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A Left Translation Preserving a Finsler Connection Invariant

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Recently T. NAGANO and T. AIKOU [2] introduced an interesting transformation of Finsler connections, called a *collinear change* and showed the following: A change $F\Gamma = (F_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k) \to F\bar{\Gamma} = (\bar{F}_j{}^i{}_k, \bar{N}^i{}_j, \bar{C}_j{}^i{}_k)$ of Finsler connections is collinear if and only if we have $\bar{F}_j{}^i{}_k = F_j{}^i{}_k + C_j{}^i{}_r B^r{}_k, \bar{N}^i{}_j = N^i{}_j - B^i{}_j, \ \bar{C}_j{}^i{}_k = C_j{}^i{}_k$ and the tensor field $B^i{}_j$ is h– and v–covariant constant. The property of covariant constance of the tensor $B^i{}_j$ is worthy of notice, although its existence in relation to the Finsler metric is an essential future problem.

On the other hand, the last section of the present paper is specially concerned with a linear connection and is the revival of the manuscript which was presented to Professor L. TAMÁSSY in 1979. This gives a solution of an interesting problem proposed by him to the author while the author was a visiting professor at Debrecen University.

Now Tamássy's problem is extended to Finsler connections in the present paper and it is interesting to observe that our main result corresponds just to the property of the tensor $B^i{}_j$, although the real reason of such a correspondence remains open.

The terminology and notation are those of the author's monograph [1]. The contents of the monograph are often quoted by putting an asterisk.

§1. Left Translation

Let $L(M) = (L, M, \pi_L, G(n))$ be the linear frame bundle over an n–dimensional differentiable manifold M and $\pi_T : T \to M$ be the projection of the tangent bundle $T(M) = (T, M, \pi_T, V, G(n))$ over the M. Then the Finsler bundle $F(M)$ of M is the induced bundle $\pi_T^{-1} L(M) = (F, T, \pi_1, G(n))$ over the total space T (*§6). We denote by $\underline{\beta} : L \times G(n) \to L$ and $\beta : F \times G(n) \to F$ the actions of the $G(n) = GL(n, \underline{R})$ on the total spaces L and F respectively. Their right–fixed mappings (*Example 1.3) $\underline{\beta}_g$ and β_g are called the right translations by $g \in G(n)$.

Now we consider a mapping $\tilde{\lambda}: F \to G(n)$ which satisfies the relation

$$\tilde{\lambda} \circ \beta_g = I_{g^{-1}} \circ \tilde{\lambda}, \tag{1.1}$$

where $I_{g^{-1}}$ is the inner automorphism of G(n) by $g^{-1} : h \in G(n) \mapsto g^{-1}hg$. From $\tilde{\lambda}$ we get a transformation of the total space F :

$$\Lambda : u \in F \mapsto u \cdot \tilde{\lambda}(u), \tag{1.2}$$

that is, $\Lambda(u) = \beta(u, \tilde{\lambda}(u))$.

Definition 1. The transformation Λ of the total space F of the Finsler bundle $F(M)$ which is given by a mapping $\tilde{\lambda}: F \to G(n)$ as above is called a *left translation* of F.

From (1.1) and (1.2) it follows immediately that a left translation Λ commutes with any right translation β_g :

$$\Lambda \circ \beta_g = \beta_g \circ \Lambda. \tag{1.3}$$

Here we shall pay attention to the following. The standard fibre V of the $T(M)$ is the n–dimensional real vector space with a fixed base $B = \{e_a\}$. The tensor space $V_1^1 = V_* \otimes V$ of (1,1)-type has the base $\{e_a^b = e^b \otimes e_a\}$ and $w \in V_1^1$ is written as $w = w_b^a e_a^b$ in terms of components w_b^a. We consider the subspace $V_G = \{w \in V_1^1 \mid \det(w_b^a) \neq 0\}$, called the *G-subspace.* Then we get the mapping $i : g = (g_b^a) \in G(n) \mapsto g_b^a e_a^b \in V_G$, by which $G(n)$ may be identified with the G–space V_G.

Then the above mapping $\tilde{\lambda} : F \to G(n)$ gives rise to

$$\lambda = i \circ \tilde{\lambda} : F \to V_G. \tag{1.4}$$

It is observed that from (1.1) we have $\lambda(ug) = i(g^{-1}\tilde{\lambda}(u)g) = \{(g^{-1})_c^a(\tilde{\lambda}(u))_d^c g_b^d\} e_a^b$ for $g = (g_b^a)$, which is equal to $g^{-1} \cdot \lambda(u)$ from *(2.14). Therefore we get

$$\lambda \circ \beta_g = g^{-1} \cdot \lambda. \tag{1.5}$$

Consequently *Definition 6.2 shows that λ is a Finsler tensor field of (1,1)-type.

Definition 2. The mapping λ which is obtained from the $\tilde{\lambda}$ in Definition 1 by (1.4) is called the *characteristic tensor field* of the left translation Λ.

We are concerned with a local coordinate system (x^i) of the base manifold M and the canonical coordinate system (x^i, y^i, z^i_a) of the F. The components $(\lambda^i{}_j)$ of the characteristic tensor field λ at $u = (x^i, y^i, z^i_a)$ are given by

$$\lambda^i{}_j \;=\; z^i_a \lambda^a_b(u)(z^{-1})^b_j, \tag{1.6}$$

where $\lambda(u) = \lambda^a_b(u)e^b_a$ and $\lambda^i{}_j$ are functions of (x^i, y^i) (cf. *(6.12)). The λ is regular, i.e., $\det(\lambda^i{}_j) \neq 0$.

We shall deal with the left translation Λ in terms of the coordinates (x^i, y^i, z^i_a). It is observed that $\tilde{\lambda}(u) = i^{-1} \circ \lambda(u) = (\lambda^a_b(u))$ and $\Lambda(u) = u \cdot \tilde{\lambda}(u) = (y, z \cdot \tilde{\lambda}(u)) = (x^i, y^i, z^i_b \lambda^b_a(u))$. Therefore we have

$$\Lambda \,:\, u = (x^i, y^i, z^i_a) \mapsto (x^i, y^i, \lambda^i{}_j(x, y) z^j_a). \tag{1.2$'$}$$

It is noted here that the right translation $\beta_g \,:\, u \mapsto ug$ is written as $ug = (x^i, y^i, z^i_b g^b_a)$ for $g = (g^a_b) \in G(n)$. The expression (1.2′) shows the origin of the name "left translation".

§2. Invariant Finsler connection

We consider a Finsler connection $F\Gamma = (\Gamma^h, \Gamma^v)$, a pair of distributions in the F satisfying certain conditions (*Definition 9.2). In particular they are invariant by any right translation β_g. A left translation Λ of F is such that $\Lambda(u) = \beta(u, \tilde{\lambda}(u))$, and so its differential Λ' is written as

$$\Lambda' = \beta'_g \;+\; {}_u\beta' \circ \tilde{\lambda}' \,, \qquad (g = \tilde{\lambda}(u)), \tag{2.1}$$

where ${}_u\beta \,:\, g \in G(n) \mapsto ug \in F$ is the left-fixed mapping of the group action β. Then at a point $u \in F$ we get

$$\begin{aligned} \Lambda'(\Gamma^h_u) &= \Gamma^h_{ug} + {}_u\beta' \circ \tilde{\lambda}'(\Gamma^h_u), \\ \Lambda'(\Gamma^v_u) &= \Gamma^v_{ug} + {}_u\beta' \circ \tilde{\lambda}'(\Gamma^v_u). \end{aligned} \tag{2.2}$$

The first terms of the right-hand sides are h–and v–horizontal respectively, while the second terms are both vertical because $\pi_1 \circ {}_u\beta$ is a constant mapping.

We are further concerned with the second terms. The h– and v–horizontal subspaces Γ^h_u and Γ^v_u are respectively spanned by the h– and

v–basic vectors $B^h(v)_u$ and $B^v(v)_u$ corresponding to each vector $v \in V$ (cf. *§9). As to $B^h(v)_u$ it is observed that

$$_u\beta' \circ \tilde{\lambda}'(B^h(v)_u) = {}_{ug}\beta' \circ {}_{g^{-1}}\tau' \circ (i^{-1})' \circ \lambda'(B^h(v)_u), \quad g = \tilde{\lambda}(u),$$

where ${}_{g^{-1}}\tau$ is the left-fixed mapping of the group multiplication $\tau : (g, h) \in G(n) \times G(n) \mapsto gh \in G(n)$. Further *(1.5), *(1.6), *Definitions 1.12 and 9.5 show

$$\lambda'(B^h(v)_u) = I_{i(g)} \circ d\lambda(B^h(v)_u) = I_{i(g)}(B^h(v)_u(\lambda)) = I_{i(g)}(\nabla^h \lambda(v)).$$

Thus we obtain

$$_u\beta' \circ \tilde{\lambda}'(B^h(v)_u) = {}_{ug}\beta'\{{}_{g^{-1}}\tau' \circ (i^{-1})' \circ I_{i(g)}(\nabla^h \lambda(v))\}.$$

Referring to the coordinates (w^a_b) in V^1_1 and (g^a_b) in $G(n)$ it is observed that for $X = X^a_b e^b_a \in V^1_1$ the equation *(2.3′) shows

$$\begin{aligned} {}_{g^{-1}}\tau' \circ (i^{-1})' \circ I_{i(g)}(X) &= {}_{g^{-1}}\tau' \circ (i^{-1})'\{X^a_b(\partial/\partial w^a_b)_{i(g)}\} = \\ &= {}_{g^{-1}}\tau'\{X^a_b(\partial/\partial g^a_b)_g\} = X^a_b(g^{-1})^c_a(\partial/\partial g^c_b)_e\,, \end{aligned}$$

which is a tangent vector of $G(n)$ at the unit e. From $g = \tilde{\lambda}(u)$ we have $(g^{-1})^c_a = (\lambda^{-1})^c_a$, and $(\partial/\partial g^c_b)_e = E^b_c$ constitute the standard base of the Lie algebra $G'(n)$ (cf. *(2.9)). Consequently we obtain

$$_u\beta' \circ \tilde{\lambda}'(B^h(v)_u) = {}_{ug}\beta'\{(\lambda^{-1})^c_a \lambda^a_{b|d} v^d E^b_c\}, \quad v = v^d e_d,$$

where $\lambda^a_{b|d}$ are components of the h–covariant derivative $\nabla^h\lambda$ in the tensor space V^1_2. The above is nothing but the fundamental vector $Z(A)_{ug}$ corresponding to $A = \lambda^{-1} \cdot \nabla^h\lambda(v)$ in terms of the matrix multiplication (cf. *(6.4′)).

An analogous procedure is applicable to the $B^v(v)_u$. Consequently, paying attention to *(8.7) and *(9.8), the exact expressions of (2.2) are established:

$$\begin{aligned} \Lambda'(B^h(v)_u) &= B^h(g^{-1}v) + Z(\lambda^{-1} \cdot \nabla^h\lambda(v)), \\ \Lambda'(B^v(v)_u) &= B^v(g^{-1}v) + Z(\lambda^{-1} \cdot \nabla^v\lambda(v)), \end{aligned} \tag{2.3}$$

at $u \in F$, where $g = \tilde{\lambda}(u)$.

Definition 3. A left translation Λ is called *tensorially affine* with respect to a Finsler connection $F\Gamma = (\Gamma^h, \Gamma^v)$ if it preserves $F\Gamma$ invariant: $\Lambda'(\Gamma^h) = \Gamma^h$, $\Lambda'(\Gamma^v) = \Gamma^v$.

It is well-known that a fundamental vector field $Z(A)$, corresponding to a non-zero A, does not vanish at every point of the F. Therefore (2.3) shows

Theorem. *A left translation is tensorially affine with respect to a Finsler connection if and only if the characteristic tensor field is $h-$ and $v-$covariant constant.*

§3. The Case of Linear Connections

The concept of left translation is similarly defined in the linear frame bundle $L(M)$. First a mapping $\tilde{\underline{\lambda}} : L \to G(n)$ is assumed to satisfy

$$\tilde{\underline{\lambda}} \circ \underline{\beta}_g = I_{g^{-1}} \circ \tilde{\underline{\lambda}}, \tag{3.1}$$

where $\underline{\beta}_g$ is the right translation by $g \in G(n)$ of the total space L. Then the *left translation* is defined as

$$\underline{\Lambda} : z \in L \mapsto z \cdot \tilde{\underline{\lambda}}(z) = \underline{\beta}(z, \tilde{\underline{\lambda}}(z)). \tag{3.2}$$

It is easy to show that $\underline{\Lambda}$ commutes with any right translation. Next we give $\underline{\lambda} = i \circ \tilde{\underline{\lambda}} : L \to V_1^1$ and the relation

$$\underline{\lambda} \circ \underline{\beta}_g = g^{-1} \cdot \underline{\lambda} \tag{3.3}$$

is easily shown, that is, $\underline{\lambda}$ is a tensor field of (1.1)-type in the ordinary sense. It is called the *charasteristic tensor field* of $\underline{\Lambda}$. If $\underline{\lambda}$ has components $\underline{\lambda}^i{}_j$ in a coordinate system (x^i) of the M, then in terms of the canonical coordinate system (x^i, z^i_a) in L it is seen that

$$\underline{\Lambda} : Z = (x^i, z^i_a) \mapsto (x^i, \underline{\lambda}^i{}_j(x) z^j_a). \tag{3.2'}$$

Assume that a linear connection $\underline{\Gamma}$ is given. In terms of the basic vector field $\underline{B}(v)$ of the connection we have analogously

$$\underline{\Lambda}'(\underline{B}(v)) = \underline{B}(g^{-1}v) + \underline{Z}(\underline{\lambda}^{-1} \cdot \nabla\underline{\lambda}(v)), \tag{3.4}$$

at $z \in L$, where $g = \tilde{\underline{\lambda}}(z)$ and $\underline{Z}(A)$ is the fundamental vector field of the L. Therefore we obtain a theorem which is quite similar to the above theorem.

In case of a linear connection $\underline{\Gamma}$ we get the associated connection $\underline{\Gamma}^*$ in the tangent bundle $T(M)$ by means of the associated mapping $\alpha_v : z \in L \mapsto zv \in T$ for $v \in V$. Thus $\underline{\Lambda}$ gives rise to a transformation $\underline{\Lambda}^*$ of the total space $T : y = zv \in T \mapsto \underline{\Lambda}(z)v$, which is well-defined at every point y. In terms of the canonical coordinate system (x^i, y^i) in T we have

$$\underline{\Lambda}^* : y = (x^i, y^i) \mapsto (x^i, \underline{\lambda}^i{}_j(x)y^j). \tag{3.5}$$

Therefore a left translation may be regarded as a rotation of the tangent vector of the base manifold M.

References

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