Publ. Math. Debrecen
38 / 1–2 (1991), 103–109

Invariant submanifolds of a trans–Sasakian manifold

By D. CHINEA and P. S. PERESTELO (Islas Canarias)

S. TANNO [T] and K. YANO and S. ISHIHARA [YI] have proved that any invariant submanifold of a Sasakian manifold is minimal. The authors [YI] have obtained conditions for an invariant submanifold of a normal contact manifold to be totally geodesic in the case of codimension 2, and M. KON [K] have studied the case of codimension ≥ 2. He also studied invariant submanifolds satisfying the condition $\tilde{R}(X,\xi)h = 0$.

The purpose of this paper is to show that similar results hold true for a more general class of manifolds, namely the class of trans-Sasakian manifolds, [O]. In §1 we recall definitions and some properties of almost contact metric manifolds. In §2 we prove that any invariant submanifold of a trans-Sasakian manifold is also trans-Sasakian and minimal. We obtain conditions for an invariant submanifold of a trans-Sasakian manifold to be totally geodesic. Finally, we study invariant submanifolds satisfying the condition $\tilde{R}(X,\xi)h = 0$ in a trans-Sasakian manifold.

1. Preliminaries

A (2n + 1)–dimensional real differentiable manifold M of class C^∞ is said to have a (φ,ξ,η)–sructure, or an almost contact structure, if it admits a field φ of endomorphisms of the tangent spaces, a vector field ξ, and a 1-form η satisfying

$$\eta(\xi) = 1,$$

$$\varphi^2 = -I + \eta \otimes \xi,$$

where I denotes the identity transformation, [B].

Denote by $\mathcal{X}(M)$ the Lie algebra of C^∞ vector fields on M. A (paracompact) manifold M with a (φ,ξ,η)-structure admits a Riemannian metric g such that

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

This work was supported by the "Consejería de Educación del Gobierno de Canarias".

where $X, Y \in \mathcal{X}(M)$. Then M is said to have a (φ, ξ, η, g) -structure or an almost contact metric structure and g is called a compatible metric. The 2-form Φ on M defined by

$$\Phi(X, Y) = g(X, \varphi Y)$$

is called the fundamental 2-form of the almost contact metric structure. If ∇ is the Riemannian connection of g, then

$$(\nabla_X \eta)Y = g(Y, \nabla_X \xi),$$

$$(\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \varphi)Z).$$

An almost contact metric structure (φ, ξ, η, g) is said to be
normal ($|N|$) if $(\nabla_X \varphi)Y - (\nabla_{\varphi X}\varphi)\varphi Y + \eta(Y)\nabla_{\varphi X}\xi = 0$;
α -Sasakian ($|\alpha S|$) if $(\nabla_X \varphi)Y = \alpha[g(X, Y)\xi - \eta(Y)X], \quad \alpha \in R$;
α -Kenmotsu ($|\alpha S|$) if $(\nabla_X \varphi)Y = \alpha[g(\varphi X, Y)\xi - \eta(Y)\varphi X], \quad \alpha \in R$;
Trans-Sasakian ($|tS|$) if $(\nabla_X \Phi)(Y, Z) = \frac{1}{2n}[(g(X, Y)\eta(Z) - g(X, Z)\eta(Y))$
$\delta\Phi(\xi) + (g(X, \varphi Y)\eta(Z) - g(X, \varphi(Z)\eta(Y))\delta\eta]$;
where δ denotes the coderivative on M.

Remark. If (φ, ξ, η, g) is trans-Sasakian and $\delta\Phi(\xi) = \delta\eta = 0$ then (φ, ξ, η, g) is cosymplectic, [B].

The structure (φ, ξ, η, g) is said to be Kenmotsu if it is 1-Kenmotsu, and Sasakian if it is 1-Sasakian. The relations among these structures are represented in the following diagram (where $\rightarrow$ denotes strict inclusion),

$$\begin{matrix} |\alpha S| \searrow \\ |\alpha K| \nearrow \end{matrix} |tS| \rightarrow |N|.$$

If M is a trans-Sasakian manifold, then it is α-Sasakian if $\alpha = \frac{\delta\Phi(\xi)}{2n}$ and $\delta\eta = 0$, and it is α -Kenmotsu if $\alpha = \frac{\delta\eta}{2n}$ and $\delta\Phi(\xi) = 0$. It follows that if M is trans-Sasakian, then

$$\nabla_X \xi = -\frac{1}{2n}[(X - \eta(X)\xi)\delta\eta + \varphi X \delta\Phi(\xi)]. \tag{1.1}$$

Moreover, the Riemann curvature tensor R satisfies the following property,

[ChG]:

$$
\begin{aligned}
(1.2)\qquad & \\
R(X,Y,Z,W) =& R(X,Y,\varphi Z,\varphi W) \\
&+\left(\left(\frac{\delta\Phi(\xi)}{2n}\right)^2-\left(\frac{\delta\eta}{2n}\right)^2\right)\{-g(X,Z)g(Y,W)+g(X,W) \\
&g(Y,Z)+g(X,\varphi Z)g(Y,\varphi W)-g(X,\varphi W)g(Y,\varphi Z)\} \\
&-\frac{1}{2n}[X(\delta\eta)\{g(Y,Z)\eta(W)-g(Y,W)\eta(Z)\}- \\
&-Y(\delta\eta)\{g(X,Z)\eta(W)-g(X,W)\eta(Z)\}]+\frac{1}{2n}[X(\delta\Phi(\xi)) \\
&\{g(Y,\varphi W)\eta(Z)-g(Y,\varphi Z)\eta(W)\}-Y(\delta\Phi(\xi)) \\
&\{g(X,\varphi W)\eta(Z)-g(X,\varphi Z)\eta(W)\}].
\end{aligned}
$$

For an extensive study of these structures we refer to [B], [ChG], [JV], [O].

2. Invariant submanifolds of a trans–Sasakian manifold

A (2r +1)-dimensional submanifold M of a (2n +1)-dimensional almost contact metric manifold $\tilde{M}$ with structure $(\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is said to be invariant if $\tilde{\varphi}X$ is tangent to M for any tangent vector field X to M, and $\tilde{\xi}$ is always tangent to M. It is well known that any invariant submanifold M of an almost contact metric manifold $\tilde{M}$ is also an almost contact metric manifold with the induced structure (φ,ξ,η,g), where $\varphi X=\tilde{\varphi}X$, $X\in\mathcal{X}(M)$, ξ,η and g are the restrictions of $\tilde{\xi},\tilde{\eta}$ and $\tilde{g}$ to M (see [YI]). Moreover, for any vector fields X,Y,Z on M, we have

$$(2.1)\qquad \Phi(X,Y)=\tilde{\Phi}(X,Y),$$

$$(2.2)\qquad (\tilde{\nabla}_X\tilde{\varphi})Y=(\nabla_X\varphi)Y+h(X,\varphi Y)-\tilde{\varphi}(h(X,Y)),$$

$$(2.3)\qquad (\tilde{\nabla}_X\tilde{\Phi})(Y,Z)=(\nabla_X\Phi)(Y,Z),$$

$$(2.4)\qquad (\tilde{\nabla}_X\tilde{\eta})Y=(\nabla_X\eta)Y,$$

where $\tilde{\Phi}$ and $\tilde{\nabla}$ (resp. Φ and ∇) denote the fundamental 2-form and the Riemannian connection in $\tilde{M}$ (resp. M), and h is the second fundamental form of M.

Let $\tilde{\delta}$ and δ denote the coderivatives on $\tilde{M}$ and M, respectively, and let $\mathcal{X}(M)^\perp$ denote the set of all vector fields orthogonal to M. Then from (2.3) and (2.4) we have

Lemma 2.1. *If M is an invariant submanifold of a trans-Sasakian manifold $\tilde{M}$, then,*

$$\tilde{\delta}\tilde{\Phi}(X) = \frac{n}{r}\delta\Phi(X), \qquad X \in \mathcal{X}(M), \tag{2.5}$$

$$\tilde{\delta}\tilde{\eta} = \frac{n}{r}\delta\eta. \tag{2.6}$$

Now, from (2.3) and lemma 2.1 we obtain

Proposition 2.1. *Any invariant submanifold M with induced structure (φ, ξ, η, g) of a trans-Sasakian manifold $\tilde{M}$ is also trans-Sasakian.*

Proposition 2.2. *Let M be an invariant submanifold of a trans–Sasakian manifold $\tilde{M}$. Then we have*

$$h(X, \varphi Y) = \tilde{\varphi}(h(X, Y)), \tag{2.7}$$

$$h(\varphi X, \varphi Y) = -h(X, Y), \tag{2.8}$$

$$h(X, \xi) = 0, \tag{2.9}$$

for any vector fields X and Y on M.

Proof. Making use of the definition of trans-Sasakian manifold we obtain that $\tilde{\delta}\tilde{\Phi}(N) = 0$ for any $N \in \mathcal{X}(M)^{\perp}$, and moreover, from (2.2) we have

$$\tilde{g}((\tilde{\nabla}_X\tilde{\varphi})Y, N) = \tilde{g}(h(X, \varphi Y) - \tilde{\varphi}(h(X, Y), N) = -\frac{1}{2n}\tilde{g}(X, Y)\tilde{\delta}\tilde{\Phi}(N) = 0,$$

for any $X, Y \in \mathcal{X}(M)$ and $N \in \mathcal{X}(M)^{\perp}$. Thus we obtain (2.7).

Also, from (2.7), we have $h(X, \varphi Y) = \tilde{\varphi}(h(X, Y)) = h(Y, \varphi X)$. Finally, using this relation we deduce (2.8) and (2.9).

From the identity (2.8) we conclude:

Theorem 2.1. *Any invariant submanifold M of a trans–Sasakian manifold $\tilde{M}$ is minimal.*

Corollary 2.1. *Any invariant submanifold M of an α–Sasakian, or of an α–Kenmotsu manifold $\tilde{M}$ is minimal.*

Let M be an invariant submanifold of a trans-Sasakian manifold $\tilde{M}$ and R and $\tilde{R}$ be the Riemannian curvature tensors of M and $\tilde{M}$, respectively. Then, using the equation of Gauss, [KN], and proposition 2.2, we have

$$R(X, \varphi X, X, \varphi X) = \tilde{R}(X, \varphi X, X, \varphi X) - 2\tilde{g}(h(X, X)h(X, X)), \tag{2.10}$$

for any vector field X on M.

Let K (resp. $\tilde{K}$) be a φ (resp. $\tilde{\varphi}$) –sectional curvature of M (resp. $\tilde{M}$).

Then, from (2.10), we obtain

Proposition 2.3. *Let M be an invariant submanifold of $\tilde{M}$. Then the $\varphi-$ and $\tilde{\varphi}$–sectional curvatures satisfy the inequality $K \leq \tilde{K}$, with equality holding if and only if M is totally geodesic.*

Next, we shall give some conditions for an invariant submanifold of a trans-Sasakian manifold to be totally geodesic.

Proposition 2.4. *Let M be an invariant submanifold of a trans-Sasakian manifold $\tilde{M}$. Then M is totally geodesic if and only if $(\tilde{\nabla}_X \tilde{\nabla}_Y h)(\xi, \xi) = 0$ for any vector field X and Y on M.*

PROOF. From (1.1) we obtain that

$$\varphi X = -\frac{2n}{\delta\Phi^2(\xi) + \delta^2\eta}(\delta\Phi(\xi)\nabla_X\xi + \delta\eta\nabla_{\varphi X}\xi).$$

Thus, using the definition of the covariant derivative for the second fundamental form h of M, (see [KN], p.25), we have

$$h(X,Y) = -h(\varphi X, \varphi Y) = -\frac{2n}{\delta\Phi^2(\xi) + \delta^2\eta}[\delta\Phi(\xi)(\tilde{\nabla}_Y h)(\varphi X, \xi) + \delta\eta(\tilde{\nabla}_{\varphi Y} h)(\varphi X, \xi)].$$

Now, using this relation and definition of the covariant derivative of h we obtain

$$h(\varphi X, \varphi Y) = \frac{1}{2}\left(\frac{2n}{\delta\Phi^2(\xi) + \delta^2\eta}\right)^2 [\delta\Phi(\xi)^2(\tilde{\nabla}_X\tilde{\nabla}_Y h)(\xi,\xi) + \delta\eta\delta\Phi(\xi) ((\tilde{\nabla}_{\varphi X}\tilde{\nabla}_Y h)(\xi,\xi) + (\tilde{\nabla}_X\tilde{\nabla}_{\varphi Y} h)(\xi,\xi)) + \delta\eta^2(\tilde{\nabla}_{\varphi X}\tilde{\nabla}_{\varphi Y} h)(\xi,\xi)]$$

which proves our assertion.

We now put

$$\tilde{R}(X,Y)\alpha = [\tilde{\nabla}_X, \tilde{\nabla}_Y]\alpha - \tilde{\nabla}_{[X,Y]}\alpha,$$

for a normal a bundle valued symmetric 2-form α. Then we have

$$(\tilde{R}(X,Y)h)(V,W) = R^{\perp}(X,Y)(h(V,W)) - -h((R(X,Y)V,W) - h(V,R(X,Y)W), \tag{2.11}$$

for any vector fields X, Y, V and W on M, where $R^{\perp}(X,Y) = [D_X, D_Y] - D_{[X,Y]}$, and D is the linear connection in the normal bundle $T(M)^{\perp}$.

Proposition 2.5. *Let M be an invariant submanifold of a trans-Sasakian manifold $\tilde{M}$. Then M is totally geodesic if and only if $\tilde{R}(X,\xi)h = 0$, for any vector field X on M.*

PROOF. If $\tilde{R}(X,\xi)h = 0$, from (2.11), we have

$$R^{\perp}(X,\xi)(h(V,W)) = h(R(X,\xi)V,W) + h(V,R(X,\xi)W)$$

Then, if we put $V = \xi$, from (2.9),

$$h(R(X,\xi)\xi,W) = 0. \tag{2.12}$$

On the other hand, since M is trans-Sasakian, using (1.2),

$$R(X,\xi,\xi,W) = \left(\left(\frac{\delta\Phi(\xi)}{2n}\right)^2 - \left(\frac{\delta\eta}{2n}\right)^2\right)\{g(X-\eta(X)\xi, W-\eta(W)\xi)\}$$
$$-\frac{1}{2n}\xi(\delta\eta)g(X-\eta(X)\xi, W-\eta(W)\xi) - \frac{1}{2n}\xi(\delta\Phi(\xi))g(X,\varphi W).$$

Then, we have

$$R(X,\xi)\xi = \left(\left(\frac{\delta\Phi(\xi)}{2n}\right)^2 - \left(\frac{\delta\eta}{2n}\right)^2 - \frac{1}{2n}\xi(\delta\eta)\right)(X-\eta(X)\xi) + \frac{1}{2n}\xi(\delta\Phi(\xi))\varphi X. \tag{2.13}$$

From (2.12) and (2.13), we obtain

$$\left(\left(\frac{\delta\Phi(\xi)}{2n}\right)^2 - \left(\frac{\delta\eta}{2n}\right)^2 - \frac{1}{2n}\xi(\delta\eta)\right)h(X,W) + \frac{1}{2n}\xi(\delta\Phi(\xi))h(\varphi X,W) = 0,$$

and, replacing X by φX, we deduce that $h(X,Y) = 0$, which shows that M is totally geodesic.

Finally, from propositions 2.4 and 2.5 we obtain

Theorem 2.2. *Let M be an invariant submanifold of a trans-Sasakian manifold $\tilde{M}$. Then, the following conditions are equivalent:*
(i) *M is totally geodesic;*
(ii) $(\tilde{\nabla}_X\tilde{\nabla}_Y h)(\xi,\xi) = 0$;
(iii) $\tilde{R}(X,\xi)h = 0$;
(iv) $\tilde{R}(X,Y)h = 0$,
where X and Y are arbitrary vector fields on M.

References

[B] D. E. BLAIR, Contact manifolds in Riemannian geometry, *Lecture Notes in Math.* **509** (1976), *Springer.*

[ChG] D. CHINEA and C. GONZALEZ, Relaciones de curvatura en las variedades trans-Sasakianas, *XII Jornadas Luso-Espanholas de Matemática, Braga*, 1987.

[JV] D. JANSSENS and L. VANHECKE, Almost contact structures and curvature tensors, *Kodai Math. J.* **4** (1981), 1–27.

[Ke] K. KENMOTSU, A class of almost contact Riemannian manifolds, *Tôhoku Math. J.* **24** (1972), 93–103.

[KN] S. KOBAYASHI and K. NOMIZU, Foundations of differential geometry, vol. II., *John Wiley and Sons, New York*, 1969.

[K] M. KON, Invariant submanifolds of normal contact metric manifolds, *Kodai Math. Sem. Rep.* **25** (1973), 330–336.

[O] J. OUBIÑA, New classes of almost contact metric structures, *Publicationes Mathematicae* **32** (1985), 187–193, *Debrecen.*

[T] S. TANNO, Isometric inmersion of Sasakian manifolds in spheres, *Kodai Math. Sem. Rep.* **21** (1969), 448–458.

[YI] K. YANO and S. ISHIHARA, Invariant submanifolds of an almost contact manifold, *Kodai Math. Sem. Rep.* **21** (1969), 350–364.

DEPARTAMENTO DE MATEMÁTICA FUNDAMENTAL
FACULTAD DE MATEMÁTICAS
UNIVERSIDAD DE LA LAGUNA
ISLAS CANARIAS — ESPAÑA

(Received January 3, 1988)