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Generating functions of some products of arithmetical functions

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1. Introduction

The Dirichlet series generating function of the product of two divisor sum functions was given by S. RAMANUJAN in a paper published in 1916 [10]:

$$\sum \sigma_a(n)\sigma_b(n)n^{-s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}, \tag{1}$$

which holds for $\min(R(s), R(s-a), R(s-b), R(s-a-b)) > 1$. Here, and throughout the paper, the unadorned summation sign indicates the sum as n runs over the set of positive integers, and $R(s)$ denotes the real part of s. A proof of (1) was given by B. M. WILSON [15]: it is the proof which appears in [8, pp. 197–198]. Other proofs have been given by R. VAIDYANATHASAMY [14, p. 611] and by P. BUNDSCHUH [2].

The generating functions of other products of arithmetical functions have been obtained by several authors, notably S. CHOWLA [3], M. M. CRUM [4], D. REDMOND and R. SIVARAMAKRISHNAN [11] and A. MERCIER [9]. In Section 2 we will find the generating functions of products of two functions from a wide class of functions which includes Klee's function and generalized Dedekind functions, and products in which one factor is from this class and the other is a Gegenbauer function.

A Gegenbauer function $\varrho_{a,r}$ is defined by

$\varrho_{a,r}(n)$ =the sum of the a –th powers of the divisors d of n such that
n/d is an r–th power

(see [5, p. 298]). Thus $\varrho_{a,1} = \sigma_a$. The generating function of the product $\varrho_{a,r}\varrho_{b,r}$ is

$$\frac{\zeta(rs)\zeta(r(s-a))\zeta(r(s-b))\zeta(s-a-b)}{\zeta(r(2s-a-b))},$$

where the appropriate Dirichlet series converges absolutely to this function in a suitable half-plane. This is equivalent to a result proved by CRUM [4, p. 11], who considered another arithmetical function which is a Gegenbauer function in disguise. In Section 3 we will use a formula due to D. M. KOTELYANSKIĬ [7] to extend Crum's result by obtaining the generating function of $\varrho_{a,r}\varrho_{b,tr}$, where t is a positive integer.

The Dirichlet product of two arithmetical functions f and g will be denoted by $f \star g$. The function ζ_a is defined by $\zeta_a(n) = n^a$, and we set $\zeta = \zeta_0$. The arithmetical function ζ has the Riemann zeta function ζ as its generating function: there is little possibility of confusing the two uses of the Greek letter zeta.

We will make use of an identity due to R. D. von STERNECK (see [5, pp. 151 – 152]) which, for two arithmetical functions f and g, states that

$$(f \star \zeta)(g \star \zeta) = h \star \zeta, \text{ where} \tag{2}$$

$$h(n) = \sum_{[a,b]=n} f(a)g(b) \quad \text{for all } n.$$

$[a,b]$ denotes the least common multiple of a and b, and the sum in (2) is over all ordered pairs of positive integers a and b with $[a,b] = n$. The function h in (2) will be denoted by $[f,g]$. Bundschuh used (2) in his proof of (1).

2. The function $\phi_{r,\alpha,a}$

Let r and α be positive integers and let a be a complex number. The arithmetical function $\phi_{r,\alpha,a}$ is defined by

$$\phi_{r,\alpha,a}(n) = \sum_{d|n} \mu_r(d)^\alpha (n/d)^a \quad \text{for all } n,$$

where

$$\mu_r(n) = \begin{cases} \mu(n^{1/r}) & \text{if } n \text{ is an } r\text{–th power} \\ 0 & \text{otherwise} \end{cases}$$

The function μ_r is the r–th convolute of the Möbius function μ. Convolutes of arithmetical functions, which were first studied systematically by VAIDYANATHASWAMY [14], occur frequently and naturally (see [8, p. 53]). If α is odd then $\mu_r^\alpha = \mu_r$, and if α is even then μ_r^α is the r–th convolute of μ^2. Thus, we can restrict α to the values 1 and 2.

The function $\phi_{r,\alpha,a}$ was introduced by B.C. BERNDT [1] as a convenient device for considering simultaneously several well–known functions. $\phi_{1,1,1}$ is Euler's function ϕ : $\phi_{1,1,a}$ is the function ϕ_a, and therefore Jordan's function J_a when a is a positive integer; $\phi_{r,1,1}$ is the function Φ_r, generally referred to as Klee's function because V. L. Klee gave a summary of its properties [6], but actually studied many years earlier (see [5, p. 134]). $\phi_{1,2,1}$ is Dedekind's function ψ, and $\phi_{1,2,a}$ and $\phi_{r,2,1}$ are the generalizations ψ_a and Ψ_r of ψ introduced by D. SURYANARAYANA [12], [13]. Definitions of all of these functions are given in [8] (see the index, p. 362).

Theorem 1. *If* $r \le t$, *then for* $\min(R(s-a), R(s-b), R(s-a-b)) > 1$,

$$\sum \phi_{r,\alpha,a}(n)\phi_{t,\beta,b}(n)n^{-s}$$

$$= \zeta(s-a-b)\prod_p \left(1 + (-1)^\alpha \frac{p^{rb}}{p^{rs}} + (-1)^\beta \frac{p^{ta} + (-1)^\alpha p^{(t-r)a}}{p^{ts}}\right).$$

PROOF. Since $\phi_{r,\alpha,a} = \mu_r^\alpha \star \zeta_a$, we have

$$\begin{aligned}\phi_{r,\alpha,a}\phi_{t,\beta,b} &= \zeta_{a+b}(\zeta_{-a}\mu_r^\alpha \star \zeta)(\zeta_{-b}\mu_t^\beta \star \zeta)\\ &= \zeta_{a+b}(h \star \zeta) = \zeta_{a+b}h \star \zeta_{a+b},\end{aligned}$$

where $h = [\zeta_{-a}\mu_r^\alpha, \zeta_{-b}\mu_t^\beta]$. For all primes p and all $\gamma \ge 1$,

$$h(p^\gamma) = \sum_{\max(i,j)=\gamma} p^{-(ai+bj)}\mu_r(p^i)^\alpha \mu_t(p^j)^\beta.$$

Thus, $h(p^\gamma) = 0$ unless $\gamma = r$ or t. For $r < t$,

$$h(p^r) = (-1)^\alpha p^{-ra},\ (\zeta_{a+b}h)(p^r) = (-1)^\alpha p^{rb},$$

and

$$\begin{aligned}h(p^t) &= (-1)^\beta p^{-tb} + (-1)^{\alpha+\beta} p^{-(ra+tb)},\\ (\zeta_{a+b}h)(p^t) &= (-1)^\beta p^{ta} + (-1)^{\alpha+\beta} p^{(t-r)a}.\end{aligned}$$

For $r = t$,

$$\begin{aligned}h(p^r) &= (-1)^\alpha p^{-ra} + (-1)^\beta p^{rb} + (-1)^{\alpha+\beta} p^{-r(a+b)},\\ (\zeta_{a+b}h)(p^r) &= (-1)^\alpha p^{rb} + (-1)^\beta p^{ra} + (-1)^{\alpha+\beta}.\end{aligned}$$

The proof is completed by noting that in the statement of the theorem, the Dirichlet series on the left–hand side, and the series for the zeta function and the infinite product on the right–hand side, converge absolutely in the given half–plane.

Several examples of special interest: if a is a positive integer then for $R(s) > a + 2$,

$$\sum J_a(n)\Phi_t(n)n^{-s} = \zeta(s-a-1)\prod_p \left(1 - \frac{p}{p^s} - \frac{p^{ta} - p^{(t-1)a}}{p^{ts}}\right);$$

if $r \le t$ then for $R(s) > 3$,

$$\sum \Phi_r(n)\Phi_t(n)n^{-s} = \zeta(s-2)\prod_p \left(1 - \frac{p^r}{p^{rs}} - \frac{p^t - p^{t-r}}{p^{ts}}\right);$$

for $R(s) > 3$,

$$\sum \psi(n)^2 n^{-s} = \zeta(s-2)\prod_p \left(1 + \frac{2p+1}{p^s}\right).$$

The next theorem is a generalization of a result of REDMOND and SIVARAMAKRISHNAN [11].

Theorem 2. *In a suitable half-plane and for t a positive integer,*

$$\sum \varrho_{a,r}(n)\phi_{tr,\beta,b}(n)n^{-s} = \zeta(s-a-b)\zeta(r(s-b))$$
$$\prod_p \left(1 + (-1)^\beta \frac{\sum\limits_{k=0}^{t} p^{(t-k)ra}}{p^{trs}} - (-1)^\beta \frac{p^{rb}\sum\limits_{k=0}^{t-1} p^{(t-k)ra}}{p^{(t+1)rs}}\right).$$

PROOF. Recall the definition of a Gegenbauer function $\varrho_{a,r}$ from Section 1,

$$\varrho_{a,r}(n) = \sum_{d|n} \nu_r(d)(n/d)^a \quad \text{for all } n,$$

where

$$\nu_r(n) = \begin{cases} 1 & \text{if } n \text{ is an } r\text{-th power} \\ 0 & \text{otherwise.} \end{cases}$$

The function ν_r is the r-th convolute of the arithmetical function ζ. Since $\mu_r^{-1} = \nu_r$, the function $\varrho_{a,r}$ is related to $\phi_{r,1,a}$ as the function $\sigma_a = \varrho_{a,1}$ is related to $\phi_a = \phi_{1,1,a}$.

We have

$$\varrho_{a,r}\phi_{tr,\beta,b} = \zeta_{a+b}(\zeta_{-a}\nu_r \star \zeta)(\zeta_{-b}\mu^{\beta}_{tr} \star \zeta) =$$
$$= \zeta_{a+b}(h \star \zeta) = \zeta_{a+b}h \star \zeta_{a+b},$$

where $h = [\zeta_{-a}\nu_r, \zeta_{-b}\mu^{\beta}_{tr}]$. For all primes p and all $\gamma \geq 1$,

$$(\zeta_{a+b}h)(p^{\gamma}) = \sum_{\max(i,j)=\gamma} p^{(\gamma-i)a}p^{(\gamma-j)b}\nu_r(p^i)\mu^{\beta}_{tr}(p^j).$$

This is equal to zero unless γ is a multiple of r. For $1 \leq k \leq t-1, (\zeta_{a+b}h)(p^{kr}) = p^{krb}$,

$$(\zeta_{a+b}h)(p^{tr}) = p^{trb} + (-1)^{\beta}\sum_{k=0}^{t} p^{(t-k)ra}$$

and for $k > t$, $(\zeta_{a+b}h)(p^{kr}) = p^{(k-t)rb}(p^{trb} + (-1)^{\beta})$. The Dirichlet series of the function $\zeta_{a+b}h$ is therefore, equal to

$$\prod_p \left(1 + \sum_{k=1}^{t-1} p^{-kr(s-b)} + \left(p^{trb} + (-1)^{\beta}\sum_{k=0}^{t} p^{(t-k)ra}\right)p^{-trs} \right.$$
$$\left. + \sum_{k=t+1}^{\infty} p^{(k-t)rb}(p^{trb} + (-1)^{\beta})p^{-krs}\right)$$

The Dirichlet series of $\varrho_{a,r}\phi_{tr,\beta,b}$ is $\zeta(s-a-b)$ times this expression, and after some manipulation, we obtain the right–hand side of the formula of the theorem.

With $r = 1$ and $\beta = b = 1$ we obtain

$$\sum \sigma_a(n)\Phi_t(n)n^{-s} =$$

$$= \zeta(s-a-1)\zeta(s-1)\prod_p \left(1 - \frac{\sum\limits_{k=0}^{t} p^{(t-k)a}}{p^{ts}} + \frac{\sum\limits_{k=0}^{t-1} p^{(t-k)a+1}}{p^{(t+1)s}}\right),$$

and the generating function of $\sigma_a\Psi_t$ is obtained by changing the signs before the two fractional terms on the right–hand side. In particular, with $r = 1,\ t = 1,\ \beta = 2$ and $b = 1$,

$$\sum \sigma_a(n)\psi(n)n^{-s} = \zeta(s-a-1)\zeta(s-1)\prod_p \left(1 + \frac{p^a + 1}{p^s} - \frac{p^{a+1}}{p^{2s}}\right).$$

3. Kotelyanskiĭ's theorem

Starting from a result of N. P. Romanov, which was obtained using Hilbert space methods, D. Kotelyanskiĭ [7] obtained a result which can be stated as follows.

Theorem 3. *If f and g are arithmetical functions and $h = [f, g]$, then*

$$\sum h(n)n^{-s} = \sum \phi_s(n)n^{-2s} \sum_{j=1}^{\infty} f(nj)j^{-s} \sum_{k=1}^{\infty} g(nk)k^{-s} \tag{3}$$

in any half plane in which the various Dirichlet series converge absolutely.

In fact, there is a short, simple proof of Theorem 3 which makes the result accessible to students in a beginning course in the theory of numbers. By (2), $h = (f \star \zeta)(g \star \zeta) \star \mu$. Hence, the left – hand side of (3) is equal to

$$\sum n^{-s} \sum_{d|n} \left(\sum_{a|d} f(a) \right) \left(\sum_{b|d} g(b) \right) \mu(n/d).$$

As a and b run independently over the positive integers, for each pair of values of a and b there is one term containing $f(a)g(b)$ as a factor for each divisor d of each multiple $[a, b]m$ of the least common multiple $[a, b]$ of a and b such that $[a, b]|d$. Thus, the quadruple sum is equal to

$$\sum_{a=1}^{\infty} \sum_{b=1}^{\infty} f(a)g(b) \sum_{m=1}^{\infty} [a, b]^{-s} m^{-s} \sum_{e|m} \mu(m/e),$$

and by the defining property of the Möbius function, this is equal to

$$\sum_{a=1}^{\infty} \sum_{b=1}^{\infty} f(a)g(b)[a, b]^{-s}, \tag{4}$$

On the other hand, the right–hand side of (3) is equal to

$$\sum_{a=1}^{\infty} \sum_{b=1}^{\infty} (f(a)a^{-s})(g(b)b^{-s}) \sum_{d|(a,b)} \phi_s(d) =$$

$$= \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} f(a)g(b)a^{-s}b^{-s}(a, b)^{s},$$

which is equal to (4).

There is an unfortunate misprint on page 457 of [7], where the exponent on n in the right–hand side of (3) is s rather than $2s$: it is corrected in Mathematical Reviews, vol. 15, page 779.

KOTELYANSKIĬ used (3) to give a proof of Ramanujan's formula (1). We will use it to obtain a generalization of (1), a result even more general than the one due to Crum which is mentioned in Section 1.

When we began work on this paper, we used Kotelyanskiĭ's result to prove Theorem 1 and 2. Then we noticed that the simple and direct proofs given in Section 2 allowed us to side step Theorem 3. In those proofs, $(\zeta_{a+b}h)(p^k)$ has a bounded number of terms for k large. However, if we set out to use the same method to find the generating function of the product $\varrho_{a,r}\varrho_{b,t}$, we find that we must consider $(\zeta_{a+b}h)(p^{km})$, where $m = [r,t]$ and $h = [\zeta_{-a}\nu_r, \zeta_{-b}\nu_t]$, and that it has an increasing number of terms for large, increasing values of k. This leads to a formidable problem of manipulation in simplifying the generating function. It is much easier to use Theorem 3 to prove the following theorem which, although it does not give the generating function of $\varrho_{a,r}\varrho_{b,t}$ for all r and t, does generalize Crum's result.

Theorem 4. *For* $\min(R(s), R(s-a), R(s-b), R(s-a-b)) > 1$, *and* t *a positive integer,*

$$\sum \varrho_{a,r}(n)\varrho_{b,tr}(n)n^{-s} = \zeta(s-a-b)\zeta(r(s-b))\zeta(tr(s-a))\zeta(trs)$$

$$\prod_p \left(1 - \frac{(p^{tra}-1)p^{r(a+b)} + p^{rs}(p^{ra}-p^{tra}}{(p^{ra}-1)p^{(t+1)rs}}\right)$$

PROOF. Since

$$\frac{\varrho_{a,r}(n)}{n^a} = \sum_{d|n} \frac{\nu_r(d)}{d^a} \quad \text{for all n,}$$

we have, from Theorem 3,

$$\sum \varrho_{a,r}(n)\varrho_{b,tr}(n)n^{-(s+a+b)} = \tag{5}$$

$$= \zeta(s)\sum_{n=1}^{\infty} \frac{\phi_s(n)}{n^{2s+a+b}} \sum_{j=1}^{\infty} \frac{\nu_r(nj)}{j^{s+a}} \sum_{k=1}^{\infty} \frac{\nu_{tr}(nk)}{k^{s+b}}.$$

For a positive integer n, let $Q_r(n)$ be the smallest positive integer such that $nQ_r(n)$ is an r–th power. If nj is an r–th power then $Q_r(n)|j$

and $j/Q_r(n)$ is an r-th power. Thus, $\nu_r(nj) = 1$ when and only when $nj = nQ_r(n)i^r$ for some i. Therefore,

$$\sum_{j=1}^{\infty} \frac{\nu_r(nj)}{j^{s+a}} = \sum_{i=1}^{\infty} \frac{1}{Q_r(n)^{s+a} i^{r(s+a)}}$$
$$= \frac{1}{Q_r(n)^{s+a}} \sum_{i=1}^{\infty} \frac{1}{i^{r(s+a)}} = \frac{\zeta(r(s+a))}{Q_r(n)^{s+a}}.$$

If we substitute for the two inner sums in the right–hand side of (5), and replace s by $s-a-b$, we obtain

$$\sum \varrho_{a,r}(n)\varrho_{b,tr}(n)n^{-s}$$
$$= \zeta(s-a-b)\zeta(r(s-b))\zeta(tr(s-a)) \sum_{n=1}^{\infty} \frac{\phi_{s-a-b}(n)}{(nQ_r(n))^{s-b}(nQ_{tr}(n))^{s-a}},$$

which holds in the stated half–plane. If we denote the sum on the right–hand side by S, then in that half plane,

$$S = \prod_p \left(1 + \sum_{j=1}^{\infty} \frac{\phi_{s-a-b}(p^j)}{(p^jQ_r(p^j))^{s-b}(p^jQ_{tr}(p^j))^{s-a}} \right).$$

Let S_p be the factor of S corresponding to the prime p. For $\alpha = 1, 2, \dots$, $(p^jQ_{tr}(p^j))^{s-a} = p^{\alpha tr(s-a)}$ for j in the interval $(\alpha-1)tr+1 \leq j \leq \alpha tr$, and for $\beta = 1, \dots, t$, $(p^jQ_r(p^j))^{s-b} = p^{((\alpha-1)t+\beta)r(s-b)}$ for j in the subinterval $((\alpha-1)t+\beta-1)r+1 \leq j \leq ((\alpha-1)t+\beta)r$. If we sum over the values of j in the subinterval, and then over the values of β, we find after some manipulation that

$$S_p = 1 + \frac{p^{r(s-a-b)}-1}{p^{r(s-a-b)}} \; \frac{p^{tra}-1}{p^{ra}-1} \sum_{\alpha=1}^{\infty} \frac{1}{p^{\alpha trs}}$$
$$= \frac{1}{1-p^{-trs}} \left(1 - \frac{(p^{tra}-1)p^{r(a+b)} + p^{rs}(p^{ra}-p^{tra}}{(p^{ra}-1)p^{(t+1)rs}} \right).$$

Thus, we obtain the formula of the theorem.

When $t = 1$, Theorem 4 is Crum's result, and with $r = 1$ we have

$$\sum \sigma_a(n)\varrho_{b,t}(n)n^{-s} =$$
$$= \zeta(s-a-b)\zeta(s-b)\zeta(t(s-a))\zeta(ts)$$
$$\prod_p \left(1 - \frac{(p^{ta}-1)p^{a+b} + p^s(p^a-p^{ta}}{(p^a-1)p^{(t+1)s}} \right).$$

References

[1] B. C. BERNDT, A new method in arithmetical functions and contour integration, *Canad. Math. Bull.* **16** (1973), 381–387.

[2] P. BUNDSCHUH, Aufgabe 635. Zweite Lösung, *Elem. Math.* **26** (1971), 140–141.

[3] S. D. CHOWLA, On some identities involving zeta–functions, *J. Indian Math. Soc.* **17** (1928), 153–163.

[4] M. M. CRUM, On some Dirichlet series, *J. London Math. Soc.* **15** (1940), 10–15.

[5] L. E. DICKSON, History of the Theory of Numbers, Volume I, *Chelsea Publ. Co., New York*, 1952.

[6] V. L. KLEE, A generalization of Euler's function, *Amer. Math. Monthly* **55** (1948), 358–359.

[7] D. M. KOTELYANSKIĬ, On N. P. Romanov's method of obtaining identities for arithmetic functions, *Ukrain Mat. Ž.* **5** (1953), 453–458. (Russian)

[8] P. J. MCCARTHY, Introduction to Arithmetical Functions, *Springer–Verlag* , *New York*, 1986.

[9] A. MERCIER, Remarques sur les fonctions spécialement multiplicatives, *Ann. Sci. Math. Québec* **6** (198[illegible] 99–107.

[10] S. RAMANUJAN, Some formulae in the analytic theory of numbers, *Messenger Math.* **45** (1916), 81–84; Collected Papers, *New York*, 1962, pp. 133–135.

[11] D. REDMOND and R. SIVARAMAKRISHNAN, Some properties of specially multiplicative functions, *J. Number Theory* **13** (1981), 210–227.

[12] D. SURYANARAYANA, A generalization of Dedekind's ψ – function, *Math. Student* **37** (1969), 81–86.

[13] D. SURYANARAYANA, Extensions of Dedekind's ψ – function, *Math. Scand.* **26** (1970), 107–118.

[14] R. VAIDYANATHASWAMY, The theory of multiplicative arithmetic functions , *Trans. Amer. Math. Soc.* **33** (1931), 579–662.

[15] B. M. WILSON, Proofs of some formulae enunciated by Ramanujan, *Proc. London Math. Soc.* **21** (1923), 235–255.

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