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A note on stationarity of bilinear models

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1. Introduction

Bilinear stochastic processes in discrete time have been studied intensively by several authors. A good list of references of this field can be found in SUBBA RAO and GABR (1984). One of the most important problem is the stationarity of such processes. A sufficient assumption for the assymptotic stationarity was given by SUBBA RAO and GABR (1981). This assumption proved to be sufficient for the strictly stationarity M.B. RAO, SUBBA RAO and A.M. WALKER (1983) as well as for the second order stationarity Gy. TERDIK (1985).

In this paper we construct the transfer function system for bilinear realizable processes and show that the assumption mentioned above is necessary and sufficient for the second order stationarity.

2. Bilinear realizable processes

We consider a second order stationary stochastic series $y_t, t \in \mathcal{Z} = \{0, \pm 1, \pm 2, \cdots\}$ which is measurable with respect to the σ – algebra $\mathcal{B}_t$ generated by a Gaussian white noise series $v_s, s \leq t, (Ev_s = 0, Ev_s^2 = \sigma^2)$ that is y_t is *physically realizable* . We assume that if T_s is the shift transformation for v_t, i.e., $T_s v_t = v_{t+s}, t, s \in \mathcal{Z}$ then it is also shift transformation for y_t. In that case $y_t, t \in \mathcal{Z}$ is referred as *subordinated* to $v_t, t \in \mathcal{Z}$. The series y_t is called *bilinear realizable* if there exist $m \times m$ matrices A, D and m-dimensional vectors b,c such that y_t is connected to v_t by the following state space equations

$$(2.1) \quad x_t = Ax_{t-1} + Dx_{t-1}v_{t-1} + bv_t + f_0, \quad t = \{0, \pm 1, \pm 2, \cdots\},$$
$$y_t = c^T x_t,$$

where c^T denotes the transpose of c and $f_0 = -\sigma^2 Db$ keeping $Ex_t = 0$. We should mention that the realization (2.1) is not unique at all. It is easy to see that a lower triangular bilinear model y_t i.e.

$$\sum_{m=0}^{P} a_m y_{t-m} = \sum_{m=0}^{Q} b_m v_{t-m} + \sum_{m=0}^{R}\sum_{n=0}^{S} c_{m,m+n} y_{t-m-n} v_{t-m}$$

is bilinear realizable ($a_0 = b_0 = 1$, $c_{0,0} = 0$).

If that lower triangular bilinear y_t is second order stationary and subordinated to v_t then one can put it into the Wiener–Itô expansion, see Terdik and Subba Rao (1987),

$$y_t = \sum_{r=0}^{\infty} \int_{\mathcal{D}^r} g_r(w_{(r)}) e^{it\Sigma w_{(r)}} \mathcal{W}(dw_{(r)}); \quad \mathcal{D}^r = [-\pi, \pi]^r,$$

where the integrals are r–fold Wiener–Itô stochastic integrals with respect to the Gaussian stochastic measure $\mathcal{W}(dw)$, $E\mathcal{W}(dw) = 0$, $E \mid \mathcal{W}(dw) \mid^2 = \frac{\sigma^2 dw}{2\pi}$ of v_t i.e.

$$v_t = \int_{\mathcal{D}} e^{itw} \mathcal{W}(dw).$$

The $w_{(r)}$ denotes the vector $(w_1, w_2, \cdots, w_r)$; $w_k \in [-\pi, \pi]$ and the $\Sigma w_{(r)} = \sum_{k=1}^{r} w_k$. The trasfer functions g_r are given by the following recursive way

$$g_1(w_1) = \frac{\beta(w_1)}{\alpha(w_1)};$$
$$g_r(w_{(r)}) = \frac{\gamma(\Sigma w_{(r-1)}, w_r)}{\alpha(\Sigma w_{(r)})} g_{r-1}(w_{(r-1)}); \quad r \geq 2,$$

where

$$\alpha(w) = \sum_{k=0}^{P} a_m e^{-imw}; \qquad \beta(w) = \sum_{k=0}^{Q} b_k e^{-imw},$$
$$\gamma(w, \lambda) = \sum_{m=0}^{R}\sum_{n=0}^{S} c_{m,m+n} e^{i(m+n)w + m\lambda}.$$

In case $g_2(w_{(2)}) \not\equiv 0$, i.e. y_t is not linear then it has infinite many nonzero transfer functions, say its degree is infinite. The model (2.1) is more general in that sense because it can produce finite degree nonlinear processes as well.

3. Wiener–Itô representation for the bilinear stationary states

Let us suppose that the state vector process x_t in (2.1) is also physically realizable and subordinated to v_t. Moreover all eigenvalues of the matrix A (of linearity) are inside of the unite circle. Then the stationary solution for the equation (2.1) can be given as follows. Under the above assumption x_t has Wiener–Itô representation

$$x_t = \sum_{r=1}^{\infty} \int_{\mathcal{D}^r} e^{it\Sigma w_{(r)}} f_r(w_{(r)}) \mathcal{W}(dw_{(r)}), \quad f_0 = 0,$$

where the vector functions $f_r(w_{(r)})$ are uniquely determined up to permutation of their variables. We must note that this f_0 is not the same as f_0 in (2.1). $f_0 = 0$ follows from $Ex_t = 0$.

The diagram formula (see TERDIK–SUBBA RAO (1987)) and (2.1) give for the product,

$$x_{t-1}v_{t-1} = \sum_{r=2}^{\infty} \int_{\mathcal{D}^r} f_{r-1}(w_{(r-1)}) e^{i(t-1)\Sigma w_{(r)}} \mathcal{W}(dw_{(r)}) + \sigma^2 b.$$

We get now from the uniqueness of the r–dimensional transfer function that

$$\begin{aligned} f_1(w_{(1)}) &= (I - Ae^{-iw_1})^{-1} b, \\ f_r(w_{(r)}) &= (I - Ae^{-i\Sigma w_{(r)}})^{-1} De^{-i\Sigma w_{(r)}} f_{r-1}(w_{(r-1)}); \quad r \geq 2. \end{aligned} \tag{3.1}$$

4. Assumptions for stationarity

Let us consider the degree N polynomial bilinear model y_t, i.e., y_t is bilinear realizable and for the transfer functions

$$\begin{aligned} g_N(w_{(N)}) &\not\equiv 0 \\ g_r(w_{(r)}) &\equiv 0; \quad r > N. \end{aligned}$$

To study the stationarity of this model is extremely simple because it has N strictly proper recognizable regular transfer functions and it is stationary if and only if all the denominators of the transfer functions has no root on the unit circle. In case it is physically realizable then the poles of the transfer function are inside of the unit circle.

The stationarity of the state variables in the model (2.1) needs more attention. In one hand from the stationarity of the state variables x_t clearly follows the stationarity of the model y_t. On the other hand for a N-degree stationary bilinear model y_t there always exists stationary state variables x_t. This follows from the construction of the state space and the linear representation theory.

Let us now consider the general case. The transfer function system for the state varibles x_t is given by (3.1). The question is that under what condition will be all the components of the $Ex_t \otimes x_t$ finite, where $\otimes$ denotes the tensor product. For that purpose let us regard the transfer functions

$$\begin{aligned} f_1(w_{(1)}) &= (I - Ae^{-iw_1})^{-1}b, \\ f_2(w_{(2)}) &= (I - Ae^{-i\Sigma w_{(2)}})^{-1}De^{-i\Sigma w_{(2)}}f_1(w_{(1)}) \\ &= \tilde{f}_2(w_{(2)}) + \hat{f}_2(w_{(2)}), \end{aligned}$$

where

$$\begin{aligned} \tilde{f}_2(w_{(2)}) &= (I - Ae^{-i\Sigma w_{(2)}})^{-1}De^{-i\Sigma w_{(2)}}b, \\ \hat{f}_2(w_{(2)}) &= \sum_{k=1}^{\infty}(I - Ae^{-i\Sigma w_{(2)}})^{-1}De^{-i\Sigma w_{(2)}}A^k e^{-ikw_1}b \end{aligned} \tag{4.1}$$

There is no matter of fact of the convergence as all eigenvalues of A are inside of the unit circle. We get for $k \geq 3$ that

$$f_k(w_{(k)} = \tilde{f}_k(w_{(k)}) + \hat{f}_k(w_{(k)}), \tag{4.2}$$

where

$$\begin{aligned} \tilde{f}_k(w_{(k)}) &= (I - Ae^{-i\Sigma w_{(k)}})^{-1}De^{-i\Sigma w_{(k)}}\tilde{f}_{k-1}(w_{(k-1)}), \\ \hat{f}_k(w_{(k)}) &= (I - Ae^{-i\Sigma w_{(k)}})^{-1}De^{-i\Sigma w_{(k)}}\hat{f}_{k-1}(w_{(k-1)}). \end{aligned}$$

It can be shown that (4.2) is the orthogonal decomposition of $f_k(w_{(k)})$. Let us now consider the expectation of the tensor product of the k^{th} term, for $k \geq 3$

$$E \int_{\mathcal{D}^k} f_k(w_{(k)}) e^{it\Sigma w_{(k)}} \mathcal{W}(dw_{(k)}) \otimes \int_{\mathcal{D}^k} f_k(w_{(k)}) e^{it\Sigma w_{(k)}} \mathcal{W}(dw_{(k)})$$

$$= k! \int_{\mathcal{D}^k} \operatorname{sym} f_k(w_{(k)}) \otimes \operatorname{sym} f_k(-w_{(k)}) \frac{\sigma^{2k}}{(2\pi)^k} dw_{(k)}$$

$$= 2 \int_{\mathcal{D}^k} \tilde{f}_k(\Sigma w_{(2)}, w_{(3,k)}) \otimes \tilde{f}_k(-\Sigma w_{(2)}, -w_{(3,k)}) \frac{\sigma^{2k}}{(2\pi)^k} dw_{(k)}$$

$$+ \int_{\mathcal{D}^k} \hat{f}_k(w_{(k)}) \otimes \hat{f}_k(-w_{(k)}) \frac{\sigma^{2k}}{(2\pi)^k} dw_{(k)}$$

$$= \int_{\mathcal{D}^k} f_k(w_{(k)}) \otimes f_k(-w_{(k)}) \frac{\sigma^{2k}}{(2\pi)^k} dw_{(k)}$$

$$+ \int_{\mathcal{D}^k} \tilde{f}_k(\Sigma w_{(2)}, w_{(3,k)}) \otimes \tilde{f}_k(-\Sigma w_{(2)}, -w_{(3,k)}) \frac{\sigma^{2k}}{(2\pi)^k} dw_{(k)},$$

where $w_{(\ell,k)} = (w_\ell, w_{\ell+1}, \cdots, w_k),\ k \geq \ell$. From (3.1) we get

$$\int_{\mathcal{D}^k} f_k(w_{(k)}) \otimes f_k(-w_{(k)}) \frac{\sigma^{2k}}{(2\pi)^k} dw_{(k)}$$

$$= \int_{\mathcal{D}^k} \sum_{p=0}^{\infty} (A e^{-i\Sigma w_{(k)}})^p D e^{-i\Sigma w_{(k)}} f_{k-1}(w_{(k-1)})$$

$$\otimes \sum_{r=0}^{\infty} (A e^{i\Sigma w_{(k)}})^r D e^{i\Sigma w_{(k)}} f_{k-1}(-w_{(k-1)}) \frac{\sigma^{2k}}{(2\pi)^k} dw_{(k)}$$

$$= \int_{\mathcal{D}^k} \sum_{p=0}^{\infty} (A \otimes A)^p (D \otimes D) f_{k-1}(w_{(k-1)}) \otimes f_{k-1}(-w_{(k-1)}) \frac{\sigma^{2k}}{(2\pi)^k} dw_{(k)}$$

$$= \sigma^2 (I - A \otimes A)^{-1} (D \otimes D) \int_{\mathcal{D}^{k-1}} f_{k-1}(w_{(k-1)})$$

$$\otimes f_{k-1}(-w_{(k-1)}) \frac{\sigma^{2(k-1)}}{(2\pi)^{k-1}} dw_{(k-1)}.$$

In similar way we get

$$\int_{\mathcal{D}^k} \tilde{f}_k(w_{(k)}) \otimes \tilde{f}_k(-w_{(k)}) \frac{\sigma^{2k}}{(2\pi)^k} dw_{(k)} = \sigma^2 (I - A \otimes A)^{-1} (D \otimes D)$$

$$\int_{\mathcal{D}^{k-1}} \tilde{f}_{k-1}(w_{(k-1)}) \otimes \tilde{f}_{k-1}(-w_{(k-1)}) \frac{\sigma^{2(k-1)}}{(2\pi)^{k-1}} dw_{(k-1)}.$$

We used the property $(AB)\otimes(CD) = (A\otimes C)(B\otimes D)$ of the tensor product. From this it follows that $Ex_t \otimes x_t$ is finite if and only if all eigenvalues of the matrix $\sigma^2(I - A\otimes A)^{-1}(D\otimes D)$ are less then 1 in absolute value, i.e., $\varrho(\sigma^2(I - A\otimes A)^{-1}D\otimes D) < 1$ where $\varrho(A)$ is usual notation of the spectral radius of the matrix A.

Theorem. *Let us suppose that the m-dimensional state space variables x_t fulfill the following bilinear state equation*

$$x_t = Ax_{t-1} + Dx_{t-1}v_{t-1} + bv_t + f_0, \quad t = \{0, \pm1, \pm2, \cdots\} \tag{4.3}$$

where the noise process v_t is Gaussian i.i.d., $Ev_t = 0, Ev_t^2 = \sigma^2$, A and D are $m\times m$ matrices, $b \in R^m, f_0 = -\sigma^2 Db$ and all eigenvalues of A are inside the unit circle. Moreover x_t is physically realizable and subordinated to $v_t, t \in \mathcal{Z}$ and the transfer functions $f_r(w_{(r)}), r \in \mathcal{Z}_+$ are different from zero in $L^2[-\pi,\pi]^r$. Then the necessary and sufficient condition for the stationarity of x_t is that all eigenvalues of the matrix

$$\sigma^2(I - A\otimes A)^{-1}(D\otimes D)$$

be less then 1 in modulus. In that case

$$\begin{aligned} Ex_t\otimes x_t = [I - \sigma^2(I - A\otimes A)^{-1}(D\otimes D)]^{-1}\sigma^2(I - A\otimes A)^{-1}) \\ (D\otimes D + I)(b\otimes b) + \sigma^2(I - A\otimes A)^{-1}(b\otimes b). \end{aligned}$$

So we have a necessery and sufficient condition for existence of weakly stationary solution of the bilinear state space equation (2.1). A sufficient condition of the strictly stationarity for the vector valued bilinear process x_t defined by (4.3) was given by Rao M.B. at al. (1983), as the eigenvalues of the matrix $A\otimes A + \sigma^2 D\otimes D$ be inside of the unit circle. The following lemma shows that this condition is equivalent to the condition

$$\varrho(\sigma^2(I - A\otimes A)^{-1}(D\otimes D)) < 1.$$

So we get an important result namely the strictly stationarity of the model (2.1) follows from the weakly stationarity of this model.

Lemma. *Let $A, D \in \mathcal{R}^{m\times m}$ and $\varrho(A) < 1$. Then the following statements are equivalent*

(*i*) $\varrho(A\otimes A + \sigma^2 D\otimes D) < 1$

(*ii*) $\varrho(\sigma^2(I - A\otimes A)^{-1}(D\otimes D)) < 1$.

PROOF. It is enough to put $\sigma = 1$ by replacing σD with D. The proof is broken down into two steps.

1. From (ii) follows (i).
Put

$$P_A(\lambda) = \mathrm{Det}(A - \lambda I),$$

for $A \in \mathcal{R}^{m\times m}$ and λ complex number. If $|\lambda| \geq 1$ then

$$P_{A\otimes A+D\otimes D}(\lambda) = (-1)^m P_{A\otimes A}(\lambda) P_{(\lambda I - A\otimes A)^{-1}D\otimes D}(1).$$

Therefore it is enough to prove that

$$P_{(\lambda I - A\otimes A)^{-1}D\otimes D}(1) \neq 0, \tag{4.4}$$

if $|\lambda| \geq 1$. One can get by the spectral radius formula that

$$\lim_{n\to\infty} \sqrt[n]{\|[(I - A\otimes A)^{-1}D\otimes D]^n\|} = \varrho((I - A\otimes A)^{-1}D\otimes D) < 1,$$

so there exist $N \in \mathcal{N}$ that

$$\|[(I - A\otimes A)^{-1}D\otimes D]^N\| < 1. \tag{4.5}$$

Denote

$$M_N(\lambda) = [(\lambda I - A\otimes A)^{-1}D\otimes D]^N,$$

if $|\lambda| \geq 1$. It is easy to see that

$$M_N(\lambda) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_N=0}^{\infty} \lambda^{-\Sigma k_j - N}(\prod_{j=1}^{N}(A^{k_j}D)) \otimes (\prod_{j=1}^{N}(A^{k_j}D)).$$

For every vector $v \in \mathcal{C}^{m^2}$ there exists a matrix $V \in \mathcal{C}^{m\times m}$ such that $v = \mathrm{Vec}(V)$. By the singular decomposition (see Gantmacher(1958), ch. IX, theorem 9)

$$V = U_1 S U_2^*,$$

where $*$ is the notation of the Hermite-transpose and U_1 and U_2 are orthogonal matrices i.e. $U_i U_i^* = I, i = 1,2,$ and S is diagonal with the singular values of V. Moreover $\|V\|^2 = 1$ if and only if tr $S^2 = 1$. Now, let us consider the following quadratic form

$$\begin{aligned} Q_\lambda(v) =& v^* M_N(\lambda) v = ((U_2\otimes \overline{U}_1)\mathrm{Vec}(S))^T M_N(\lambda)((\overline{U}_2\otimes U_1)\mathrm{Vec}(S)) \\ =& \sum_{\substack{k_l\geq 0\\ l=1,2,\cdots N}} \lambda^{-\Sigma k_j - N} \\ & \sum_{i,j=1}^{m} s_i s_j [(e_i^T U_1^T \prod_{l=1}^{N}(A^{k_l}D)\overline{U}_1 e_j)(e_i^T \overline{U}_2^T \prod_{l=1}^{N}(A^{k_l}D)U_2 e_j)], \end{aligned}$$

where the singular values s_i in S are nonnegative and we used the fact that

$$\mathrm{Vec}(S) = \sum_{i=1}^{n} s_i e_i \otimes e_i,$$

where e_i are the unite vectors in $\mathcal{R}^m$. Since $| ab | \leq \frac{1}{2}(| a |^2 + | b |^2)$ for every $a, b \in \mathcal{C}$ we get

$$| Q_\lambda(v) | \leq \frac{1}{2} \sum_{\substack{k_l \geq 0 \\ l=1,2,\cdots N}} | \lambda |^{-\Sigma k_l - N} \sum_{i,j=1}^{m} s_i s_j [| e_i^T \bar{U}_1^T \prod_{l=1}^{N} (A^{k_l} D) U_1 e_j |^2$$

$$+ | e_i^T \bar{U}_2^T \prod_{l=1}^{N} (A_{k_l} D) U_2 e_j |^2].$$

By $| \lambda |^{-\Sigma k_l - N} < 1$ if $| \lambda | \geq 1$ we get

$$| Q_\lambda(v) | \leq \frac{1}{2}(Q_1(v_1) + Q_1(v_2)),$$

where $v_i = \mathrm{Vec}(U_i S U_i^*), i = 1, 2$, and $\|v_i\|^2 = 1, i = 1, 2$. The assumption (4.5) implies that

$$| Q_1(v) | < 1,$$

if $\|v\|^2 = 1$ so for every $| \lambda | \geq 1$ we get

$$| Q_\lambda(v) | < 1,$$

if $\|v\|^2 = 1$ and so

$$\varrho((\lambda I - A \otimes A)^{-1} D \otimes D) < 1$$

what we got replacing v with eigenvectors of $M_N(\lambda)$ in Q_λ which justifies that

$$P_{(I\lambda - A\otimes A)^{-1} D\otimes D}(1) \neq 0,$$

what we want to prove.
2. From (i) follows (ii).
If $| \lambda | \geq 1$ then

$$P_{(I-A\otimes A)^{-1}(D\otimes D)}(\lambda) = (-\lambda)^m P^{-1}_{A\otimes A}(1) P_{\frac{1}{\lambda} D\otimes D + A\otimes A}(1).$$

Therefore it is enough to prove that

$$P_{\frac{1}{\lambda} D\otimes D + A\otimes A}(1) \neq 0$$

if $| \lambda | \geq 1$. The reader can prove this as (4.4).

References

[1] T. Subba Rao and M. M. Gabr, An introduction to bispectral analysis and bilinear time series models. In lecture notes in statistics, (ed. D. Brillinger et al.), *Springer Verlag, New York.*

[2] T. Subba Rao and M. M. Gabr, The properties of bilinear time series models and their usefulness in forecasting- with examples, *Indian Stat. Ins.* (1981), *Calcutta.*

[3] M. B. Rao and T. Subba Rao and A. M. Walker, On existence of some bilinear time series models, *Jour. Time Series Anal.* **4** (1983), 95–110.

[4] Gy. Terdik, Transfer functions and conditions for stationarity of bilinear models with gaussian residuals, *Proc. R. Soc. London A* **400** (1985), 315–330.

[5] Gy. Terdik and T. Subba Rao, On Wiener Itô representation and the best linear predictors for bilinear time series, *J. Appl. Prob.* **26** (1990), 274–286.

[6] F. R. Gantmacher, Matrizenrechnung Teil I: Allgemeine Theorie, (Hochschulbücher für Math.,Band 36), *VEB Deutscher Verlag der Wissenschaften, Berlin.*

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