms is isomorphic to $\mathrm{Aut}(A)$. Also

if β is an automorphism of B, then β induces an automorphism $\beta^\star$ of $W = AW_rB$ by

$$(bf)^{\beta^\star} = b^\beta f^{\beta^\star} \quad \text{for all} \quad b \in B,\ f \in A^B, \quad \text{where}$$

$f^{\beta^\star}(x) = f(x^{\beta^{-1}})$ for all $x \in B$. The group $B^\star$ of all such automorphisms is isomorphic to $\mathrm{Aut}(B)$. (cf. C. HOUGHTON [1]).

We shall see in the following propositions the implication on $\alpha^\star, \beta^\star$ the property of α, β being central and on α, β the property of $\alpha^\star, \beta^\star$ being central.

Proposition 2.1. *If* $\alpha \in \mathrm{Aut}_c(A)$ *then* $(bf)^{\alpha^\star} = (bf)g$, *where* $g \in Z(A^B)$.

PROOF. If $f \in A^B$ then $f^{\alpha^\star}(x) = (f(x))^\alpha = f(x)z_{f(x)}$ where $z_{f(x)} \in Z(A)$. Then, for the element $g \in A^B$ with

$$g(x) = z_{f(x)} \quad \text{for all} x \in B$$

we have $f^{\alpha^\star}(x) = f(x) \cdot g(x) = (fg)(x)$ for all $x \in B$. So $f^{\alpha^\star} = fg$ and $(bf)^{\alpha^\star} = (bf)g$, where $g \in Z(A^B)$.

Proposition 2.2. *The automorphism* $\alpha^\star$ *is central if and only if* $\alpha = 1$.

PROOF. Let $a \in A$ and $f \in A^B$ with $f(x) = a$ for some $x \in B$ and $f(y) = 1$ for all $y \in B,\ y \neq x$. Then

$$a^\alpha = (f(x))^\alpha = f^{\alpha^\star}(x) = (f \cdot z_f)(x) = f(x) \cdot z_f(x) = a \cdot z_f(x),$$

where $z_f(x) \in Z(W) = Z(D)$ and D is the diagonal of A^B.

$$1 = f(y) = (f(y))^\alpha = f^{\alpha^\star}(y) = (f \cdot z_f)(y) = f(y) \cdot z_f(y) = z_f(y)$$

for all $y \in B,\ y \neq x$.
Since $z_f \in D$ we have that $z_f(x) = 1$ and so

$$a^\alpha = a. \quad \text{Hence} \quad \alpha = 1.$$

Proposition 2.3. *The automorphism* $\beta^\star$ *is central if and only if* $\beta = 1$.

PROOF. From [3] we have that $\beta^\star = k \cdot i$, with $k \in K_c \leq K$ and $i \in I_c \leq I_1$, where K is the group of all the automorphisms of $W = AW_rB$ which leave the group B elementwise fixed and I_1 is the group of all the inner automorphisms which are induced by the elements of the group A^B. But it is known that the group $B^\star$ intersects trivially the group $K \cdot I_1$ (cf. C. HOUGHTON [1]). So $\beta^\star = 1$ and $\beta = 1$.

Now, if $\beta \in \mathrm{Aut}_c(B),\ \beta \neq 1$ then we cannot state a Proposition like Proposition 2.1.

3. The group K_c

In this section we study the relation of the group K_c to the groups A and B.

If $k \in K_c$ then $k = \alpha^\star h$ with $\alpha^\star \in A^\star$ and $h \in H$, where H is the group of all the automorphisms of W which leave the groups B and the diagonal D elementwise fixed. (cf. C. HOUGHTON [1]. Theorem 3.3). For an element $f \in D$ we have that

$$f^{\alpha^\star} = f^{kh^{-1}} = (fz_f)^{h^{-1}} = f \cdot z_f, \quad z_f \in Z(W),$$

since the automorphism h leaves the diagonal D elementwise fixed. Hence, the corresponding automorphism α of A is central. So, there is map $\varphi : K_c \to \mathrm{Aut}_c(A)$ such that if $k = \alpha^\star h \in K_c$ then $k\varphi = \alpha$. The map φ is a homomorphism. In fact, for the elements $k = \alpha^\star h$ and $k_1 = \alpha_1^\star h_1$ of K_c we have

$$(kk_1)\varphi = (\alpha^\star h \alpha_1^\star h_1)\varphi = (\alpha^\star \alpha_1^\star h' h_1)\varphi = \alpha\alpha_1 = (k\varphi) \cdot (k_1\varphi).$$

(This is because the group K is the semidirect product of the group H by the group $A^\star$).

Now, if $k = \alpha^\star h \in \mathrm{Ker}\,\varphi$ then $\alpha = 1$, so $\alpha^\star = 1$ and $k = h \in K_c \cap H$. If we define $H_c = K_c \cap H$, then it is clear that $H_c \leq \mathrm{Ker}\,\varphi$ so $H_c = \mathrm{Ker}\,\varphi$. Hence

$$K_c/H_c \leq \mathrm{Aut}_c(A) \quad \text{by an isomorphism.}$$

Proposition 3.1. *If $K_c = K \cap \mathrm{Aut}_c(W)$ and $H_c = K_c \cap H$ then H_c is normal in K_c and the factor group K_c/H_c isomorphic to a subgroup of $\mathrm{Aut}_c(A)$.*

In the following we study the group H_c, assuming that the groups A and B are finite. If $B = \{x_1, x_2, \ldots, x_k\}$, then we write

$$A^B = A_{x_1} \times A_{x_2} \times \cdots \times A_{x_k}, \quad \text{where for any} \quad i = 1, 2, \ldots, k \quad \text{the}$$

group A_{x_i} consists of those elements $f_{x_i} \in A^B$ with

$$f_{x_i}(x_i) = a \in A \quad \text{and}$$
$$f_{x_i}(x_j) = 1 \quad \text{for all} \quad x_j \neq x_i.$$

If $h \in H_c$, $h \neq 1$, then for any $f_{x_i} \in A_{x_i}$ we have that

$$f_{x_i}^h = f_{x_i} z_{f_{x_i}}, \quad \text{where } z_{f_{x_i}} \in Z(W).$$

For any $x_j \neq x_i$ there is an element $b_{ij} \in B$ such that $x_j = x_i b_{ij}$. Hence,

$$f_{x_j}^h = f_{x_i b_{ij}}^h = f_{x_i}^{b_{ij} h} = f_{x_i}^{h b_{ij}} = (f_{x_i} z_{f_{x_i}})^{b_{ij}} = f_{x_i}^{b_{ij}} z_{f_{x_i}} = \\ = f_{x_i b_{ij}} z_{f_{x_i}} = f_{x_j} z_{f_{x_i}}$$

because it is easy to see that $f_{x_i}^{b_{ij}} = f_{x_i b_{ij}}$ for any $i \neq j$ and it is known that the automorphisms which are induced by the elements of the group B, leave elementwise fixed the diagonal D. Hence, the automorphism h multiplies the elements f_{x_i}, $i = 1, 2, \ldots, k$ by the same element of the center of W. If now, $f \in D$ with $f(x) = a$ for all $x \in B$ then

$$f = f_{x_1} f_{x_2} \ldots f_{x_k}, \quad \text{where} \quad f_{x_i}(x_i) = a \quad \text{and} \\ f_{x_i}(x_j) = 1, \ x_i \neq x_j \\ \text{for all} \quad i = 1, 2, \ldots, k$$

Then

$$f^h = f_{x_1}^h f_{x_2}^h \ldots f_{x_k}^h = (f_{x_1} z_f)(f_{x_2} z_f) \ldots (f_{x_k} z_f) = \\ = f \cdot z_f^k, \quad \text{where} \quad z_f \in Z(W). \quad \text{Since} \quad f^h = f \quad \text{for all} \quad f \in D,$$

we have that $z_f^k = 1$.
Thus, we have proved the following proposition.

Proposition 3.2. *If $W = AW_rB$ and the groups A and B are finite with $(|Z(A)|, |B|) = 1$ then $H_c = 1$.*

It remains the case $(|Z(A)|, |B|) = d$, $d \neq 1$. In this case we consider the subgroup C of $Z(A)$ which consists of all the elements whose order divides the integer d. We shall prove the following proposition.

Proposition 3.3. *There is a one–one correspondence between the group H_c and the group $\mathrm{Hom}(A, C)$.*

Proof. Let $h \in H_c$ and $f_{x_i} \in A_{x_i}$ for some fixed $x_i \in B$. Then

$$f_{x_i}^h = f_{x_i} z_{f_{x_i}}, \quad \text{where} \quad z_{f_{x_i}} \in Z(W).$$

If $f_{x_i}(x_i) = a \in A$ and $z_{f_{x_i}}(x) = z_a \in Z(A)$ for all $x \in B$, then we define the map $\varphi : A \to C$ with $a\varphi = z_a$ for all $a \in A$. The map φ is a homomorphism. Indeed, if $a, a' \in A$ we take the elements f_{x_i}, f'_{x_i} so that

$$f_{x_i}(x_i) = a, \ f'_{x_i}(x_i) = a'. \quad \text{Then}$$

$$(aa')\varphi = [f_{x_i}(x_i)f'_{x_i}(x_i)]\varphi = [(f_{x_i}f'_{x_i})(x_i)]\varphi = z_a z_{a'} = (a\varphi)(a'\varphi)$$

because $(f_{x_i}f'_{x_i})^h = f^h_{x_i}f'^h_{x_i} = f_{x_i}f'_{x_i}z_{f_{x_i}}z_{f'_{x_i}}$, where

$$z_{f_{x_i}}(x) = z_a\,,\ z_{f'_{x_i}}(x) = z_{a'} \quad \text{for all} \quad x \in B.$$

Thus, we see that there is map

$$\vartheta : H_c \to \mathrm{Hom}(A, C) \quad \text{with} \quad h^\vartheta = \varphi \quad \text{for all} \quad h \in H_c.$$

We shall see that the map ϑ is onto the group $\mathrm{Hom}(A, C)$. Let $\varphi \in \mathrm{Hom}(A, C)$, with $a\varphi = z_a \in C$ for all $a \in A$. We define the map $h : A^B \to A^B$ by the rule:

$$\text{if} \quad f_{x_i} \in A_{x_i} \quad \text{with} \quad f_{x_i}(x_i) = a, \quad \text{then}$$

we define

$$f^h_{x_i} = f_{x_i}z_{f_{x_i}}, \quad \text{where} \quad z_{f_{x_i}} \in D \quad \text{with}$$

$$z_{f_{x_i}}(x) = a\varphi \quad \text{for all} \quad x \in B \quad \text{and} \quad i = 1, 2, \ldots, k.$$

First, we can see that the restriction of h to each A_{x_i} is a homomorphism. Indeed, if

$$f_{x_i}, g_{x_i} \in A_{x_i}, \quad \text{then}$$

$$(f_{x_i}g_{x_i})^h = f_{x_i}g_{x_i}z_{f_{x_i}g_{x_i}}, \quad \text{where}$$

$$z_{f_{x_i}g_{x_i}}(x) = [(f_{x_i}g_{x_i})(x_i)]\varphi = [f_{x_i}(x_i)g_{x_i}(x_i)]\varphi = [f_{x_i}(x_i)]\varphi[g_{x_i}(x_i)]\varphi =$$

$$= z_{f_{x_i}}(x)z_{g_{x_i}}(x) = (z_{f_{x_i}}z_{g_{x_i}})(x) \quad \text{for all} \quad x \in B. \quad \text{Thus,}$$

$$z_{f_{x_i}g_{x_i}} = z_{f_{x_i}}z_{g_{x_i}} \quad \text{and consequently}$$

$$(f_{x_i}g_{x_i})^h = (f_{x_i}z_{f_{x_i}})(g_{x_i}z_{g_{x_i}}) = f^h_{x_i}g^h_{x_i}.$$

Now, if $f, g \in A^B$ with $f = f^1_{x_1}f^2_{x_2}\cdots f^k_{x_k}$, $g = g^1_{x_1}g^2_{x_2}\cdots g^k_{x_k}$ where $f^i_{x_i}, g^i_{x_i} \in A_{x_i}$, $i = 1, 2, \ldots, k$, then

$$\begin{aligned}(fg)^h &= (f^1_{x_1}g^1_{x_1})^h \ldots (f^k_{x_k}g^k_{x_k})^h = (f^1_{x_1})^h(g^1_{x_1})^h \ldots (f^k_{x_k})^h(g^k_{x_k})^h = \\ &(f^1_{x_1}z_{f^1_{x_1}})(g^1_{x_1}z_{g^1_{x_1}}) \ldots (f^k_{x_k}z_{f^k_{x_k}})(g^k_{x_k}z_{g^k_{x_k}}) = \\ &= (f^1_{x_1} \ldots f^1_{x_k})(z_{f^1_{x_1}} \ldots z_{f^k_{x_k}})(g^1_{x_1} \ldots g^k_{x_k})(z_{g^1_{x_1}} \ldots z_{g^k_{x_k}}) = \\ &= f^h g^h. \quad \text{Hence, the map } h \text{ is an endomorphism of } A^B.\end{aligned}$$

We shall now see that the endomorphism h leaves the diagonal D elementwise fixed. If $f \in D$, then

$$f = f_{x_1} f_{x_2} \dots f_{x_k}, \quad \text{so} \quad f^h = f^h_{x_1} f^h_{x_2} \dots f^h_{x_k} =$$
$$=(f_{x_1} z_{f_{x_1}}) \dots (f_{x_k} z_{f_{x_k}}) = (f_{x_1} \dots f_{x_k})(z_{f_{x_1}} \dots z_{f_{x_k}}) = f \cdot z^k = f,$$

because we know that $z_{f_{x_1}} = \dots = z_{f_{x_k}} = z \in Z(W)$, with $z^d = 1$, since $z(x) \in C$ for all $x \in B$ and d divides k.

Now we shall prove that h is an epimorphism. If $f_{x_i} \in A_{x_i}$ we consider the element $f \in A^B$ with $f = f^1_{x_1} f^2_{x_2} \dots f^k_{x_k}$, so that

$$f^i_{x_i}(x_i) = f_{x_i}(x_i)[f^{-1}_{x_i}(x_i)]\varphi \quad \text{and}$$
$$f^j_{x_j}(x_j) = [f^{-1}_{x_i}(x_i)]\varphi \quad \text{for all} \quad j \neq i.$$

Then, we have that

$$f = f_{x_i} z_f, \quad \text{where} \quad z_f \in Z(W) \quad \text{with}$$
$$z_f(x) = [f^{-1}_{x_i}(x_i)]\varphi.$$

Thus, $f^h = (f_{x_i} z_f)^h = f^h_{x_i} z_f = f_{x_i} z_{f_{x_i}} z_f = f_{x_i}$, because $z_f \in D$ and $(z_{f_{x_i}} z_f)(x) = z_{f_{x_i}}(x) z_f(x) = [f_{x_i}(x_i)]\varphi[f^{-1}_{x_i}(x_i)]\varphi = [(f_{x_i} f^{-1}_{x_i}(x_i)]\varphi = 1$, for all $x \in B$.
From the above we conclude that h is an epimorphism of A^B. Since the group $W = AW_rB$ is finite, h is an automorphism of A^B. In order to prove that $h \in H_c$, it remains to check whether $hb = bh$ holds for all the automorphisms b which are induced by the elements of B. (ch. C. Houghton [1], 3.4). Let $f_{x_i} \in A_{x_i}$ for some $x_i \in B$. Then

$$f^{hb}_{x_i} = (f_{x_i} z_{f_{x_i}})^b = f^b_{x_i} z_{f_{x_i}} = f_{x_{ib}} z_{f_{x_i}} = f_{x_{ib}} z_{f_{x_{ib}}} =$$
$$= f^h_{x_{ib}} = f^{bh}_{x_i}$$

Now, it is clear that $h^\vartheta = \varphi$, so ϑ is onto the group $\mathrm{Hom}(A, C)$. Also, it is not difficult to see that ϑ is a one–one map of H_c onto $\mathrm{Hom}(A, C)$, but not a homomorphism.

Let now $W = AW_rB$, where the group B contains an element of infinite order. It is known that in this case the derived subgroup of W is the base group A^B. (cf. P. Neumann [2]). Hence, the group K_c must be trivial and theorem 3.3 in [3] gives:

Proposition 3.4. *If* $W = AW_rB$ *and the group* B *contains an element of infinite order then*

$$\mathrm{Aut}_c(W) = I_c \quad \textit{with} \quad I_c \cong \frac{Z_2(W)}{Z(W)}.$$

4. The group I_c

It is known that if $W = AW_rB$ with A of exponent $\neq 2$, when $|B| = 2$, then

$$\mathrm{Aut}_c(W) = K_c \times I_c, \quad \text{where} \quad I_c \cong \frac{Z_2(W)}{Z(W)}.$$

(cf. J. Panagopoulos [3]).

In this section we study the case : when is the group I_c trivial, so that

$\mathrm{Aut}_c(W) = K_c$? This problem is equivalent to the

following: under what conditions does the group $Z_2(W)$ coincide with the group $Z(W)$?

If we apply the procedure of C. Ruiz de Velasco and M. Torres in [4] to the standard wreath product $W = AW_rB$, we conclude that the group $Z_2(W)$ is isomorphic to the direct product $Z(A) \times \mathrm{Hom}(B, Z(A))$. Let us describe this isomorphism: if $f \in Z_2(W)$ then $f_b = [b, f] \in Z(W)$, for all $b \in B$. We define the map $\varphi_f : B \to Z(A)$ by the rule

$$\varphi_f(b) = f_b(b) \quad \text{for all} \quad b \in B.$$

The map φ_f is a homomorphism. Now, for an arbitrary element $c \in B$ we define the map $F_c : Z_2(W) \to Z(A) \times \mathrm{Hom}(B, Z(A))$ by

$$F_c(f) = (f(c), \varphi_f) \quad \text{for all} \quad f \in Z_2(W).$$

It is proved that the map F_c is an isomorphism of $Z_2(W)$ onto $Z(A) \times \mathrm{Hom}(B, Z(A))$. We see that if $f \in Z(W)$ then

$$F_c(f) = (f(c), \varphi_f) \quad \text{with} \quad \varphi_f(b) = f_b(b) = [b, f](b) = 1 \quad \text{for all} \quad b \in B.$$

Hence, $\varphi_f = 1$ when $f \in Z(W)$, which means that

$$[Z(W)]F_c \leq Z(A).$$

Conversely if $(z, 1) \in Z(A) \times \mathrm{Hom}(B, Z(A))$, then

$$F_c(f) = (z, 1), \quad \text{where} \quad f(x) = z \quad \text{for all} \quad x \in B.$$

This gives that $[Z(W)]F_c = Z(A)$. Hence, we see that $Z_2(W) = Z(W)$ if and only if $\mathrm{Hom}(B, Z(A)) = 1$.

As a matter of fact we have proved the following.

Proposition 4.1. *Let $W = AW_rB$ where A is not of exponent 2, when $|B| = 2$. Then*

$$\mathrm{Aut}_c(W) = K_c \quad \textit{if and only if} \quad \mathrm{Hom}(B, Z(A)) = 1.$$

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