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Splitting theorems for nonassociative rings

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To the memory of Professor O. Steinfeld.

1. Introduction

In the algebra various direct decomposition theorems are known asserting that a certain kind of algebras decomposes into a direct sum of two (or more) uniquely determined distinguished subalgebras of diverse properties. Such theorems will be referred to as splitting theorems. In section 2 we shall discuss the general theory of splitting theorems in terms of Plotkin radicals. A natural Plotkin radical is introduced for alternative rings in section 3 and a splitting theorem is proved for semiprime alternative MPR–rings which generalizes a result of SLATER [19] and ZHEVLAKOV [24] (MPR means d.c.c. on principal right ideals). Section 4 is devoted to the splitting of the maximal torsion ideal of a not necessarily associative ring and two criteria are proved for the splitting. Applying the results of section 4, in section 5 it is proved that *every alternative MPR–ring splits with respect to the torsion radical.* This result generalizes the AYOUB – HUYNH Theorem [1], [7] on the splitting of associative MPR–rings and WIDIGER'S Theorem [23] of the splitting of alternative artinian rings. Finally a sufficient condition is given for the splitting of Jordan rings with respect to the maximal torsion ideal.

Next, as a motivation for the investigations we present a sample of splitting theorems.

1) If $(\mathcal{T}, \mathcal{F})$ is a *centrally splitting torsion theory*, then every module is a direct sum of its maximal $\mathcal{T}$–module and $\mathcal{F}$–module, (cf. for instance [2], [12], [13]).

2) Every *associative artinian ring* is a direct sum of its maximal torsion ideal and of a uniquely determined torsionfree ideal ([21]). This statement is true also for *associative MPR–rings* ([1], [7]) and for *alternative artinian rings* ([23]).

3) Every *right and left artinian ring* is a direct sum of two uniquely determined ideals I and K such that $|I| < \alpha$ and $|A/K| < \alpha$ where α is an arbitrarily given infinite cardinal ([8]).

4) The Jacobson radical of every member of *an associative ring variety* is a direct summand if and only if the variety satisfies the identities

$$x^k y = y x^k = x^k y^n$$

for some integers $k \geq 1$ and $n \geq 2$ (see [22]). As M. V. VOLKOV has pointed out, also the Jacobson semisimple summand is a uniquely determined ideal (private communication).

5) Every *autodistributive ring* A (and algebra over the field of two elements) is a direct sum of the ideals

$$I = \{a \in A : a^2 = a\}$$

and

$$N = \{b \in A : b \cdot b^2 = 0\}$$

(cf. [10], [11], [14]).

6) Every *weakly distributive algebra* decomposes into a direct sum of three uniquely determined ideals which are a lattice ordered group (the multiplication defines the meet in the lattice) a commutative regular ring and an algebra of characteristic 3 (see [16]).

7) Let $\{P, Q\}$ be a partition of the prime numbers. Every *torsion abelian group* is a direct sum of its maximal P–subgroup and Q–subgroup, and these subgroups are uniquely determined.

8) In the variety of complemented semigroups satisfying the identities IR and IV of [4] every semigroup is a direct product of an idempotent semigroup and of a cancellative semigroup, and the factors are uniquely determined subsemigroups.

9) In the variety of *complemented semigroups* satisfying the identities IR and BV (or IV and BV) of [4] every semigroup is a direct product of two uniquely determined subsemigroups, one is a boolean ring, the other the "positive cone of an ℓ–group" (in German: Verbandsgruppenkern; cf. [4] Korollar 2 and Satz 15).

10) A "*divisibility semigroup*" (in German: Teilbarkeitshalbgruppe) satisfying the identity of [5] Satz 2, is a direct product of a distributive lattice and of a lattice–ordered group.

Let us observe that in many cases of the above examples the decomposition theorem holds only in a rather restricted subclass of the variety considered (e.g. artinian rings, or torsion abelian groups). There are also instances for decomposition theorems in which only one component is a uniquely determined subalgebra, we mention here two such examples.

11) Every *abelian group* is a direct sum of its maximal divisible subgroup and of *a* reduced subgroup.

12) Every *right group* is a direct product of its maximal right zero subsemigroup and of a group ([6]).

2. General theory

Let $\mathcal{V}$ be a variety of nonassociative rings (or Ω–groups, etc.). A subclass $\mathcal{A}$ of $\mathcal{V}$ is called a *universal class*, if $\mathcal{A}$ is closed under taking homomorphic images and ideals. A subclass ρ of a universal class $\mathcal{A}$ is called a *Plotkin radical* [17] (or pretorsion class in [3]), if ρ is homomorphically closed and every $A \in \mathcal{A}$ has a unique largest ρ–ideal ρA, called the *ρ–radical of A*. The class

$$\sigma = \{A \in \mathcal{A} : \rho A = 0\}$$

is referred to the *semisimple class of ρ*. Clearly $\rho \cap \sigma = 0$. A class $\gamma \subseteq \mathcal{V}$ is said to be *hereditary,* if it is closed under taking ideals. The *Kurosh–Amitsur radicals* are precisely those Plotkin radicals ρ which are *closed under extensions,* that is, $I \triangleleft A \in \mathcal{A}$, $I \in \rho$ and $A/I \in \rho$ imply $A \in \rho$, or equivalently, $\rho(A/\rho A) = 0$ for all $A \in \mathcal{A}$. The semisimple class σ of a Kurosh–Amitsur radical ρ is always *regular,* that is, if $0 \neq I \triangleleft A \in \sigma$, then I has a nonzero homomorphic image in σ.

Proposition 1. *Let ρ be a hereditary Plotkin radical in a universal class $\mathcal{A} \subseteq \mathcal{V}$ and σ its semisimple class. The ρ–radical ρA of A is a direct summand of A if and only if there exists an ideal K of A such that $K \in \sigma$ and $A = \rho A + K$.*

The necessity is obvious. For the sufficiency we have to prove only that $\rho A \cap K = 0$. Since ρ is hereditary, $\rho A \cap K \triangleleft \rho A \in \rho$ implies $\rho A \cap K \in \rho$. Hence by $\rho A \cap K \triangleleft K \in \sigma$ it follows $\rho A \cap K \subseteq \rho K = 0$, and so $A = \rho A \boxplus K$ is a direct sum.

We say that an object $A \in \mathcal{A}$ *splits with respect to the radical ρ* if A is a direct sum $A = \rho A \boxplus \sigma A$ such that σA is the unique largest σ–ideal of A. In this case we also say that the *ρ–radical ρA of A splits off.*

Theorem 1. *Let ρ be a hereditary Plotkin radical in a universal class $\mathcal{A} \subseteq \mathcal{V}$ and σ the corresponding semisimple class. An object $A \in \mathcal{A}$ splits with respect to ρ if and only if*

1) there exists an ideal K of A such that $K \in \sigma$ and $A = \rho A + K$,

2) for any two maximal σ–ideals I and L of A $\quad I/(I \cap L) \in \sigma$.

PROOF. *Necessity.* 1) follows from Proposition 1 and 2) becomes trivial.

Sufficiency. First we prove that K is a maximal σ–ideal of A. Suppose that an ideal M properly contains K. Then in view of Proposition 1 we have $A = \rho A \boxplus K$, and so it follows $M = R \boxplus K$ with a suitable ideal $0 \neq R \subseteq \rho A$. Since ρ is hereditary, we have $R \in \rho$, and therefore $0 \neq R \subseteq \rho M$, that is, $M \notin \sigma$. Hence K is a maximal σ–ideal of A. Let I be any other maximal σ–ideal of A. Now we have

$$I/(I \cap K) \simeq (I + K)/K \triangleleft A/K \simeq \rho A \in \rho.$$

Hence condition 2), the hereditariness of ρ and $\rho \cap \sigma = 0$ yield $I = K$.

Proposition 2. *If ρ is a hereditary Plotkin radical in $\mathcal{A}$ with semisimple class σ, then every object $A \in \mathcal{A}$ has maximal σ–ideals .*

PROOF. Let $I_1 \subseteq \ldots \subseteq I_\alpha \subseteq \ldots$ be an ascending chain of ideals of $A \in \mathcal{A}$ such that $I_\alpha \in \sigma$ for each index α. Putting $I = \cup I_\alpha$, we have

$$I_\alpha \cap \rho I \subseteq \rho I \in \rho,$$

and so the hereditariness of ρ implies $I_\alpha \cap \rho I \in \rho$ for all indices α. Hence $(I_\alpha \cap \rho I) \subseteq \rho I_\alpha = 0$ for all α, yielding $\rho I = 0$, that is, $I \in \sigma$. Now an application of Zorn's Lemma yields the assertion.

Proposition 3. *The semisimple class σ of a Plotkin radical ρ is closed under extensions.*

PROOF. Since ρ is homomorphically closed, for every ideal I of $A \in \mathcal{A}$ the relation

$$\rho A/(\rho A \cap I) \simeq (\rho A + I)/I \subseteq \rho(A/I)$$

must be true. Hence, if $A/I \in \sigma$, we get $\rho A \subseteq I$. If also $I \in \sigma$ holds, then by $\rho A \in \rho$ it follows $\rho A \subseteq \rho I = 0$, that is, $A \in \sigma$.

Proposition 4. *Let ρ be a Plotkin radical with semisimple class σ. If every $A \in \mathcal{A}$ is a direct sum $A = \rho A \boxplus K$ with an ideal $K \in \sigma$, then ρ is a Kurosh–Amitsur radical in $\mathcal{A}$.*

PROOF. For every A we have $A/\rho A \simeq K \in \sigma$, and so $\rho(A/\rho A) = 0$ holds.

Proposition 5. *If σ is a homomorphically closed class, then condition 2) of Theorem 1 is trivially fulfilled.*

Proposition 6. *If ρ is a hereditary Plotkin radical in $\mathcal{A}$ and its semisimple class σ is homomorphically closed, then every $A \in \mathcal{A}$ has a unique largest σ–ideal, and σ is a Kurosh–Amitsur radical class.*

Let us observe that if σ is a Plotkin semisimple class which is also a Plotkin radical, then σ is a Kurosh–Amitsur radical.

PROOF. Let I be an arbitrary σ–ideal of A. By Proposition 2 A has a maximal σ–ideal K. Since σ is homomorphically closed, by $I \in \sigma$ we get

$$(I + K)/K \simeq I/(I \cap K) \in \sigma.$$

Hence by Proposition 3 it follows $(I+K) \in \sigma$, and therefore the maximality of K implies $I \subseteq K$.

Now the second assertion follows immediately from Proposition 3.

In his [11] Theorem 2.1 Gardner proved the equivalence of the following statements:

1) in the variety $\mathcal{V}\rho$ is a radical semisimple class such that its semisimple class σ is a radical class with semisimple class ρ,

2) ρ and σ are subvarieties and every $A \in \mathcal{V}$ can be decomposed as a direct sum $A = R \boxplus S$ where $R \in \rho$ and $S \in \rho$ are uniquely determined ideals.

Our next result is a counterpart of Gardner's Theorem for universal classes.

Theorem 2. *Let ρ be a hereditary Plotkin radical with semisimple class σ in a universal class $\mathcal{A} \subseteq \mathcal{V}$. Then the following conditions are equivalent:*

I) σ is a (Plotkin) radical with semisimple class ρ,

II) σ is homomorphically closed and every $A \in \mathcal{A}$ splits with respect to ρ.

PROOF. $I \Rightarrow II$. Since both ρ and σ are Kurosh–Amitsur radicals by Proposition 3, and σ is homomorphically closed, we have

$$A/(\rho A + \sigma A) \simeq \frac{A/\rho A}{(\rho A + \sigma A)/\rho A} \in \sigma$$

and

$$A/(\rho A + \sigma A) \simeq \frac{A/\sigma A}{(\rho A + \sigma A)/\sigma A} \in \rho$$

which implies $A/(\rho A + \sigma A) = 0$, that is, $A = \rho A + \sigma A$. Hence by Proposition 5 we can apply Theorem 1 establishing II.

$II \Rightarrow I$. By Proposition 6 σ is a Kurosh–Amitsur radical class. If $\sigma A = 0$, then by the unique decomposition $A = \rho A \boxplus \sigma A$ it follows $A = \rho A \in \rho$, and conversely. Hence

$$\rho = \{A \in \mathcal{A} : \sigma A = 0\}$$

is the semisimple class of σ.

In the variety $\mathcal{V}$ of all associative rings every radical semisimple class $\rho \neq \mathcal{V}$ consists of idempotent rings and so only the radicals $\rho = 0$ and $\rho = \mathcal{V}$ satisfy condition I in Theorem 2.

3. Alternative rings and the maximal nuclear ideal

In view of Proposition 4 one gets the impression that in the reasonable cases the Plotkin radical ρ has to be a Kurosh–Amitsur radical. This is, however, not so, and in this section we shall give an example for a very natural Plotkin radical in the variety $\mathcal{V}$ of *all alternative rings* which splits off in the subclass $\mathcal{B} \subseteq \mathcal{V}$ of semiprime MPR–rings. The class $\mathcal{B}$ is, of course, not a universal class (cf. Proposition 2 and Theorem 2).

A not necessarily associative ring A will be called an *MPR–ring*, if A satisfies the d.c.c. on principal right ideals. As is well–known, a ring A is *alternative*, if A satisfies

$$x^2y = x(xy) \quad \text{and} \quad (xy)y = xy^2$$

for all $x, y \in A$. The associator (x, y, z) of the elements $x, y, z \in A$ is defined by

$$(x, y, z) = (xy)z - x(yz).$$

An element $u \in A$ is said to be ***nuclear***, if the associator (u, x, y) vanishes for all $x, y \in A$. The set $\mathcal{N}(A)$ of all nuclear elements of A is the ***nucleus*** of A. A has a unique maximal ideal $\mathcal{U}(A)$ in $\mathcal{N}(A)$ which is called the ***maximal nuclear ideal*** of A.

Every alternative ring A possesses another distinguished ideal, the ***associator ideal*** $\mathcal{D}(A)$, which is the ideal of A generated by all associators. As is well–known $\mathcal{U}(A)\mathcal{D}(A) = \mathcal{D}(A)\mathcal{U}(A) = 0$. For details we refer to [25].

As in the associative case, an alternative ring A is said to be ***semiprime***, if $I \triangleleft A$ and $I^2 = 0$ imply $I = 0$.

Propositions 7. *If M is a minimal right ideal of a semiprime alternative ring A, then $M^2 \neq 0, M^2 = M$ and $M = eA$ for a nuclear idempotent e of A.*

PROOF. The first assertion follows immediately from [18] Lemma (3.3) which states that a trivial right ideal generates a trivial ideal. The rest is [20] Proposition 3.3 (b) and (c).

The sum of all minimal right ideals of a ring A is called the ***right socle*** of A and will be denoted by Soc A.

Proposition 8. *If A is an alternative semiprime MPR–ring, then $A =$ Soc A.*

PROOF. Let $a \neq 0$ be an arbitrary element of A. We shall prove that a is contained in a finite sum of minimal right ideals. Since A is an MPR–ring, there exists a minimal right ideal M_1 of A which is contained in the principal right ideal $[a)$ of A. By Proposition 7 there exists a nuclear idempotent e_1 such that $M_1 = e_1A$. Take the element $a_2 = a - e_1a$. We have obviously

$$[a) = M_1 + [a_2).$$

Let $x \in M_1 \cap [a_2)$ be an arbitrary element. Now x has the forms

$$x = e_1b, \qquad b \in A$$

and

$$x = ma_2 + \sum_{i=1}^{n}(\dots((a_2c_{i1})c_{i2})\dots)c_{ik_i}$$

where m is an integer and $c_{ij_i} \in A,\ j_i = 1, \dots, k_i$. Since e_1 is a nuclear idempotent, we have

$$e_1x = e_1(e_1b) = e_1^2b = x$$

and

$$e_1a_2 = e_1a - e_1(e_1a) = e_1a - e_1^2a = 0,$$

furthermore, also

$$x = e_1x = e_1(ma_2) + e_1\sum_{i=1}^{n}(\dots((a_2c_{i1})c_{i2})\dots)c_{ik_i} =$$
$$= m(e_1a_2) + \sum_{i=1}^{n}(\dots(((e_1a_2)c_{i1})c_{i2})\dots c_{ik_i}) = 0.$$

Thus $[a)$ is a direct sum

$$[a) = M_1 \oplus [a_2)$$

of the right ideals, and so $[a)$ properly contains a_2. Continuing this procedure we get a decomposition

$$[a) = M_1 \oplus \dots \oplus M_{n-1} \oplus [a_n)$$

and also a strictly descending chain

$$[a) \supset [a_2) \supset \dots \supset [a_n)$$

of principal right ideals of A. Since A is an MPR–ring, this chain has to terminate in finitely many steps. Thus $[a)$ is a finite direct sum of minimal right ideals, proving the assertion.

Proposition 9. (cf. [20] Proposition 4.15). *If A is an alternative semiprime ring, then*

$$\text{Soc } A = \text{Soc } \mathcal{U}(A) \boxplus \text{Soc } \mathcal{D}(A)$$

where Soc $\mathcal{U}(A)$ *and* Soc $\mathcal{D}(A)$ *are two–sided ideals in A.*

Theorem 3. *If A is an alternative semiprime MPR–ring, then the maximal nuclear ideal $\mathcal{U}(A)$ splits off, in fact*

$$A = \mathcal{U}(A) \boxplus \mathcal{D}(A).$$

PROOF. Since A is semiprime, $(\mathcal{U}(A) \cap \mathcal{D}(A))^2 \subseteq \mathcal{U}(A)\mathcal{D}(A) = 0$ implies $\mathcal{U}(A) \cap \mathcal{D}(A) = 0$. Hence Propositions 8 and 9 yield

$$A = \text{Soc } A = \text{Soc } \mathcal{U}(A) \boxplus \text{Soc } \mathcal{D}(A) \subseteq \mathcal{U}(A) \boxplus \mathcal{D}(A) \subseteq A.$$

Theorem 3 provides also a full description of the structure of any alternative semiprime MPR–ring A inasmuch as it reduces the description of A to that of an associative semiprime MPR–ring $\mathcal{U}(A)$ and to that of a purely alternative semiprime MPR–ring $\mathcal{D}(A)$. Hence $\mathcal{U}(A)$ is a discrete direct sum of simple rings of linear transformations of finite rank on vector spaces over division rings (cf. e.g. [15] Theorems 77.4 and 78.2 or [20], and $\mathcal{D}(A)$ is a discrete direct sum of Cayley–Dickson algebras (cf. [20])). Moreover, the right socle of an alternative semiprime MPR–ring coincides with its left socle. Thus Theorem 3 generalizes the ZHEVLAKOV – SLATER Theorem describing the structure of alternative semiprime artinian rings ([24] Theorem 3 and [19] Theorem B, or [25] Theorem 12.2.3).

As is well–known, semiprimeness is a hereditary property (Slater [18] or [25] Theorem 9.1.4). Hence the class $\beta = \{A \in \mathcal{V} \mid A$ has only 0 semiprime homomorphic image $\}$ is a Kurosh–Amitsur radical class, called the *Baer radical class* (cf. [25] p 162). Let us consider the class $\alpha = \{A \in \mathcal{V} \mid A/\beta(A)$ is associative $\}$. Since $\mathcal{N}(I) = I \cap \mathcal{N}(A)$ holds for every $I \triangleleft A$ (see [18] or [24] Theorem 9.1.1) by the heredity of semiprimeness it follows that *α is a hereditary Plotkin radical, but not a Kurosh–Amitsur radical in $\mathcal{V}$*. Furthermore, we have obviously

$$\alpha(A)/\beta(A) = \mathcal{U}(A/\beta(A)),$$

and hence $\alpha(A) = \mathcal{U}(A)$ for every semiprime ring A. Thus Theorem 3 yields immediately the following reformulation.

Corollary 1. *The Plotkin radical α splits off on the class $\mathcal{B}$ of all alternative semiprime MPR–rings.*

Let $\mathcal{J}$ denote the *quasi–regular radical* (or *Zhevlakov radical* in [25] Theorem 10.4.5).

Corollary 2. *If A is an alternative MPR–ring, then $\beta(A) = \mathcal{J}(A)$.*

PROOF. Clearly $\beta(A) \subseteq \mathcal{J}(A)$ always holds. Hence without loss of generality we may assume that $\beta(A) = 0$. Since a quasi–regular ideal does not contain nonzero idempotents, the assertion follows from Theorem 3.

4. Rings and the torsion radical

In this section we shall work in the variety $\mathcal{V}$ of *all not necessarily associative rings*. It is well–known that in $\mathcal{V}$ the class of all torsion rings forms a hereditary (Kurosh–Amitsur) radical class, and so we can speak of the *maximal torsion ideal* of a ring A. We know also that the maximal divisible subgroup of any ring $A \in \mathcal{V}$ forms an ideal, the *maximal divisible ideal* of A.

Proposition 10. *If F_+ is a torsionfree divisible subgroup and T_+ is a torsion subgroup of the additive group A_+ of a ring A, then $FT = TF = 0$.*

For the proof, which uses only the distributivity, we refer to Kertész [15] Proposition 57.7.

Proposition 11. *Let A be a ring such that $A_+ = B_+ \oplus D_+$ is a group direct sum of a reduced torsion group B_+ and of the maximal divisible subgroup D_+. Then*

1) $A_+ = T_+ \oplus F_+$ where T is the maximal torsion ideal of A and F_+ is a torsionfree divisible subgroup of A_+,

2) every quasi–cyclic subgroup of A_+ is in the annihilator of A,

3) $BD = DB = 0$,

4) $D/(T \cap D) \simeq A/T$.

PROOF. By $A_+ = B_+ \oplus D_+$ we have $T_+ = B_+ \oplus (T \cap D)_+$ and $D_+ = (T \cap D)_+ \oplus F_+$. Hence 1) is obviously true.

For the proof of 2) we refer to Kertész [15] Proposition 57.8. In view of the decomposition $D_+ = (T \cap D)_+ \oplus F_+$ statement 2) and Proposition 10 yield statement 3).

4) and its proof is Ayoub's Theorem 2.(3) in [1] where no associativity of the multiplication is used.

Theorem 4. *The following conditions are equivalent:*

$$I) \qquad A = T \boxplus K,$$
$$II) \qquad D^2 \text{ is torsionfree},$$
$$III) \qquad F^2 \text{ is torsionfree}.$$

PROOF. By the assumption on A_+ we have

$$D^2 = ((T \cap D) + F)^2 = (T \cap D)^2 + (T \cap D)F + F(T \cap D) + F^2.$$

In view of Proposition 10 and Proposition 11, 2) the first three terms of the right hand side are 0. Hence $D^2 = F^2$ always holds proving the equivalence of II and III.

$I \Rightarrow II$. Let $A = T \boxplus K$ where K is a torsionfree ideal of A. By the structure of A_+ it follows that $D_+ = (T \cap D)_+ \oplus K_+$. Hence by the previous consideration we have $D^2 = K^2 \subseteq K$, and therefore D^2 is torsionfree.

$II \Rightarrow I$. By the distributivity D^2 is clearly a divisible group, hence $D_+ = (T \cap D)_+ \oplus D^2_+ \oplus C_+$ holds with a suitable divisible torsionfree subgroup C_+. Moreover, also $A_+ = T_+ \oplus D^2_+ \oplus C_+$ holds. By Proposition 10 we get

$$(D^2 + C)A = D^2(B + D) + C(B + D) = D^2B + D^2D + CB + CD \subseteq D^2,$$

and similarly also $A(D^2 + C) \subseteq D^2$. Thus $D^2 + C$ is an ideal in A, and so in the direct decomposition of A_+ both summands are ideals of A.

It may happen that *in the direct decomposition $A = T \boxplus F$ the ideal F* is not uniquely determined. For instance, let A be a ring such that $A^2 = 0$

and $A_+ = C(p^\infty) \oplus Q_+$ where $C(p^\infty)$ is a quasi–cyclic group and Q_+ is the additive group of rational numbers. Let $a_1, \dots, a_n, \dots$ be generators of $C(p^\infty)$ subjected to $pa_1 = 0, pa_2 = a_1, \dots, pa_n = a_{n-1}, \dots$. Let us define a mapping $\phi : Q_+ \to Q_+ \oplus C(p^\infty)$ by

$$\phi\left(\frac{k}{\ell}p^r\right) = \begin{cases} \dfrac{k}{\ell}p^r & \text{if } r \geq 0 \text{ (including } k = 0) \\ \dfrac{k}{\ell}p^r + \dfrac{k}{\ell}a_{-r} & \text{if } r < 0 \end{cases}$$

where $k, \ell \neq 0$, and p are mutually relatively prime. It is easy to check that ϕ is an isomorphism onto a subgroup $Q'_+ \neq Q_+, A_+ = C(p^\infty) \oplus Q'_+$ and so the second direct component of A is not uniquely determined as an ideal.

This example is typical, as we see it from the following

Proposition 12. *Let A be a ring such that $A_+ = B_+ \oplus D_+$ where B_+ is a reduced group and D_+ is a divisible group. If A_+ does not contain a quasi–cyclic subgroup, then the maximal torsion ideal T of A splits off.*

PROOF. By the assumption we have $B_+ = T_+$. Since both T and D are uniquely determined ideals, the assertion follows.

The smallest nonzero ideal of a subdirectly irreducible ring A is called the *heart* of A. Let $\mathcal{M}$ be a homomorphically closed subclass of $\mathcal{M}$ such that for every $A \in \mathcal{M}$, it holds $A_+ = B_+ \oplus D_+$ where B_+ is a reduced torsion group and D_+ is the maximal divisible subgroup.

Theorem 5. *The following conditions are equivalent:*

I) for every ring $A \in \mathcal{M}$ the maximal torsion ideal T splits off,

II) for every ring $A \in \mathcal{M}$ D^2 is torsionfree where D denotes the maximal divisible ideal of A,

III) every subdirectly irreducible ring $A \in \mathcal{M}$ having a torsion heart, is a torsion ring.

PROOF. Theorems 1 and 4 yield the equivalence of I and II.

$I \Rightarrow III$. Let A be a subdirectly irreducible ring from the class $\mathcal{M}$ with torsion heart H. By I we have

$$A = T \boxplus F,$$

and hence also

$$F \cap H \subseteq F \cap T = 0.$$

Thus $F = 0$ and $A = T$ follows.

$III \Rightarrow II$. Assume that there are elements $a_1, \dots, a_n$, $b_1, \dots, b_n$ in the maximal divisible ideal D of a ring A such that $0 \neq \sum_{i=1}^{n} a_i b_i \in T$. Using Zorn's Lemma we can choose an ideal I of A such that I is maximal with respect to $\sum_{i=1}^{n} a_i b_i \notin I$. The ring $\overline{A} = A/I$ is clearly subdirectly irreducible, and its heart $\overline{H}$ contains the coset $\sum_{i=1}^{n} \overline{a}_i \overline{b}_i$ which is a torsion elément. Hence $\overline{H}$ has to be a torsion ring (in fact a p-torsion ring for some prime p.) Thus by III also the ring $\overline{A}$ is a torsion ring, and consequently its maximal divisible ideal $\overline{D}$ is the sum of quasi-cyclic groups and so $\overline{D}^2 = 0$. Since $\overline{a}_1, \dots, \overline{a}_n$ and $\overline{b}_1, \dots, \overline{b}_n$ are in $\overline{D}$, it follows $\sum_{i=1}^{n} \overline{a}_i \overline{b}_i = 0$ which contradicts $\sum_{i=1}^{n} \overline{a}_i \overline{b}_i \notin I$. Thus D^2 is torsionfree.

We shall make use of the following nearly trivial

Proposition 13. *If* $[a_1, \dots, a_n]$ *is any product of* n *elements and* i *is any of the integers* $1, \dots, n$, *then*

$$k[a_1, \dots, a_n] = [a_1, \dots, ka_i, \dots, a_n]$$

for every integer k. *Moreover* $[ka) = k[a)$ *holds for the principal right ideals* $[ka)$ *and* $[a), a \in A$.

PROOF. By the distributivity we have

$$(ka)b = k(ab) = a(kb)$$

for every element $a, b \in A \in \mathcal{A}$, and the assertion follows by induction. The rest is straightforward.

Proposition 14. *If* A *is an MPR–ring, then its additive group* A_+ *is a direct sum* $A_+ = B_+ \oplus D_+$ *where* B_+ *is a reduced torsion group and* D_+ *is the maximal divisible subgroup of* A_+.

PROOF. The additive group of A can be surely decomposed as $A_+ = B_+ \oplus D_+$ where D_+ is the maximal divisible subgroup in A_+ and B_+ is a reduced group. Let us observe that B_+ does not contain nonzero divisible element. Since A is an MPR–ring, the set

$$\{[b) : b \in B \text{ and } o(b) = \infty\}$$

of principal right ideals of A has a minimal element, say $[c)$. Now we have

$$[c) = [2c) = \dots = [nc) = \dots .$$

Since by Proposition 13

$$[c) = [nc) = n[c),$$

the element c is divisible. Thus by $c \in B$ we conclude that $c = 0$, and so B is a torsion group.

Proposition 15. *If A is a torsionfree MPR–ring, then every element $a \in A$ is a finite sum*

$$a = \sum_{i=1}^{r} (\dots((ab_{i2})b_{i3})\dots)b_{is_i}$$

of products of at least two factors.

PROOF. Suppose that A is an MPR–ring. Then every descending chain

$$[a) \supset \dots \supset [p^k a) \supset \dots$$

terminates at an integer $k \geq 1$ for every prime number p. Hence

$$p^k a = \ell(p^{k+1}a) + \sum_{i=1}^{r} [p^{k+1}a, c_{i2}, \dots, c_{is_i}]$$

where ℓ is an integer $s_i \geq 2$ and $[x_1, \dots, x_n]$ stands for $(\dots((x_1x_2)x_3)\dots)x_n$. Further, for $n = p^k - \ell p^{k+1} \neq 0$ we have

$$na = \sum_{i=1}^{r} [p^{k+1}a, c_{i2}, \dots, c_{is_i}].$$

Since by Proposition 14 the additive group A_+ is divisible, to each c_{i2} there exists an element $d_{i2} \in A_+$ such that $nd_{i2} = c_{i2}$, $(i = 1, \dots, r)$. Hence by Proposition 13

$$na = n \sum_{i=1}^{r} [p^{k+1}a, d_{i2}, c_{i3}, \dots, c_{is_i}]$$

(if $s_i = 2$, then the product is just $p^{k+1}a \cdot d_{i2}$). Taking into account that A is torsionfree, we get

$$a = \sum_{i=1}^{r} [p^{k+1}a, d_{i2}, c_{i3}, \dots, c_{is_i}]$$

and hence Proposition 13 yields the desired form

$$a = \sum_{i=1}^{r} [a, p^{k+1}d_{i2}, c_{i3}, \dots, c_{is_i}]$$

and also the parantheses are in the required order.

Proposition 16. *The class of all torsionfree MPR–rings is homomorphically closed.*

PROOF. Let A be a torsionfree MPR–ring and $I \triangleleft A, a \in A$. Suppose that there exists an integer $k \neq 0$ such that $ka \in I$. By Proposition 15 we have

$$a = \sum_{i=1}^{r} [a, b_{i2}, \dots , b_{is_i}]$$

where $s_i \geq 2$ for each $i = 1, \dots , r$. Since by Proposition 11, 1) and 14 A_+ is divisible, for every b_{i2} there exists an element c_{i2} such that $kc_{i2} = b_{i2}, (i = 1, \dots , r)$. Now by Proposition 13 we have

$$a = \sum_{i=1}^{r} [a, kc_{i2}, b_{i3}, \dots , b_{is_i}] = \\ = \sum_{i=1}^{r} [ka, c_{i2}, b_{i3}, \dots , b_{is_i}] \in I.$$

Hence $ka \in I$ implies $a \in I$. Thus the factor ring A/I is torsionfree, and the assertion is proved.

Theorem 6. *For a not necessarily associative MPR–ring A the following conditions are equivalent:*

I) the maximal torsion ideal T of A splits off,

II) D^2 is torsionfree where D denotes the maximal divisible ideal of A,

III) if a subdirectly irreducible factor ring $\overline{A}$ of A has a torsion heart, then $\overline{A}$ is a torsion ring.

PROOF. By Propositions 14 and 16 Theorem 5 is applicable yielding the equivalences.

In the previous considerations we followed ideas of CHRISTINE W. AYOUB who proved that every associative MPR–ring splits with respect to the torsion ideal (cf. [1] Corollary 1). In fact, she proved in [1] Theorem 4 that D^2 is torsionfree.

5. Alternative rings and the torsion radical

In this section we shall work again in the variety $\mathcal{V}$ of *all alternative rings*. The main goal of this section is to prove that every alternative MPR–ring splits with respect to its maximal torsion ideal. In proving this, we shall use Theorems 3 and 6 as well as several arguments from the book [25].

Proposition 17. ([9] Lemma 1). *If M is a right ideal of an alternative ring A, then $AM + M \triangleleft A$.*

Proposition 18. *Let A be an alternative ring with maximal divisible ideal D and maximal torsion ideal T such that $A = D + T$. Further, let $a_1, \dots, a_n \in D$ and $b_1, \dots, b_n \in A$ be elements such that $\sum_{i=1}^{n} a_i b_i \in T$. If there exists elements $d \in A$ and $t_1, \dots, t_n \in T$ such that $b_i = b_i d + t_i$ for $i = 1, \dots, n$, then $\sum_{i=1}^{n} a_i b_i = 0$.*

PROOF. We shall use the Moufang identity

$$(xy)(zx) = x(yz)x \tag{1}$$

(see for instance [25] Lemma 2.3.7.).

In view of Propositions 10 and 11 it follows $TD = DT = 0$. Hence by $A = T + D$ we have

$$(T \cap D)A = A(T \cap D) = 0, \tag{2}$$

and so also

$$a_i t_i = 0 = (da_i)t_i, \qquad i = 1, \dots, n.$$

Using repeatedly (2) and finally (1) we get

$$\begin{aligned}
&\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} a_i(b_i d + t_i) = \sum_{i=1}^{n} a_i(b_i d) = \\
&= \sum_{i=1}^{n} a_i(b_i d) - \left(\sum_{i=1}^{n} a_i b_i\right) d = -\sum_{i=1}^{n}(a_i, b_i, d) = -\sum_{i=1}^{n}(d, a_i, b_i) = \\
&= -\sum_{i=1}^{n}(da_i)b_i + d\left(\sum_{i=1}^{n} a_i b_i\right) = -\sum_{i=1}^{n}(da_i)(b_i d + t_i) = \\
&= \sum_{i=1}^{n}(da_i)(b_i d) = -d\left(\sum_{i=1}^{n} a_i b_i\right) d = 0.
\end{aligned}$$

Proposition 19. *Again, let $A = D + T$ be an alternative ring, and let M be a right ideal of A such that $M \subseteq D$. If for any finitely many elements $m_1, \dots, m_n \in M$ there exists an idempotent $e \in A$ such that $m_i e = m_i$; $i = 1, \dots, n$, then $(AM + M) \cap T = 0$.*

PROOF. Let $m + \sum_{i=2}^{n} a_i m_i$ be an arbitrary element of $AM + M$, and suppose that this element is in T. Now there exists an idempotent $e \in A$

such that $me = m$ and $m_i e = m_i$ for $i = 2, \dots, n$. Since $A = D + T$, we have $e = x + y$ where $x \in T$ and $y \in D$. By (2) it follows $mx \in DT = 0$, and so

$$m = me = mx + my = my.$$

Putting $a_1 = m$ and $m_1 = y$ we get

$$\begin{aligned} m + \sum_{i=2}^{n} a_i m_i &= \sum_{i=1}^{n} a_i m_i = \sum_{i=1}^{n} a_i (m_i e) = \\ &= \left(\sum_{i=1}^{n} a_i m_i \right) e - \sum_{i=1}^{n} (a_i, m_i, e). \end{aligned}$$

Taking into account (2) and $m + \sum_{i=2}^{n} a_i m_i \in D \cap T$, and (1) we may continue

$$\begin{aligned} m + \sum_{i=2}^{n} a_i m_i &= - \sum_{i=1}^{n} (a_i, m_i, e) = - \sum_{i=1}^{n} (e, a_i, m_i) = \\ &= - \sum_{i=1}^{n} (e a_i) m_i + \sum_{i=1}^{n} e(a_i m_i) = - \sum_{i=1}^{n} (e a_i)(m_i e) = \\ &= - e \left(\sum_{i=1}^{n} a_i m_i \right) e = 0, \end{aligned}$$

proving the assertion.

Proposition 20. *Let A be an alternative MPR–ring with Baer radical $\beta(A)$.*

1) If $e \in A$ and $e^2 - e \in \beta(A)$, then there exists an element $v \in A$ such that $v^2 = v$ and $v - e \in \beta(A)$.

2) For any finitely many elements $a_1, \dots, a_n \in A$ there exists an idempotent $e \in A$ such that $a_i e - a_i \in \beta(A), e a_i - a_i \in \beta(A)$, for $i = 1, \dots, n$.

3) If I is a prime ideal of A, then A/I is isomorphic to a direct summand of $A/\beta(A)$.

PROOF. 1) Let B denote the subring of A generated by e. The ring B is clearly associative, and $B \cap \beta(A)$ is a nil–ideal of B. Since $e^2 - e \in \beta(A)$, the element $e^2 - e$ is idempotent modulo $B \cap \beta(A)$. Thus there exists an idempotent $v \in B$ such that $v - e \in B \cap \beta(A)$, and so $v - e \in \beta(A)$.

2) By Theorem 3 $A/\beta(A)$ is a direct sum of associative simple MPR–rings and of Cayley–Dickson algebras, and so for $a_1, \dots, a_n \in A$ there

exists an element e such that $e^2 - e,\ ea_i - a_i,\ a_ie - a_i \in \beta(A)$ for every $i = 1, \dots, n$. Thus by 1) there exists an idempotent $v \in A$ such that $v - e \in \beta(A)$. Clearly

$$va_i - a_i = (v - e)a_i + ea_i - a_i \in \beta(A)$$

holds, and also $a_iv - a_i \in \beta(A)$ is valid for every $i = 1, \dots, n$.

3) The assertion is an immediate consequence of Theorem 3.

In proving the next Proposition we shall make use of the representation theory of alternative algebras as given in [25] Chapter 11. If M and N are right ideals of an alternative ring A, such that $N \subseteq M$, then the canonical action of A on the factor group M/N defines an *alternative right A – module* (cf. [25] Proposition 11.1.4).

Proposition 21. *Let A be an alternative MPR–ring, M and N right ideals of A such that $N \subseteq M$. Let us suppose that for every right ideal L of A with $N \subseteq L \subseteq M$ either $L = N$ or $L = M$. Then*

1) M/N is an indecomposable alternative right A –module,

2) $M\mathcal{J}(A) \subseteq N$ where $\mathcal{J}(A)$ is the quasi–regular radical of A,

3) if $I = \{a \in A \mid Ma \subseteq N\}$, then $I \triangleleft A$ and A/I is either a Cayley–Dickson algebra or a simple associative ring with a minimal right ideal,

4) for every finitely many elements $m_1, \dots, m_n \in M$ there exists an idempotent $e \in A$ such that $m_ie - m_i \in N, i = 1, \dots, n$.

Proof. 1) By [25] Proposition 11.1.4 it follows that the canonical mapping $\rho : A \to \mathrm{End}_Z(M/N)$ is an alternative right representation of the ring A. Hence the assertion follows directly from the definitions (see [25] Chapter 11).

2) The statement follows from [25] Theorems 10.4.5 and 11.3.4.

3) By [25] Corollary on p23 q it follows that $I \triangleleft A$. Moreover, [25] Lemma 11.3.7 yields that A/I is a prime ring. Hence the statement follows from our Proposition 20.3.

4) By [25] Theorem 11.3.2 there exists an element $u \in M$ such that $u + N$ generates M/N and either

$$(ua)b + N = u(ab) + N \tag{3}$$

or

$$(ua)b + N = u(ba) + N \tag{4}$$

for all $a, b \in A$. Hence we conclude $M = uA + N$, and so there exist elements $a_1, \dots, a_n \in A$ such that

$$m_i - ua_i \in N, \qquad i = 1, \dots, n.$$

By Proposition 20.2 there exists an idempotent $e \in A$ such that

$$a_i e - a_i,\ ea_i - a_i \in \beta(A) \qquad i = 1, \dots, n.$$

Hence using the identities (3) and (4) we have

$$m_i e + N = (ua_i)e + N = \begin{Bmatrix} u(a_i e) + N \\ u(ea_i) + N \end{Bmatrix} = ua_i + u\beta(A) + N.$$

Applying statement 2 of this Proposition, it follows

$$m_i e + N = ua_i + N = m_i + N, \qquad i = 1, \dots, n.$$

Thus $m_i e - m_i \in N$ holds for $i = 1, \dots, n$.

Let us recall that the *Loewy series* of a right ideal M of a ring A is defined as

$$\mathcal{L}_0(M) = \text{Soc } M,$$
$$\mathcal{L}_{\alpha+1}(M)/\mathcal{L}_\alpha(M) = \text{Soc } (M/\mathcal{L}_\alpha(M))$$

and

$$\mathcal{L}_\gamma(M) = \bigcup_{\alpha<\gamma} \mathcal{L}_\alpha(M)$$

for limit ordinals γ.

Proposition 22. *Let A be an alternative subdirectly irreducible MPR–ring with heart H. If H is a torsion ring, then so is A as well.*

PROOF. We prove the Proposition in four steps.

1) If $\beta(A) = 0$, then the assertion is just that of [25] Theorem 8.3.12.

2) Let us suppose that $\beta(A) \neq 0$. In this case also $\mathcal{J}(A) \neq 0$. Since the heart H is a torsion ring, there exists a prime p such that $pH = 0$. Consequently also the maximal torsion ideal T of A is a p–torsion ring.

We claim that A has no nonzero torsionfree right ideal. Let $M \neq 0$ be a torsionfree right ideal of A and N be a minimal principal right ideal of A such that $N \subseteq M$. Clearly also $N \cap T = 0$ holds, and by Proposition 14 also $N \subseteq D$ where D is the maximal divisible ideal of A. By Proposition 17 we have $AN + N \triangleleft A$, and so $H \subseteq AN + N$. Applying Proposition 21.4 and 19 for 0 and N, it follows that $(AN + N) \cap T = 0$, contradicting $0 \neq H \subseteq (AN + N) \cap T$. Thus $M \cap T \neq 0$ for every right ideal $M \neq 0$ of A.

3) Let M be a principal right ideal of A minimal with verpect to the properties $M \subseteq D$ and $M \not\subseteq T$. The right ideals $N = M \cap T$ and M satisfy clearly the conditions of Propositions 21, and hence $M\mathcal{J}(A) \subseteq N$ is valid.

We claim that $MJ = 0$ for $J = \mathcal{J}(A)$. We prove this by induction on α of the Loewy series $\mathcal{L}_\alpha(J)$.

By definition $\mathcal{L}_0(J) = \operatorname{Soc} J$. By step 2 we conclude $\operatorname{Soc} J \subseteq T$ and by $M \subseteq D$ Propositions 10 and 11 yield $M\mathcal{L}_0(J) = 0$.

Suppose that $M\mathcal{L}_\gamma(J) = 0$ for every $\gamma < \alpha$. If α is a limit ordinal, then $M\mathcal{L}_\alpha(J) = 0$. Let us consider the case when α is not a limit ordinal, and choose an arbitrary element $a \in \mathcal{L}_\alpha(J) \setminus \mathcal{L}_{\alpha-1}(J)$. Now

$$a + \mathcal{L}_{\alpha-1}(J) \in \operatorname{Soc}(J/\mathcal{L}_{\alpha-1}(J)),$$

and therefore there exist elements $a_0, a_1, \dots, a_n \in J$ such that

$$(a) \qquad a + \mathcal{L}_{\alpha-1}(J) = \sum_{i=0}^{n} a_i + \mathcal{L}_{\alpha-1}(J),$$

(b) $qa_0 \in \mathcal{L}_{\alpha-1}(J)$ for some natural number q,

(c) every element $a_i + \mathcal{L}_{\alpha-1}(J)$ is contained in an indecomposable submodule of $\mathcal{L}_\alpha(J)/\mathcal{L}_{\alpha-1}(J)$.

By Proposition 21.4 there exist idempotents $e_1, \dots, e_n \in A$ such that $a_ie_i - a_i \in \mathcal{L}_{\alpha-1}(J)$, $i = 1, \dots, n$. Now we have

$$Ma \subseteq \sum_{i=0}^{n} Ma_i$$

and by the minimality of M also $qM = M$ holds. Hence by (b) and the hypothesis it follows

$$Ma_0 = (qM)a_0 = M(qa_0) \subseteq M\mathcal{L}_{\alpha-1}(J) = 0.$$

Let $m \in M$ be an arbitrary element. Then for every $i = 1, \dots, n$ we have

$$ma_i = m(a_ie_i) - m(a_ie_i - a_i) = m(a_ie_i),$$

because

$$m(a_ie_i - a_i) \in m\mathcal{L}_{\alpha-1}(J) = 0.$$

Since $a_i \in J$, it follows $m(a_ie_i) \in mJ \subseteq N = M \cap T$. If $b_i = a_ie_i$, then

$$b_ie_i = (a_ie_i)e_i = a_i(e_i^2) = a_ie_i = b_i.$$

Hence from Proposition 18 we conclude that $mb_i = 0$ for every $i = 1, \dots, n$. Thus $Ma_i = 0$ holds for every $i = 1, \dots, n$. This together with $Ma_0 = 0$

imply $Ma = 0$, and so $M\mathcal{L}_\alpha(J) = 0$ is valid. Since A is an MPR–ring, $J = \mathcal{L}_\gamma(J)$ for an ordinal γ and by induction we get $MJ = 0$.

4) Let us consider elements $a_1, \dots, a_n \in A$ and $m_1, \dots, m_n \in M$, and suppose that $\sum_{i=1}^{n} m_i a_i \in T$. By Corollary 2 and Proposition 20.2 there exist an idempotent $e \in A$ such that $a_i e - a_i \in J$ for $i = 1, \dots, n$. Since $MJ = 0$, it follows that $m_i a_i = m_i(a_i e)$ for every $i = 1, \dots, n$. For the element $b_i = a_i e$ clearly

$$b_i e = (a_i e)e = a_i e^2 = a_i e = b_i, \qquad i = 1, \dots, n$$

holds. Hence we have

$$\sum_{i=1}^{n} m_i a_i = \sum_{i=1}^{n} m_i b_i.$$

In view of Proposition 14 for the MPR–ring A Proposition 18 is applicable with $m_1, \dots, m_n \in D$ and $b_1, \dots, b_n \in A$ and $\sum_{i=1}^{n} m_i b_i \in T$. Hence we obtain that $\sum_{i=1}^{n} m_i b_i = 0$, and therefore $MA \cap T = 0$. Since MA is a right ideal of A, by step 2 of this proof we conclude $MA = 0$. Hence by $M \subseteq D$, M_+ has d.c.c. on subgroups and so $M \subseteq T$. This contradicts $M \not\subseteq T$, consequently $D \subseteq T$ and A is a torsion ring, in fact a p–torsion ring.

Theorem 7. *Every alternative MPR–ring splits with respect to the torsion radical.*

PROOF. The statement is an immediate consequence of Theorem 6 and Proposition 22.

Remark. For associative MPR–rings the splitting of the maximal torsion ideal was proved by CHRISTINE AYOUB [1] and DINH VAN HUYNH [7]. In [23] Widiger proved the splitting of alternative artinian rings. Thus Theorem 7 generalizes both of these results.

6. Jordan rings and the torsion radical

In this section we shall consider *Jordan rings,* that is, rings which satisfy the identities

$$xy = yx$$
$$(x^2y)x = x^2(yx)$$

for all elements x and y. The second identity can be expressed also in the form

$$(x^2, y, x) = 0,$$

and by linearization one gets the identity

$$(xy, z, t) + (xt, z, y) + (yt, z, x) = 0 \tag{5}$$

for all elements x, y, z and t (cf. [25] Chapter 14, identity (22)).

Proposition 23. *Let A be a Jordan ring such that $A_+ = B_+ \oplus D_+$ where B_+ is a reduced torsion subgroup and D is the maximal divisible ideal of A. Further, let T denote the maximal torsion ideal of A. If for any finitely many elements $x_1, \dots, x_n \in A/T$ there exists an element $e \in A/T$ such that $x_i e = x_i = x_i e^2$, $i = 1, \dots, n$, then D^2 is torsionfree.*

PROOF. Substituting $x = y = e, z = a$ and $t = b$ into the identity (5), we get

$$(e^2, a, b) = -2(eb, a, e). \tag{6}$$

Let us assume that $\sum_{i=1}^{n} a_i b_i \in T$ for some elements $a_i, b_i \in D$; $i = 1, \dots, n$. Since by Proposition 11.4 it follows that there exists an element $e \in D$ such that

$$a_i e^2 = a_i + r_i$$
$$a_i e = a_i + s_i$$
$$b_i e = b_i + t_i$$

wher $r_i, s_i, t_i \in T$ for $i = 1, \dots, n$. Thus by $TD = DT = 0$ we have

$$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} (a_i + r_i) b_i = \sum_{i=1}^{n} (e^2 a_i) b_i = \sum_{i=1}^{n} (e^2 (a_i b_i) + (e^2, a_i, b_i)) =$$

$$= e^2 \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} (e^2, a_i, b_i) = \sum_{i=1}^{n} (e^2, a_i, b_i).$$

Applying (6) we get

$$\sum_{i=1}^{n} a_i b_i = -2 \sum_{i=1}^{n} (eb_i, a_i, e) = -2 \sum_{i=1}^{n} (((eb_i) a_i) e - (eb)(a_i e)) =$$

$$= -2\sum_{i=1}^{n}(((b_i+t_i)a_i)e-(b_i+t_i)(a_ie)) = -2\sum_{i=1}^{n}((b_ia_i)e-b_i(a_ie)) =$$

$$= -2\left(\sum_{i=1}^{n}a_ib_i\right)e+2\sum_{i=1}^{n}b_i(a_i+s_i) = 2\sum_{i=1}^{n}a_ib_i.$$

Hence

$$\sum_{i=1}^{n}a_ib_i = 2\sum_{i=1}^{n}a_ib_i$$

holds implying

$$\sum_{i=1}^{n}a_ib_i = 0,$$

and consequently D^2 is a torsionfree.

Let $T(A)$ denote the maximal torsion ideal of the Jordan ring A, and let $\mathcal{M}$ be the class of all Jordan rings such that

i) $A_+ = B_+ \oplus D_+$ where B_+ is a reduced torsion subgroup and D_+ the maximal divisible subgroup of A_+,

ii) for any finitely many elements $x_1,\dots,x_n \in A/T(A)$ there exists an element $e \in A/T(A)$ with $x_ie = x_i = x_ie^2$.

Clearly, condition i) is preserved under taking homomorphic images. Let I be any ideal of a ring $A \in \mathcal{M}$. Now we have $(T(A)+I)/I \subseteq T(A/I)$, and therefore there is a homomorphism φ as given below:

$$\begin{array}{ccc} A/T(A) & \xrightarrow{\varphi} & \dfrac{A/I}{T(A/I)} \\ \downarrow & & \uparrow \\ A/(T(A)+I) & \simeq & \dfrac{A/I}{(T(A)+I)/I} \end{array}$$

Let $\overline{x}_1,\dots,\overline{x}_n \in \frac{A/I}{T(A/I)}$ be arbitrary finitely many elements, and $x_1,\dots,x_n \in A/T(A)$ be elements such that $\varphi(x_i) = \overline{x}_i, i = 1,2,\dots,n$. Since $A \in \mathcal{M}$, there exists an element $e \in A/T(A)$ such that $x_ie = x_i = x_ie^2$, and so for $\overline{e} = \varphi(e)$ we have $\overline{x}_i\overline{e} = \overline{x}_i = \overline{x}_i\overline{e}^2$. Thus also the factor ring A/I satisfies condition ii), proving that the class $\mathcal{M}$ is homomorphically closed. Hence Theorem 5 is applicable, and in view of Proposition 23 we arrive at

Corollary 3. *If a Jordan ring A satisfies the requirements of Proposition 23, then the maximal torsion ideal of A splits off.*

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