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Report on the Third International Symposium on FUNCTIONAL EQUATIONS AND INEQUALITIES

September 21–27, 1986, Noszvaj, Hungary
Compiled by Gyula Maksa

The Third International Symposium on Functional Equations and Inequalities was held in Noszvaj at the Education Center of the Heves County Council from September 21 to September 27. The Symposium was organized by the Mathematical Institute of the L. Kossuth University of Debrecen.

The 82 participants came from Australia, Austria, Brasil, Bulgaria, Federal Republic of Germany, France, German Democratic Republic, Hungary, Italy, Japan, Poland and the United States of America.

The Symposium was opened by Prof. Z. DARÓCZY, who welcomed the participants to Noszvaj.

The scientific talks presented at the Symposium focused on the following subjects: *equations in one and several variables, equations on restricted domains, stability, iteration, functional differential equations, conditional equations, composite equations, interval–filling sequences and functional equations, convexity, inequalities for norms, elementary inequalities for means, quadratic inequalities, integral inequalities, recurrent inequalities.*

Every session was followed by a period devoted to remarks and open problems, these were stimulating and successful. In spite of the very tight schedule, typically twelve talks and two problems and remarks sessions per day, the participants could enjoy an extra talk of Prof. J. DHOMBRES about functional equation meetings from 1350 to 1820 and historical facts concerning functional equations and inequalities. A bus excursion to Eger and the Szalajka Valley was also organized.

The full list of participants follows:

BAINOV D., Bulgaria	MARTELLI G., Italy
BARBANTI L., Brasil	MATKOWSKI J., Poland
Mrs. BARBANTI, Brasil	MILUSHEVA S., Bulgaria
BARON K., Poland	MOSZNER Z., Poland
BEEWA D., Bulgaria	NAGY B., Hungary
BRYDAK D., Poland	NIKODEM K., Poland
CARROLL F., USA	PAGANONI L., Italy
Mrs. CARROLL, USA	PAGANONI M.S., Italy
CHOCZEWSKI B., Poland	PÁLES Zs., Hungary
CHOLEWA P.W., Poland	PATUISKA M., Poland
DARÓCZY Z., Hungary	POWAZKA Z., Poland
DHOMBRES J., France	PRESTIN J., GDR
DÖRFLER P., Austria	REICH L., Austria
EBANKS B., USA	RÉVÉSZ Sz., Hungary
FEHÉR J., Hungary	RIMÁN J., Hungary
FENYŐ I., Hungary	RONKOV A., Bulgaria
FOCHI M., Italy	ROUX D., Italy
FORTI G.L., Italy	RUSCONI D., Italy
FÖRG–ROB W., Austria	SABLIK M., Poland
GAJDA Z., Poland	SCHWAIGER J., Austria
GEORGIEWA A., Bulgaria	SEBESTYÉN Z., Hungary
GESZTELYI E., Hungary	SHIMIZU R., Japan
GRONAU D., Austria	SKOF F., Italy
Mrs. GRONAU, Austria	SKÓRNIK K., Poland
GRZASLEWICZ A., Poland	SMAJDOR A., Poland
JÁRAI A., Hungary	SMAJDOR W., Poland
JARCZYK W., Poland	STEHLING F., FRG
KÁNTOR S., Hungary	SÜMEGI L., Hungary
KOMINEK Z., Poland	SZABÓ Gy., Hungary
KOVACEC A., Austria	SZABÓ T., Hungary
KOVÁCS B., Hungary	SZÁZ Á., Hungary
KRASINSKA S., Poland	SZÉKELYHIDI L., Hungary
KRAUTER A.R., Austria	TABOR J., Poland
LACZKOVICH M., Hungary	TOTIK V., Hungary
LAJKÓ K., Hungary	TÖMÖSVÁRI M., Hungary
LAKATOS Z., Hungary	TURDZA E., Poland
LÁNG D., Hungary	UHRIN B., Hungary
LOSONCZI L., Hungary	VERDES E., Hungary
LOVE E., Australia	VINCZE E., Hungary
Mrs. LOVE, Australia	VOLKMANN P., FRG
MAKSA Gy., Hungary	ZDUN M.C., Poland

The meeting was closed by Prof. L. REICH, who expressed the thanks of the participants to Prof. Z. DARÓCZY and to the organizers.

The abstracts of the talks follow in chronological order of presentation.

JARCZYK, W., **Continuous functions additive on their graphs.**

In the well known book "Some aspects of functional equations" J.G. DHOMBRES mentioned the equation

$$(\star) \qquad f(x+f(x)) = f(x) + f(f(x))$$

which is the equation of additivity of the unknown function on its graph. He found all continuous idempotent solutions mapping the real line $\mathbf{R}$ into itself. In 1983 G.L. FORTI proved that every differentiable at the origin and continuous solution $f : \mathbf{R} \to \mathbf{R}$ of equation $(*)$ is linear, i.e. of the form $f(x) = f'(0)x$. Unfortunately the assumption of differentiability of a solution makes it sometimes impossible to apply this nice result to other problems (e.g. in iteration theory). Recently I have succeeded in finding the general continuous solution of equation $(*)$. Namely we have the following

Theorem. *If $f : \mathbf{R} \to \mathbf{R}$ is a continuous solution of equation $(*)$ then there exist non–negative numbers c_- and c_+ such that*

$$f(x) = \begin{cases} c_- x & x \in (-\infty, 0) \\ c_+ x & x \in [0, \infty), \end{cases}$$

or there exists a negative number c such that

$$f(x) = cx, \qquad x \in \mathbf{R}$$

The proof of Theorem is completely elementary but very long. It makes use of quite fundamental properties of real continuous functions (e.g. Darboux property) and some results concerning the recurrent equation

$$a_{n,k} = a_{n+1,k} + a_{n+1,k+1}.$$

There are simple examples showing that the assumption of the continuity of a solution in the Theorem cannot be replaced by the assumption of measurability, continuity at a point or monotonicity.

CARROLL, F. W., **Algebraic differential equations and functional equations.**

Suppose that S is a rational function with $S(0) = 0$ and $S'(0) = s$, with $0 < |s| < 1$. Let $\varphi(z)$ be a Koenig's solution of Schroeder's equation

$\varphi(S(z)) = s\varphi(z)$. The assumption that φ satisfies an algebraic differential equation (ADE) implies that certain other functional equations

$$(E) \qquad \psi(S(z)) = s^{\ell}(S'(z))^m \psi(z) \quad \text{or}$$
$$(F) \qquad \psi(S(z))(S'(z))^2 = \psi(z)S'(z) + \lambda S''(z), \qquad \lambda \neq 0,$$

are satisfied by rational functions, viz., quotients of coefficients in a minimal ADE. Consideration of the incidence matrix of a directed graph whose vertices are the poles of ψ shows that, except in special cases, (E) and (F) fail to have rational solutions. There will be discussion of connections with the classical work of RITT, as well as with the results of BOSHERNITZAN and RUBEL on coherent families of polynomials.

TURDZA, E., On the stability of the equation $\varphi(f(x)) = g(x)\varphi(x) + F(x)$.

Let the functions g, f, F be defined on a topological space X, and take values on the real line $\mathbf{R}$, a topological space X, and a vector space Y. The equation

$$(1) \qquad \varphi(f(x)) = g(x)\varphi(x) + F(x)$$

is called iteratively stable in the class of continuous functions defined on $X' \subset X$, if there exists a constant $K > 0$ such that for every absolutely continuous neighbourhood U_0 and every continuous solution $\psi : X' \to Y$ of the system of inequalities

$$\psi(f^n(x)) - G_n(x)\psi(x) - F_n(x) \in U_0$$

where

$$G_n(x) := \prod_{i=0}^{n-1} g(f^i(x)), \; F_n(x) = G_n(x) \sum_{i=0}^{n-1} \frac{F(f^i(x))}{G_{i+1}(x)}$$

there is a continuous solution $\varphi : X' \to Y$ of equation (1), such that

$$\psi(x) - \varphi(x) \in KU_0 \qquad \text{for } x \in X'.$$

Some conditions for the stability of equation (1) will be given.

NIKODEM, K., **Continuity of K–convex set–valued functions.**

Let X and Y be arbitrary real topological vector spaces. Assume that D is a convex and open subset of X and K is a cone in Y. A set–valued function $F : D \to 2^Y$ is said to be
— Jensen K–convex (or midpoint K–convex) iff

$$\frac{1}{2}\left[F(x)+F(y)\right] \subset F\left(\frac{x+y}{2}\right)+K$$

for all $x, y \in D$;
— K–continuous at a point $x_0 \in D$ iff for every neighbourhood W of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x) \subset F(x_0)+W+K \quad \text{and} \quad F(x_0) \subset F(x)+W+K$$

for all $x \in (x_0+U) \cap D$;
— K–upper bounded on a set $A \subset D$ iff there exists a (topologically) bounded set $B \subset Y$ such that $\bigcup_{x \in A} F(x) \subset B-K$.

We say that a set–valued function $F : D \subset X \to 2^Y$, where X is a real complete metrizable and separable topological vector space, is Christensen measurable iff for every open set $W \subset Y$ the set $\{x \in D : F(x) \cap W \neq \emptyset\}$ is Christensen measurable. By $B(Y)$ we denote the family of all non–empty and bounded subsets of Y.

The following theorems hold true.

Theorem 1. *If a set–valued function $F : D \to B(Y)$ is Jensen K–convex and K–upper bounded on a subset of D with a nonempty interior, then it is K–continuous on D.*

Theorem 2. *If a set–valued function $F : D \subset X \to B(Y)$, where X is a real complete metrizable and separable topological vector space, is Christensen measurable, then it is K–continuous on D.*

SZÁZ, Á., **Mild continuities of linear relations.**

To extend standard results on continuities of linear functions, we prove the following

Theorem. *If f is a linear relation from a preseminormed space $X(\mathcal{P})$ into another $Y(\mathcal{Q})$, then the following properties are equivalent:*

(i) *f is lower semiperfectly mildly $(\mathcal{R}^*_{\mathcal{P}}, \mathcal{R}_{\mathcal{Q}})$ – continuous;*

(ii) *f is mildly $(\mathcal{R}^*_{\mathcal{P}}, \mathcal{R}_{\mathcal{Q}})$ – continuous;*

(iii) *$R^{\varepsilon}_{q*f} \in \mathcal{R}^*_{\mathcal{P}}$ for all $q \in \mathcal{Q}$ and $\varepsilon > 0$;*

(iv) *$q * f$ is $(\mathcal{R}^*_{\mathcal{P}}, \mathcal{R}_{|\,|})$ – continuous for all $q \in \mathcal{Q}$;*

(v) *$q * f$ is $((\mathcal{R}_{\mathcal{P}})_0, \mathcal{R}_{|\ |})$ – continuous for all $q \in \mathcal{Q}$;*

(vi) *$R^{\varepsilon}_{q*f} \in (\mathcal{R}_{\mathcal{P}})_0$ for all $q \in \mathcal{Q}$ and $\varepsilon > 0$;*

(vii) *f is mildly $((\mathcal{R}_{\mathcal{P}})_0, \mathcal{R}_{\mathcal{Q}})$ – continuous;*

$(viii)$ *$f^{-1}(V) \in \mathcal{T}_{\mathcal{R}_{\mathcal{P}}}$ for all $V \in \mathcal{F}_{\mathcal{R}_{\mathcal{Q}}}$.*

Moreover, if in particular $X(\mathcal{P})$ and $Y(\mathcal{Q})$ are seminormed spaces, then the following property is also equivalent to the former ones:

(ix) *for each $q \in \mathcal{Q}$ there exist $p \in \mathcal{P}$ and $M > 0$ such that $q * f \le Mp$.*

The proof relies mainly upon the fact that $R^{\varepsilon}_{q*f}(x) = f^{-1}(R^{\varepsilon}_{q}(y))$ whenever $y \in f(x)$.

SMAJDOR, W., **Quadratic selections of subquadratic setvalued maps.**

Let $(G, +)$ be an abelian group in which the division by 2 is performable and let Y be a topological vector space. If F from G to Y with compact, convex and non–empty values is subquadratic s.v.f., then there exists a quadratic selection of F.

Let $(G, +)$ be an abelian group and let Y be a Banach space. If F from G to Y with closed, convex and non–empty values is subquadratic and $c := \sup_{x \in G} \operatorname{diam} F(x) < \infty$, then there exists a unique quadratic selection $f : G \to Y$ of F.

SZABÓ, GY., **Remarks on orthogonality spaces.**

We consider a real orthogonality space $(X, \perp)$ in the RÄTZ sense (see [1]) and an abelian group $(Y, +)$. The only known example of X with a non–trivial even orthogonally additive mapping $F : X \to Y$ (i.e. $F(x + y) = F(x) + F(y)$ whenever $x \perp y$) is a real inner product space with the natural orthogonality and $F(x) = \ell(\|x\|^2)$ with some additive $\ell : \mathbf{R} \to Y$ (see [1], [2], [3]). In this note we take some further steps towards the discovery of the general structure of such orthogonality spaces and mappings.

[1] J. RÄTZ, On orthogonally additive mappings, *Aequationes Math.* **28** (1985), 35–49.

[2] J. RÄTZ, On orthogonally additive mappings, II, *To appear in Publicationes Math.*.

[3] Gy. SZABÓ, On mappings, orthogonally additive in the Birkhoff–James sense, *Aequationes Math.* **30** (1986), 93–105.

KOMINEK, Z., **Some properties of decompositions of a group.**

Let $(G,+)$ be a commutative group divisible by two. We study some relations between the sets: $A+A, A'+A', A'+A, A-A, A'-A'$ and $A'-A$, where A' denotes the complement of A and $A \mp B$ denotes the set of all sums (differences) $a \mp b$ with $a \in A$ and $b \in B$. For example we have the following theorems.

Theorem 1. *For every subset A of G one of the following conditions $A+A=G$ or $A'+A'=G$ or $A'+A=G$ holds.*

Theorem2. *If for some subset A of G one has $A-A \neq G$ and $A'-A' \neq G$ then $A-A=A'-A'$ and $A'=A+x$ iff $x \notin A-A$. Moreover, in this case $A+A=A'+A'=A'+A=G$.*

Theorem 3. *If $A-A \neq G$ for some subset A of G then $A'+A'=G$.*

Some examples which illustrate our results and their connections with the celebrated Steinhaus theorem will also be presented.

SMAJDOR, A., **Multi–valued iteration semigroups.**

Let X be a real linear space. A multi–valued function F from X into X has at most one increasing iteration semigroup $\{F^t : t>0\}$ such that the function

$$(t,x) \mapsto F^t(x) \tag{1}$$

from $(0,+\infty)\times X$ into X is positive homogeneous.

Let $X=\mathbf{R}^n$ and let F be a multi–valued function from X into X with compact and convex values. Then F has an increasing iteration semigroup of convex functions from X into X with compact values such that the function (1) is continuous and positive homogeneous if and only if

$$F^2(2x)=2F(x)\supset F(2x)$$

and

$$\bigcap_{n=1} 2^{-n}F^{2^n+1}(2^n x)=F(x)$$

for all $x \in X$.

PRESTIN, J., **Inequalities for trigonometric polynomials in Lipschitz norms.**

We consider Bernstein– and Nikolskiĭ–type inequalities in the following Lipschitz norm

$$\|g\|_{p,r,\beta}=\|g^{(r)}\|_{p,\beta}+\sum_{k=0}^{r}\|g^{(k)}\|_p$$

with

$$\|g\|_{p,\beta} = \sup_{h>0} h^{-\beta} \|g(\cdot + h) - g(\cdot)\|_p$$

and $1 \le p \le \infty$, $0 \le \beta \le 1$, $r \in N_0$.
For trigonometric polynomials of degree less or equal n we prove

$$\|p_n^{(1)}\|_{p,m,\alpha} \le 4n^{1+m+\alpha-r-\beta} \|p_n\|_{p,r,\beta}$$

if $n > 1$ and $1 + m + \alpha \ge r + \beta$.
With these estimates we get equivalence theorems for the best approximation in Lipschitz norms as well as for interpolation processes.

PÁLES, ZS., **Inequalities for sums and differences of powers.**

In the lecture we investigate the inequalities

$$(1) \qquad 1 \le (x^a + y^a)^\alpha (x^b + y^b)^\beta (x^c + y^c)^\gamma (x^d + y^d)^\delta$$

and

$$(2) \qquad 1 \le \left| \frac{x^a - y^a}{a} \right|^\alpha \left| \frac{x^b - y^b}{b} \right|^\beta \left| \frac{x^c - y^c}{c} \right|^\gamma \left| \frac{x^d - y^d}{d} \right|^\delta .$$

We give necessary and sufficient conditions concerning the real parameters $a, b, c, d, \alpha, \beta, \gamma, \delta$ in order that (1) and (2) be valid for all positive values x and y.

LACZKOVICH, M., **Decompositions into periodic functions belonging to a given Banach space of functions.**

If a function $f : \mathbf{R} \to \mathbf{R}$ has a decomposition

$$(1) \qquad f = f_1 + f_2 + \ldots + f_n$$

such that f_i is periodic mod a_i for every $i = 1, \ldots, n$ then it is easy to see that

$$(2) \qquad \Delta_{a_1} \Delta_{a_2} \ldots \Delta_{a_n} f = 0$$

where $\Delta_a f(x) = f(x+a) - f(x)$. A family F of real functions is said to have the decomposition property (d.pr.) if for every $f \in F$ (2) implies that there is a decomposition (1) such that $f_i \in F$ and f_i is periodic mod a_i for every $i = 1, \ldots, n$. We show that some Banach spaces of functions (e.g. L_p for $p > 1$, the space of bounded functions, or L_∞) have the d.pr. (We remark that the family of all real functions or the class of continuous functions do not have the d.pr.) These results are special cases of more general theorems concerning families of functions $f : A \to \mathbf{R}$, where translations are replaced by commuting mappings $T_i : A \to A$.

GAJDA, Z., **On functional equations related to homogeneous linear differential equations.**

We replace all the differential operators occurring in Heaviside's form of the homogeneous linear differential equation by difference operators of the same order. As a result we obtain the following functional equation:

$$(1) \qquad (A_{m_1}\Delta_{h_1}^{n_1}B_{m_1}\dots A_{m_k}\Delta_{h_k}^{n_k}B_{m_k})f(x)=0,$$

where A_{m_i} and B_{m_i} denote operators defined by

$$A_{m_i}f(x):=m_i(x)f(x),\ B_{m_i}f(x):=m_i(x)^{-1}f(x).$$

The following theorem contains our main result concerning equation (1).

Theorem. *Let G be an Abelian group and let X stand for a linear space over a field K of characteristic zero. Suppose that $m_1,\dots,m_k : G\to K\setminus\{0\}$ are pairwise different homomorphisms of G into the multiplicative group of the field K. Then a function $f : G\to X$ satisfies equation (1) if and only if it has the form*

$$f=\sum_{i=1}^{k}m_ip_i,$$

where $p_i : G\to X$ is a generalized polynomial of degree less than n_i $(i=1,\dots,k)$.

SZÉKELYHIDI, L., **On addition theorems.**

In the talk we deal with addition theorems of a general type. A complex valued function f on an Abelian group (or semigroup) is said to have a polynomial addition theorem, if there exist functions $g_1,\dots,g_n$, $h_1,\dots,h_m$ such that the function value of f at $x+y$ can be computed by the values of $g_1,\dots,g_n$ at x, and by the values of $h_1,\dots,h_m$ at y using only "polynomial operations", that is, addition and multiplication. Special cases of polynomial addition theorems lead to classical functional equations. Here we characterize all functions having a polynomial addition theorem.

DÖRFLER, P., **On an inequality of Markov type.**

For any polynomial f with complex coefficients we define

$$\|f\| := \left\{ \int_a^b |f(t)|^2 w(t)dt \right\}^{1/2} ,$$

where $w : (a, b) \to \mathbf{R}$ is a positive and integrable function with all moments finite. It is well–known that there exists a constant γ_n, not depending on f, such that $\|f'\| \leq \gamma_n \|f\|$ for all f; deg $f \leq n$. In my talk I consider the analogous inequality for derivatives of higher order and compute the best possible γ_n. This constant turns out to be the largest singular value of a certain matrix. Some examples are given.

LOSONCZI, L., **On some quadratic inequalities.**

Inequalities of the form

$$\alpha \sum_{j=0}^{n} |x_j|^2 \leq \sum |x_j \pm x_{j+k}|^2 \leq \beta \sum_{j=0}^{n} |x_j|^2 \tag{1}$$

are studied where $x_0, \dots, x_n$ are real or complex variables, α, β are constants, $1 \leq k \leq n$, the summation in the middle can be understood in four different ways:

(i) $\quad \sum_{j=0}^{n-k}$

(ii) $\quad \sum_{j=0}^{n}$ with $x_{n+1} = \dots = x_{n+k} = 0$,

(iii) $\quad \sum_{j=-k}^{n-k}$ with $x_{-1} = \dots = x_{-k} = 0$,

(iv) $\quad \sum_{j=-k}^{n}$ with $x_{n+1} = \dots = x_{n+k} = 0 = x_{-1} = \dots = x_{-k}$

and either the plus or the minus sign is taken. Since the cases (ii) and (iii) are essentially the same we obtain six inequalities from (1). The exact constants α, β are found in all cases. This is based on the determination of the eigenvalues of suitable Hermitian matrices.

GESZTELYI, E., **On the capacity of the memory of processors.**

With the development of computer technics, many new concepts came to light and became generally known, such as Read Only Memory, Random Acces Memory, Microprocessors, Interface, etc. The question about the function and role of these devices arises naturally. If it is not desirable to go into details of the electric circuits of semiconductor chips then the correct answer may only be given in the framework of a mathematical theory.

I do not wish to deal here with the theory of processors. I want to speak only about a part of this theory which may be interesting for specialists of the theory of functional equations.

Let S be a nonvoid set. By a memory $\mathcal{M}$ over S we mean a triple $\mathcal{M} = (S, C, s)$ where C is a set of partitions of S and $s \in S$ is a variable on S. S is called the set of states of $\mathcal{M}$. C is called the core of $\mathcal{M}$, a value of s is called the instant state of $\mathcal{M}$. A partition $P \in C$ will be called a memory cell of $\mathcal{M}$. If C and every $P \in C$ are finite then $\mathcal{M}$ is said to be finite.

One can describe partitions of S in a convenient way by means of equivalence relations defined on S. For any function $f : S \to H$ we define a binary relation (f) as follows: $x\,(f)\,y \iff f(x) = f(y)$. Clearly, (f) is an equivalence relation on S. (f) will be called the "identity according to f". It is an interesting fact that any equivalence relation is the identity according to a suitable function $f : S \to H$. Thus any partition of S may be described by means of functions defined on S. If a partition P is the classification generated by the identity according to f then we say that $f : S \to H$ is a generator function of P. If for the element $h \in H$ and the instant state s the relation $h = f(s)$ holds then we say that h is stored in the memory cell P at the instant state s.

Let $\mathcal{M} = (S, \{P_1, \dots, P_n\}, s)$ be a finite memory and $f_1, \dots, f_n$ be the generator functions of the memory cells $P_1, \dots, P_n$, respectively ($f_i : S \to H_i$, $i \in \{1, \dots, n\}$). The memory $\mathcal{M}_0 = \{S, \{P_0\}, s\}$ is said to be the vector form of $\mathcal{M}$ if P_0 is the partition of S generated by the vector–valued function $F_0 : S \to H_1 \times \ldots \times H_n$ for which

$$(\star) \qquad F_0(s) = [\, f_1(s), \dots, f_n(s)\,].$$

P_0 is called the vector cell of $\mathcal{M}$.

The capacity of a finite memory $\mathcal{M} = (S, C, s)$ is defined as the capacity of the core C. The capacity of the core is defined in two steps:

(1) By the capacity of a memory cell P we mean: $\mathrm{cap}(P) = \log_2 |P|$

(2) By the capacity of the core $C = \{P_1, \dots, P_n\}$ we mean the capacity of the vector cell P_0 generated by the function $(\star)$.

Theorem. *For every finite memory* $(S, \{P_1, \dots, P_n\}, s)$ *we have*

$$\mathrm{cap}\{P_1, \dots, P_n\} \le \mathrm{cap}(P_1) + \ldots + \mathrm{cap}(P_n)$$

where the equality relation holds exactly in case of an independent memory ($\mathcal{M}$ is independent if the content may be changed in any memory cell without disturbing the contents of other cells in $\mathcal{M}$).

Remark. It would be interesting to give a characterization of the capacity memory like the characterizations of the measure of information given in J. ACZÉL, Z. DARÓCZY : Measures of information and their characterizations, *Academic Press* 1975. New York, London, Toronto.

The whole theory of processors is contained in the paper E. GESZTELYI: On a mathematical theory of processors, *to appear.*

PAGANONI, M.S., **Report of a joint paper with L. PAGANONI.**

We consider the functional equation

$$\varphi(x+y)-\varphi(x)-\varphi(y)=f(x)f(y)h(x+y)$$

and we find all its holomorphic solutions f, h, φ defined in a neighbourhood of the origin.

FOCHI, M., **Functional equations on orthogonal vectors.**

Starting from some recent results on the orthogonally additive functions i.e. on the solutions of the conditioned Cauchy functional equation

$$f(x+y)=f(x)+f(y) \qquad x \perp y \tag{1}$$

we investigate, in the class of real functions on an inner product space H, other functional equations, related to Cauchy's one, postulating for all pairs of orthogonal vectors:

$$f(x+y)=f(x)+f(y) \tag{2}$$

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{3}$$

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y) \tag{4}$$

$$f(x+\alpha y)+f(\alpha x-y)=f(x)+f(y)+f(\alpha x)+f(\alpha y) \tag{5}$$

In particular we study for each of these the relationships between the class of the solutions of the equation postulated on the whole space and that of the solutions of the conditioned one.

We also treat the same type of problems related to functional equations on the A-orthogonal vectors for the class of functions defined on a Hilbert space X, A being a selfadjoint operator on X.

SKOF, F., **On approximately quadratic functions on a restricted domain.**

1. Suppose $f : D_f \subset \mathbf{R} \to X$, X being a Banach space, and

$$(1) \quad \|f(x+y)+f(x-y)-2f(x)-2f(y)\| < \delta \quad \text{for } (x,y) \in E \subset \mathbf{R}^2,$$

where E is a proper subset of $\mathbf{R}^2$ and δ is a positive real number.

If E is a (bounded) neighbourhood of the origin, we prove that f can be uniformly approached, near the origin, by a quadratic function.

Then we consider unbounded subsets $E \subset \mathbf{R}^2$ having the property $E_x \cup E_y \cup E_{x+y} \cup E_{x-y} = \mathbf{R}$, and $f : D_f = \mathbf{R} \to X$. We show some classes of E such that there exists some function f for which the condition (1) holds on E but does not hold on $\mathbf{R}^2$. On the other hand we prove that (1) is valid on $\mathbf{R}^2$ for every f satisfying (1) on E, whenever E is the complement of a bounded set in $\mathbf{R}^2$.

2. It is natural to relate (1) with the non–homogeneous quadratic equation

$$(2) \quad f(x+y)+f(x-y)-2f(x)-2f(y) = g(x,y) \quad \text{for } (x,y) \in E \subset \mathbf{R}^2,$$

E being a restricted domain.

By some slight changes in a procedure introduced by I. FENYŐ and G.L. FORTI (1981) to solve the non–homogeneous Cauchy equation, we can give the explicit solution of (2), assuming g in a suitable class, when E is a neighbourhood of the origin. The same result is valid if $E = \mathbf{R}^2$.

3. In the end, we suggest a possible definition of "quadratic quasi–extension" of a given function (in connection with DARÓCZY's and LOSONCZI's definition, 1967), and we state an existence and uniqueness theorem of quasi–extensions.

CHOLEWA, P.W., **Almost approximately trigonometric functions.**

Let $(G,+)$ be an Abelian group and let f_1 and f_2 be two conjugate, linearly invariant, proper σ–ideals. Assume that $f : G \to \mathbf{C}$ ($\mathbf{C}$ denotes the set of all complex numbers) and there exist a $\delta \geq 0$ and $M \in f_2$ such that the inequality

$$|f(x+y)+f(x-y)-2f(x)\cdot f(y)| \leq \delta$$

holds true for all $(x,y) \in G^2 \setminus M$. Moreover, let $\sup\limits_{x \in G} \text{ess}|f(x)| = \infty$. Then there exists a function $c : G \to \mathbf{C}$ such that

$$c(x+y)+c(x-y) = 2c(x)c(y) \qquad \text{for all } x,y \in G$$

and

$$\{x \in G : c(x) \neq f(x)\} \in f_1.$$

Under the additional assumption that the group G is uniquely 2–divisible the following is true.

Let $g : G \to \mathbf{C}$ be such function that the inequality

$$|g(x+y)g(x-y) - g^2(x) - g^2(y)| \leq \delta$$

holds true for some $\delta \geq 0$ and all $(x,y) \in G^2 - \mathbf{N}$, where $N \in f_2$. Let $\sup\limits_{x \in G} \text{ess}|g(x)| = \infty$. Then there exists a function $s : G \to \mathbf{C}$ such that

$$s(x+y)s(x-y) = s^2(x) - s^2(y) \qquad \text{for all } x, y \in G$$

and

$$\{x \in G : s(x) \neq g(x)\} \in f_1.$$

MILUSHEVA, S. D., BAINOV, D. D., **Foundations of the average method for a class of non–linear integro–differential equations with impulses.**

In this work we give a foundation for the average method for non–linear integro–differential equations of the kind

$$\dot{x}(t) = \varepsilon X(t, x(t), \int\limits_{t-\omega}^{t} \psi(t, s, x(s))ds), \qquad t > 0$$

$$x(t) = \varphi(t, \varepsilon), \quad -\omega \leq t \leq 0$$

with the impuls influence. Here $x \in \mathbf{R}^n$, $\omega = \text{const}$, and $\varepsilon > 0$ is a small parameter.

REICH, L., **Holomorphic integrals of Jabotinsky's differential equations.**

In iteration theory the differential equation (introduced by E. JABOTINSKY)

$$\text{(J)} \qquad G(F(x)) = F'(x)g(x)$$

plays a certain role. Here $g(x)$ is a power series $g(x) = d_m x^m + \dots$, $m \geq 1$, $d_m \neq 0$, whereas a solution $F(x)$ is a power series $F(x) = \varrho x + c_2 x^2 + \dots$. We will study the structure of the set of all formal solutions, and show that, if $g(x)$ is convergent, every formal solution too is convergent. Furthermore the dependence on a certain parameter will be investigated. Eventually, we will consider (J) as an equation for the pair (F, G). ("Jabotinsky–correspondences").

GRONAU, D., **Some differential equations connected to iteration theory.**
(Joint work with J. ACZÉL, Waterloo/Canada).

In connection with the translation equation

$$F(F(x,s),t) = F(x,s+t) \tag{T}$$

three differential equations

$$\frac{\partial F(x,t)}{\partial t} = \frac{\partial F(x,t)}{\partial x} \cdot G(x) \tag{1}$$

$$\frac{\partial F(x,t)}{\partial t} = G(F(x,t)) \tag{2}$$

$$\frac{\partial F(x,t)}{\partial x} \cdot G(x) = G(F(x,t)) \tag{3}$$

arise together with a functional boundary condition

$$G(x) \left. \frac{\partial F(x,t)}{\partial t} \right|_{t=0} . \tag{4}$$

They are satisfied by the differentiable solutions of (T) and the initial condition

$$F(x,0) = x \tag{I}$$

These equations are attributed in the literature (Targonski) to E. JABOTINSKY who seems to have been the first who treated these equations in connection with the theory of analytic iteration.

GRONAU asked, whether the converse is true that all solutions of each of these "Jabotinsky differential equations", possibly with some further initial and/or boundary conditions added, are also solutions of the translation equation. In this paper we give counterexamples but also partial positive answers to these questions.

For Banach space valued functions we show that the initial condition (I) implies that every solution of (1) or (2) is also a solution of the translation equation (T) and (4) holds, too, supposed that the Cauchy problem for (1) or (2), respectively, has a unique solution.

Equation (3) has not such a close relation to the translation equation (T) as it will be shown in several propositions and examples which yield a general representation of the solutions of (3) in the real one dimensional case. The examples show that in general the solutions of (3) are not solutions of (T) and they are in general not even solutions of the weaker functional equation

$$F(F(x,t),s) = F(F(x,s),t) \tag{C}$$

which describes a commutativity condition of F.

SABLIK, M., **On a conditional translation equation.**

Let $F : [0, +\infty) \times [a, b] \to [a, b]$, where $[a, b]$ is an interval of reals, be a function continuous in the second variable and such that $F(0, \cdot) = \mathrm{id}$. We are interested in the following question : does F satisfying

$$(1) \qquad F(k(t) + \ell(t), x) = F(k(t), F(\ell(t), x)), \quad t \geq 0, \ x \in [a, b],$$

with some functions $k, \ell : [0, +\infty) \to [0, +\infty)$, constitute an iteration semigroup, i.e. does (1) imply the translation equation

$$(\mathrm{T}) \qquad F(t + s, x) = F(t, F(s, x)), \quad s, t \geq 0, \ x \in [a, b] \ ?$$

We give some sufficient conditions guaranteeing a positive answer to this problem. An important role in our approach is played by Jabotinsky's equation

$$(\mathrm{J}) \qquad G(x) \cdot \frac{\partial F}{\partial x}(t, x) = G(F(t, x)),$$

with $G : (a, b) \to \mathbf{R}$ given by $G(x) = \dfrac{\partial F}{\partial t}(0, x)$.

RONKOV, A., **Integral equations and inequalities of Volterra type for functions defined in partially ordered spaces.**

Linear integral operators of Volterra type acting in $L_2(T, B)$ where T is a partially ordered connected topological space and B is a Banach space, as well as the corresponding equations and inequalities are considered.

FORTI, G.L., **Some consequences of the stability of the Cauchy equation and their application for solving some alternative equation.**

Let G be a group and B a Banach space and assume that the couple (G, B) has the property of stability of homomorphisms, that is for every function $f : G \to B$ such that $\|f(xy) - f(x) - f(y)\| \leq K$ for all x, y in G and for some K, there exists $\varphi \in \mathrm{Hom}(G, B)$ such that $\|f(x) - \varphi(x)\| \leq K$ for all $x \in G$. In this hypothesis we prove some propositions connecting the set of values assumed by the function $h = f - \varphi$ and those assumed by the difference $f(xy) - f(x) - f(y)$.

This enables us to solve equations of the form $f(xy) - f(x) - f(y) \in V$ where V is a given finite set.

VINCZE, E., **Über eine Verallgemeinerung des Funktionalgleichungssystems der Wirtschaftlichkeit.**

In dem Vortrag wird das folgende System von Funktionalgleichungen vollständig gelöst:

$$\begin{aligned} G(sx,sy,sz) &= G(x,y,z), \\ G(x,sy,sz) &= H_1[G(x,y,z),s], \\ G(sx,y,sz) &= H_2[G(x,y,z),s], \\ G(sx,sy,z) &= H_3[G(x,y,z),s], \end{aligned}$$

wobei die Funktionen

$$\begin{aligned} &G : \mathbf{R}_+^3 \to R_{ab} := (a,b) \subset \mathbf{R}_+ := (0,+\infty); \\ &H_1, H_2, H_3 : R_{ab} \times \mathbf{R}_+ \to R_{ab} \end{aligned}$$

in allen Veränderlichen stetig und streng monoton sind. Das obige Gleichungssystem ist eine gemeinsame Verallgemeinerung von mehreren Systemen, die in den verschiedenen Wirtschaftlichkeitsmodellen eine grundlegende Rolle spielen.

BARON, K., **On a problem of R. SCHILLING.**

The problem concerns the functional equation

$$f(qx) = \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)] \tag{1}$$

and its solutions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(x) = 0 \quad \text{for } |x| > \frac{q}{1-q}, \tag{2}$$

where q is a fixed number from the open interval $(0,1)$. It comes from physics and has been posed by R. SCHILLING (a personal communication).

Results on solutions of equation (1) fulfilling condition (2) in different classes of functions will be presented.

FÖRG-ROB, W., **On the problem of R. SCHILLING.**

The problem concerns the functional equation

$$f(qx) = \frac{1}{4q}(f(x-1) + f(x+1) + 2f(x)) \tag{x}$$

where q is a fixed number from the interval $(0,1)$.

Some results have been found until now (SCHILLING himself, K. BARON) in the case $q \in \left(0, \frac{1}{2}\right]$. In this talk I treat the case $q = \frac{1}{3}$. A complete description of the solution under the condition

$$\text{(xx)} \qquad f(x) = 0 \quad \text{for } |x| > \frac{q}{1-q}$$

is given for $q = \frac{1}{3}$.

Furthermore, the following theorem holds:

Theorem. *Let $q = \frac{1}{3}$. Then any solution of* (x), *which is measurable on an interval of positive lenght contained in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and which fulfills* (xx), *is equal to 0* a.e.

GRZAŚLEWICZ, A., **On some solutions of the functional equation $F(x,y) \cdot F(y,z) = F(x,z)$.**

Assume $\leq$ is a linear order in the set B, $\emptyset \neq A \subset B$ and $(M, \cdot)$ is a semigroup such that $(M \setminus \{0\}, \cdot)$ is a group. We shall write $\Delta < y$ if $\Delta \subset B$, $y \in B$ and $x < y$ for all $x \in \Delta$. Let us put

$$R := \{(x,y) \in A \times B : x \leq y\}, \; R_A := \{(x,y) \in A \times A : x \leq y\}.$$

We shall consider the functional equation

$$(1) \qquad F(x,y) \cdot F(y,z) = F(x,z),$$

where F maps $A \times B$ or R into M.

Theorem 1. *A function $F : A \times B \to M$ satisfies (1) iff $F \equiv 0$ or there exists a function $f : B \to M$ such that $f(A) \subset M \setminus \{0\}$ and $F(x,y) = [f(x)]^{-1} f(y)$ for $x \in A$, $y \in B$.*

Theorem 2. *If there exists an element $a \in A$ such that $a < B \setminus A$ then a function $F : R \to M$ ($F \not\equiv 0$) is a solution of (1) such that $F(R_A) \subset M \setminus \{0\}$ iff there exists a function $f : B \to M$ such that $f(A) \subset M \setminus \{0\}$ and $F(x,y) = [f(x)]^{-1} f(y)$ for $(x,y) \in R$.*

Theorem 3. *If A is an interval in B then $F : R \to M$ is a solution of (1) iff there exists a family U of disjoint intervals of the set A such that for every $\Delta \in U$ there exist functions $f_\Delta : \Delta \to M \setminus \{0\}$ and $k_\Delta : \{y \in B : \Delta < y\} \to M$ such that*

(a) $k_\Delta(y) = 0$ *if* $\Delta < y$
and ($y \in A \setminus \Delta$ *or* $\Delta < x < y$ *for some* $x \in A \setminus \Delta$),

$$(b) \quad F(x,y) = \begin{cases} 0 & \text{if } x \in A \setminus \bigcup_{\Delta \in U} \Delta, x \le y, y \in B, \\ [f_\Delta(x)]^{-1} f_\Delta(y) & \text{if } x, y \in \Delta, x \le y, \\ [f_\Delta(x)]^{-1} k_\Delta(y) & \text{if } x \in \Delta, y \in \{y \in B : \Delta < y\}. \end{cases}$$

Theorems 1 and 2 are results of ANGELO GRZASLEWITZ and Theorem 3 is our common result.

MOSZNER, Z., **Sur la commutativité des groupes un–paramétriques des transformations affines.**

On considère le problème suivant est-ce-qu'il est vrai qu'on a l'implication suivant

$$(\star) \qquad \bigwedge_{t \in \mathbf{R}} f(t) \cdot g(t) = g(t) f(t) \Longrightarrow \bigwedge_{r,s \in \mathbf{R}} f(r) \cdot g(s) = g(s) \cdot f(r)$$

pour deux homomorphismes f et g de $(\mathbf{R}, +)$ au groupe $(A, \cdot)$ des transformations affines de $\mathbf{C}^n$, avec la superposition comme l'operation " $\cdot$ " .

On sait (les résultats avec M^me^ Z. LESZCZYŃSKA [2] et [3]) que

1) pour $n = 2, 3$ l'implication $(\star)$ a lieu pour les transformations centro–affines,
2) cet implication n' a pas lieu pour les transformationes affines déjà pour $n \ge 3$ et pour les transformationes centro–affines pour $n \ge 4$.

On peut aussi indiquer pour quelles transformationes affines cet implication $(\star)$ a lieu pour $n = 3$.

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LOVE, E. R., **Hardy's inequality with Orlicz–Luxemburg norms.**

The original Hardy's inequality may be written

$$\|Ax\| \le C\|x\|$$

where $x = (x_m)$ is a column sequence, $\|x\| = \left(\sum_{m=1}^{\infty} |x_m|^p\right)^{1/p}$, $p > 1$, $A = (a_{mn})$ is the Cesàro matrix ($a_{mn} = 1/m$ if $0 < n \le m$, $a_{mn} = 0$

otherwise) and C is independent of x. I wish to generalize this, replacing A by a much more general matrix operator and $\|\cdot\|$ by an Orlicz–type norm which is essentially due to Luxemburg.

The Orlicz–Luxemburg norm of a complex–valued sequence $x = (x_m)$ is

$$\|x\| = \inf\left\{k > 0 : \sum_{n=1}^{\infty} \lambda_n \, \Phi\left(\frac{x_n}{k}\right) \leq 1\right\},$$

where $\lambda_n > 0$ and Φ are fixed, Φ being an Orlicz function. An example is $\Phi(t) = t^p$ with $p > 1$; then $\|x\| = \left(\sum_{n=1}^{\infty} \lambda_n |x_n|^p\right)^{1/p}$, a weighted ℓ^p –norm with arbitrary weighs λ_n.

Theorem. *Let Φ be Orlicz and supermultiplicative. Let $\Lambda_m = \sum_{n=1}^{m} \lambda_n$, where $\lambda_n > 0$. Let $\alpha(t)$ be non–negative and measurable, and have a decreasing rearrangement $\overline{\alpha}(t)$, all on $(0, \infty)$. If $A = (a_{mn})$ satisfies*

$$|a_{mn}| \leq \int_{\Lambda_{n-1}/\Lambda_m}^{\Lambda_n/\Lambda_m} \alpha(t)dt, \quad \text{and} \quad C = \int_0^{\infty} \Phi^{-1}(t^{-1})\overline{\alpha}(t)dt.$$

then $\|Ax\| \leq C\|x\|$ for all x.

Example. Let $\Phi(t) = t^p$ with $p > 1$. Let

$$\begin{aligned} &\alpha(t) = 1 && \text{if } 0 < t \leq 1, \quad \alpha(t) = 0 \text{ otherwise}, \\ &a_{mn} = \Lambda_n/\Lambda_m && \text{if } 0 < n \leq m, \quad a_{mn} = 0 \text{ otherwise}. \end{aligned}$$

Then the mth element of the column Ax is

$$\sigma_m = \frac{\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m}{\lambda_1 + \lambda_2 + \ldots + \lambda_m},$$

and the conclusion of the theorem is that

$$\left(\sum_{n=1}^{\infty} \lambda_n |\sigma_n|^p\right)^{1/p} \leq \frac{p}{p-1}\left(\sum_{n=1}^{\infty} \lambda_n |x_n|^p\right)^{1/p}.$$

The case of this in which $\lambda_n = 1$ is the original Hardy's inequality.

UHRIN, B., **Some new results around integral inequalities of Henstock-Macbeath-Dinghas–type.**

Notations. $f, g : \mathbf{R}^n \to \mathbf{R}^1_+$ L–measurable functions; μ L–measure; $m_0(f) = \operatorname{ess\,sup}_x f(x)$; $s \subset \mathbf{R}^n$ k–dimensional subspace, T its orthogonal complement, $0 < k < n$;
$i(f,u) = \int_S f(x+u)dx$, $m_k(f) = \operatorname{ess\,sup}_{u\in T} i(f,u)$; for $a, b \geq 0$,
$-\infty \leq \alpha \leq +\infty, 0 \leq \lambda \leq 1, M_\alpha(a,b) = (\lambda a^\alpha + (1-\lambda)b^\alpha)^{1/\alpha}$ if $a, b > 0$ and $M_\alpha(a,b) = 0$ if $a \cdot b = 0$;
$h_\alpha(t) = \operatorname{ess\,sup}_x M_\alpha(f(x/\lambda), g((t-x)/(1-\lambda)))$,
$k_\alpha(\tau) = \operatorname{ess\,sup}_{u\in T} M_\alpha(i(f,u/\lambda)/m_k(f), i(g,(\tau-u)/(1-\lambda))/m_k(g))$;
$\ell(f,\xi) = \{x \in \mathbf{R}^n : f(x) \geq m_0(f)\cdot\xi\}$, $0 \leq \xi \leq 1$;
$\lambda A \boxplus (1-\lambda)B = \{x \in \mathbf{R}^n : \mu(\lambda A \cap (x - (1-\lambda)B)) > 0\}$.
In the talk we present two reduction theorems to gain sharp lower bounds for the integral of $h_\alpha(t)$. The first one reduces the question to lower estimations for $\mu(\lambda A \boxplus (1-\lambda)B)$, and the second one to lower estimations for the same type integral but in smaller dimensions.

(1)
$$\int_{\mathbf{R}^n} \operatorname*{ess\,sup}_{x\in\mathbf{R}^n} \min\{f(x/\lambda)/m_0(f), g((t-x)/(1-\lambda))/m_0(g)\}dt \geq$$
$$\geq \int_0^1 \mu\big(\lambda\ell(f,\xi) \boxplus (1-\lambda)\ell(g,\xi)\big)d\xi,$$

(2) if $0 < k < n$ and $\alpha + \beta \geq 0$, $\alpha\beta/(\alpha+\beta) \geq -1/k$ then

$$\int_{\mathbf{R}^n} h_\alpha(t)dt \geq M_{-\beta}(m_k(f), m_k(g)) \cdot \int_T k_\gamma(\tau)d\tau,$$

where $\gamma = (1/\alpha + 1/\beta + k)^{-1}$.
These inequalities sharpen all known inequalities of Henstock-Macbeath-Dinghas–type. They have applications in many fields of mathematics. (The results (1), (2) are further developments of those in [1], [2].)

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ROUX, D., **L_1 approximation by smoothed Fourier polynomials.**

Let $\{\alpha_n(\sigma)\}_{n\in\mathbf{Z}^N}$; $\sigma > 0$ be a sequence of multipliers of $L^1(T^N)$ $(N \geq 1)$ such that $\lim_{\sigma\to 0+} \alpha_n(\sigma) = 1$ for all $n \in \mathbf{Z}^N$. For every $f \in L^1(T^N)$ and for every integer $m \geq 0$ let us set

$$\left.\begin{aligned} P_{m,\sigma}(t) &= \sum_{|n|\leq m} \alpha_n(\sigma)\widehat{f}(n)e^{2\pi i n t} \\ P_{m,o}(t) &= \sum_{|n|\leq m} \widehat{f}(n)e^{2\pi i n t} \end{aligned}\right. \qquad t \in T^N.$$

We study the functional inequality

$$\|f - P_{m,\sigma}\|_1 \leq \|f - P_{m,o}\|_1.$$

The problem is connected with approximation theory.

EBANKS, B., **A functional equation connected with homogeneous biadditive forms and information measures.**

In the course of obtaining his results on functionally homogeneous biadditive forms, C. T. NG [1] solved the functional equation (system)

$$\text{(FE)} \qquad F(x) + M(x)G(1/x) = 0 \qquad (x \neq 0)$$

for additive F, G and multiplicative M on fields. In order to solve a certain characterization problem for information measures [2], it was necessary to solve a special case of (FE) with $G = F$ on the positive cone of $\mathbf{R}^n$ [3]. Now the general solution of (FE) on the positive cone of $\mathbf{R}^n$ is known.

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FENYŐ, I. S., **Connection between an integrodifferential equation and a functional equation.**

Let us denote by D a subset of reals (or complex numbers) with the property that $s \in D$ implies $1/s \in D$. We consider the following functional equation:

$$\text{(I)} \qquad F(s) + (\beta/s^p)F\left(\frac{1}{s}\right) = H(s)$$

where $H(s)$ is an arbitrary function defined on D, F is the unknown and p is an arbitrary number. The following theorem of alternative holds:

Theorem A. 1°. *If $\beta^2 \neq 1$, then (I) has exactly one solution which is given explicitely.*

2°. *If $\beta^2 = 1$, then (I) has solution iff H fulfills the following condition:*

$$H(s) = (\beta/s^p)H(1/s) \qquad (s \in D)$$

and the most general solution is given explicitely.

From this theorem we derive the solvability of the following integro-differential equation:

$$\text{(II)} \qquad f(t) + \beta \int_0^\infty \mathcal{J}_n(2\sqrt{tx})(t/x)^{n/2} f^{(k)}(x)dx = h(t).$$

Here h is a given function defined on $\mathbf{R}_+$, f is the unknown, $\mathcal{J}_n$ is the Bessel–function of first kind ($n > -1$) and k is a nonnegative integer. We look for the solutions of (II) in a suitable function space in which also h is lying.

Theorem B. (i) *If $\beta^2 \neq 1$ then* (II) *has exactly one solution in the considered function space which is given explicitely.*

(ii) *If $\beta^2 = 1$ then* (II) *has solutions iff h is a solution of the following equation :*

$$h(t) = \beta \int_0^\infty \mathcal{J}_n(2\sqrt{tx})(t/x)^{n/2} h^{(k)}(x)dx.$$

Also in this case the general solution is given explicitely.

PAGANONI, L., **On a functional equation concerning affine transformations.**

We consider the following functional equation

$$(\star) \qquad U(Rx+p) = \alpha(R,p)U(x) + \beta(R,p).$$

We show that if $(\star)$ has a nonconstant solution U then α and β have to be of a very special form. Then we obtain under suitable hypotheses the general solution of $(\star)$.

SHIMIZU, R., **Expansion of a completely monotone distribution.**

A probability distribution F is said to have decreasing hazard rate if the ratio $f(x)/(1-F(x))$ is non–increasing, where f is the probability density of F. If the distribution has the moments $\mu_j, j = 1, 2, \dots, K$, with $\mu_1 = 1$ then they satisfy the inequalities

$$1 \le \mu_2/2 \le \mu_3/3! \le \dots \le \mu_K/K!$$

If any one of the inequality signs becomes equality, then the distribution F is exponential: $F(x) = 1 - e^{-x}$, for $x > 0$.

A distribution F is said to be completely monotone if it can be put in the form $1 - F(x) = \int\limits_0^\infty e^{-\sigma x} dG(\sigma)$, where G is a distribution function. In statistical terminology, this means that the distribution F of a positive random variable η belongs to C if and only if there exist mutually independent positive random variables σ and X such that $\eta = \sigma X$ and X follows the exponential distribution.

Completely monotone distributions constitute a subclass C of the class D of distributions with decreasing hazard rate.

In this talk, it will be shown that the distribution F of C admits the following expansion: for any even number $k \le K$, write

$$F_k(x) = 1 - e^{-x} - \sum_{j=1}^{k-1} \frac{1}{j} \alpha_j L_{j-1}^{(1)}(x) x e^{-x}$$

where

$$\alpha_j = \sum_{p=1}^{j-1} (-1)^p \binom{j-1}{p} \varepsilon_p, \quad \varepsilon_p = \mu_p/p! - \mu_{p-1}/(p-1)!\ ,$$

and where the L's are Laguerre polynomials: $L_0^{(1)}(x) = 1$, $L_1^{(1)}(x) = x-2$, $L_2^{(1)}(x) = x^2 - 6x + 6$ and $L_3^{(1)}(x) = x^3 - 12x^2 + 36x - 24$. Then, we have

$$|F(x) - F_k(x)| \le A_k \alpha_k,$$

where the A's are positive numbers independent of the distribution G. A numerical computation shows that $A_2 = 4.23$, $A_4 = 12.29$ and $A_6 = 38.95$.

STEHLIG, F., **Functional inequalities in the theory of environmental quality indices: concepts of synergism and antagonism.**

Various attempts have been made to describe the quality of a certain area of the environment by real–valued indices $I : E \to \mathbf{R}_{(+)}$, where E is the set of (all) objects which are relevant for the environmental quality at time t. In practical applications, the set E is usually specified as a subset of $\mathbf{R}^n_{(+)}$. It has often been observed, for instance in connection with the damages caused by acid rain, that synergistic and sometimes antagonistic effects play an important role. If these effects are to be described (and measured) by environmental quality indices these indices must have certain properties which can be expressed in terms of functional inequalities. Moreover, a precise definition of synergistic and antagonistic effects can be given only by such properties of environmental indices. Different approaches are presented and their consequences with respect to the structure of the indices are analysed.

MATKOWSKI, J., **On L^p norm.**

Let $\varphi : [0,\infty) \to [0,\infty)$ be a bijection with $\varphi(0) = 0$. We prove, without any regularity conditions on φ, that if the functional $p_{\varphi,k} : \mathbf{R}^k \to [0,\infty)$, $k \geq 2$, defined by the formula

$$p_{\varphi,k}(x) := \varphi^{-1}\left(\sum_{i=1}^{k} \varphi(|x_i|)\right), \qquad x = (x_1, \dots, x_k),$$

is a norm in $\mathbf{R}^k$ then $\varphi(t) = ct^p$. Moreover the norm

$$\|x\|_{p,k} := (|x_1|^p + \ldots + |x_k|^p)^{1/p}$$

has the following property. If both φ and φ^{-1} are bounded on every interval $(0,a)$, $a < \infty$, and $\lim\limits_{t\to\infty} t^{-p}\varphi(t) > 0$ then there exists a "homogeneous regularization" $r_{\varphi,k} : \mathbf{R}^k \to [0,\infty)$ of the functional $p_{\varphi,k}$ defined by the formula

$$r_{\varphi,k}(x) := \lim_{t\to\infty} t^{-1} p_{\varphi,k}(tx)$$

and

$$r_{\varphi,k}(x) = \|x\|_{p,k}, \quad x \in \mathbf{R}^k.$$

DARÓCZY, Z., **Interval filling sequences and functional equations.**

Let Λ denote the set of those real sequences $\lambda := \{\lambda_n\}$ for which the conditions $\lambda_n > \lambda_{n+1} > 0 \ (n \in \mathbf{N})$ and $L^{(\lambda)} := \sum_{n=1}^{\infty} \lambda_n < \infty$ are satisfied. If $\lambda \in \Lambda$ then we define

$$S[\lambda] := \{\langle \varepsilon, \lambda \rangle \mid \varepsilon \in \{0,1\}^{\mathbf{N}}\}$$

where $\langle \varepsilon, \lambda \rangle := \sum_{n=1}^{\infty} \varepsilon_n \lambda_n$. The sequence $\lambda \in \Lambda$ is said to be interval filling if $S[\lambda] = [0, L^{(\lambda)}]$.

Let $\lambda \in \Lambda$ be an interval filling sequence. A mapping $\varepsilon : [0, L^{(\lambda)}] \to \{0,1\}^{\mathbf{N}}$ is an algorithm if for any $x \in [0, L^{(\lambda)}]$

$$x = \langle \varepsilon(x), \lambda \rangle$$

holds. We call the function $F : [0, L^{(\lambda)}] \to \mathbf{R}$ additive (with respect to the interval filling sequence $\lambda \in \Lambda$ and with respect to a nonvoid set A of algorithms) if for any $x \in [0, L^{(\lambda)}]$ and for any $\varepsilon \in A$ the equality

$$F(x) = F[\ \langle \varepsilon(x), \lambda \rangle\ ] = \langle \varepsilon(x), F(\lambda) \rangle \tag{1}$$

holds where $F(\lambda) := \{F(\lambda_n)\}$ and $\sum_{n=1}^{\infty} |F(\lambda_n)| < \infty$.

We present some results and problems on the functional equation (1).

CHOCZEWSKI, B., **Dirichlet's problem for a functional equation.**

(This is a joint paper by Z. POWAZKA (Kraków) and the speaker.)

Let $W \subset \mathbf{R}^n$ be a nonempty, convex and bounded set, $\partial W \neq \emptyset$ and let $F : \mathbf{R}^2 \to \mathbf{R}$, $b : \partial W \to \mathbf{R}$ be given continuous functions. The following problem will be discussed.

Problem (D). Find a continuous solution $\varphi : \overline{W} \to \mathbf{R}$ of the functional equation

$$\varphi\left(\frac{p+q}{2}\right) = F(\varphi(p), \varphi(q))$$

that satisfies the following condition : $\varphi(p) = b(p),\ p \in \partial W$.

MAKSA, GY., **Nonnegative entropies.**

It is known that there exist 1–recursive, symmetric, normalized, nonnegative entropies different from the Shannon–entropy. On the other hand, the Shannon–entropy has minimal property in the set of all 1–recursive, symmetric, normalized and nonnegative entropies.

In the talk similar problems will be discussed in connection with α–additve, nonnegative, normalized entropies which have the sum property.

ZDUN, M. C., **On C^r solutions of simultaneous Abel's equations.**

Let $f_k : (a,b) \to (a,b)$, $k = 1, \ldots, m$ be continuous bijections such that $f_i \circ f_j = f_j \circ f_i$ for $i, j = 1, \ldots, m$. Let $1 < i < m$ and suppose that for every $n, k \in \mathbf{Z}$ and $|n| + |k| \neq 0$ $f_1^n(x) \neq f_i^k(x)$ for $x \in (a,b)$. Let $x_0 \in (a,b)$. Then for every $n \in \mathbf{N}$ there is a unique $m_n \in \mathbf{Z}$ such that $f_1^{m_n - 1}(x_0) \leq f_i^n(x_0) < f_1^{m_n}(x_0)$ and there exists a limit

$$\lim_{n\to\infty} \frac{m_n}{n} =: s(f_i, f_1) \in \mathbf{Q}$$

and this limit does not depend on x_0. Put

$$A := \Bigg\{ s \in \mathbf{R} \setminus \mathbf{Q};\ \ s = a_0 + 1/(a_1 + 1/(a_2 + \ldots))),$$

$$\lim_{\beta\to\infty} \limsup_{n\to\infty} \frac{\sum\limits_{i \leq n, a_i \geq \beta} \log(1 + a_i)}{\sum\limits_{i \leq n} \log(1 + a_i)} = 0 \Bigg\}.$$

The set A is of full Lebesgue measure in $\mathbf{R}$.

If f_1 and f_i are of class C^r, $3 \leq r \leq \omega$, $f_1' \neq 0$, $f_i' \neq 0$, $s(f_i, f_1) \in A$ then there exist a function $\varphi \in C^{r-2}$ unique up to an additive constant and $c_2, \ldots, c_m \in \mathbf{R} \setminus \{0\}$ such that

(1) $\varphi(f_k(x)) = \varphi(x) + c_k$, for $k = 1, \ldots, m$, $x \in (a,b)$, where $c_1 = 1$ and $\varphi' \neq 0$. Moreover $c_i = s(f_i, f_1)$.

If $s(f_i, f_1) \notin A$, then the system (1) even for analytic f_1 and f_i may possess no C^1 solutions.

VOLKMANN, P., **Eine spezielle Klasse von Deviationsmitteln.**

Satz. *Es sei* $[a,b] \subseteq \mathbf{R}$ *und* $M : [a,b] \times [a,b] \to \mathbf{R}$. *Dann sind äquivalent:*

(A) *M ist stetig, in beiden Variablen streng wachsend, und es gilt*

$$\min\{x,y\} \leq M(x,y) = M(y,x) \leq \max\{x,y\}.$$

(B) *Es existiert* $E : [a,b] \times [a,b] \to \mathbf{R}$ *stetig mit*

$$E(x,x) \equiv 0,$$

$$x \le \overline{x},\ y \ge \overline{y},\ (x,y) \ne (\overline{x},\overline{y}) \Rightarrow E(x,y) < E(\overline{x},\overline{y}),$$

$$E(x, M(x,y)) + E(y, M(x,y)) = 0.$$

KOVAČEC, A., **Two recurrent inequalities.**

Theorem. *(a) Let* $r > 0$ *be fixed and define the function*

$$f = f(C) := \left[1 - \frac{r}{C^{1/r}(1+r)^{1+1/r}}\right].$$

Then for every $N \in \mathbf{N}$ *there holds the inequality*

$$\underbrace{(f \circ f \circ \ldots \circ f)}_{N-1-\text{times}}(1) \le x_1 \left(\sum_{i=1}^{N} x_i\right)^r + x_2 \left(\sum_{i=2}^{N} x_i\right)^r + \ldots + x_N \left(\sum_{i=N}^{N} x_i\right)^r \quad (x_i \ge 0)$$

(b) Let $p > 1$ *be fixed and define the function*

$$f = f(C) := \left[1 + \frac{C^p(p-1)^{p-1}}{p^p}\right].$$

Then for every $N \in \mathbf{N}$ *there holds the inequality*

$$\frac{x_1}{\sqrt[p]{x_1 + x_2 + \ldots + x_N}} + \frac{x_2}{\sqrt[p]{x_2 + x_3 + \ldots + x_N}} + \ldots + \frac{x_N}{\sqrt[p]{x_N}} \le$$

$$\le \underbrace{(f \circ f \circ \ldots \circ f)}_{N-1-\text{times}}(1) \left(\sum_{i=1}^{N} x_i\right)^{1-1/p}$$

The proofs are given by the "Functional equation approach to inequalities" as to be found e.g. in the works of CHUNG LIE WANG, BELLMAN and (implicitly) REDHEFFER.

Corollary. *(American Mathematical Monthly Problem* E2996, M90 (1983), 334)

$$\sum_{j=1}^{\infty} \frac{x_j}{\sqrt[p]{\sum_{i \ge j} x_i}} \le \frac{p}{p-1} \left(\sum_{j=1}^{\infty} x_j\right)^{1-1/p}$$

Another approach to inequalities of this second ("infinite") type is also discussed.

KRASINSKA, S., **The transient process and the Switching moments for the long Thomson's cable.**

The aim of the note is to derive the general form of the formula which gives the Switching Moments of the current for the Long – Thomson's cable with boundary conditions, having the following form:
$u(0,\ell) = 0$ for $\ell > 0$ and $u(t,0) = E = \text{const}$ for $t \geq 0$ and
$u(t,\ell_k) = i(t,\ell_k)R$.
The paper deals with the optimal control problem of the system with distributed parameters described by a differential equation of the parabolic type.

NAGY, B., **On spectral measures with singularities.**

It is well–known that there is a 1–1 correspondence between the class of all closed normal operators and the class of all selfadjoint spectral measures in Hilbert space or, more generally, between the class of all spectral operators of scalar type (in the sense of DUNFORD and BADE) and the class of all (non–selfadjoint) spectral measures in Banach space. As a contrast it is shown here that to every closed linear operator in a Banach space there corresponds a spectral measures with singularities that is maximal in a certain sense: however, there are such (in general unbounded) spectral measures with singularities that do not correspond to any closed operator in the sense above (even in Hilbert space). The proof of the last assertion contains elements that are similar to methods used in the theory of functional equations.

KRÄUTER, A. R., **On best possible upper bounds for the permanent of $(1,-1)$ matrices with arbitrary rank.**

Two $n\times n$ $(1,-1)$ matrices A and B are said to be equivalent $(A \sim B)$, if B can be obtained from A by a sequence of the following operations: (i) interchange any two rows or columns of A; (ii) transpose A; (iii) negate any row or column of A.

Let r be a nonnegative integer, $r \leq n-1$ and let $C(n,r) = (c_{ij})$ be the $n \times n$ $(1,-1)$ matrix with $c_{ii} = -1$ for $i = 1, \dots, r$ and $c_{ij} = 1$ otherwise. Then it is known that

$$\omega_{n,r} = \text{per}(C(n,r)) = \sum_{k=0}^{r} (-2)^k \binom{r}{k} (n-k)! \ .$$

The main topic of the talk will be a discussion of the following Conjecture ([2]). Let A be an $n \times n$ $(1,-1)$ matrix, $n \geq 5$, such that $\text{rank}\,(A) = r+1$. Then

$$|\text{per}(A)| \leq \omega_{n,r}$$

and equality occurs if and only if $A \sim C(n,r)$.

For $r = 0$ and $r = 1$ complete solutions have been given by WANG [4]. For $r = n - 1$ the conjecture as well as a partial solution is presented in [1], thus answering in part a problem posed by WANG, whereas SEIFTER [3] found a partial answer for the general case.

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SKÓRNIK, K., A remark on the Foias theorem.

The Foias theorem [2] on convolution was proved for the first time in 1961. Other proofs of this theorem can be found, for instance, in [1], [3], [4]. In all of those papers the Foias theorem was proved by applying the representation theorem for linear continuous functionals and the Hahn – Banach extension theorem so that transfinite methods were involved. J. MIKUSIŃSKI in [5], eliminated transfinite methods by using the representation theorem of functionals on L^1 (see also [6]). Similarly, in [8], the Foias theorem on convolution for continuous functions was proved.

In [7] a new proof of the Foias theorem on convolution of integrable functions is given, without use of transfinite methods and without use of the theorem on representation of functionals on L^1. Instead we use some properties of Hilbert space.

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RÉVÉSZ, SZ., **Continuous solutions of the difference equation $\Delta_{a_1}\dots\Delta_{a_n}f=0$.**

Denote by Δ_a the difference operator with step $a\in\mathbf{R}$, which operates over $\mathbf{R}^{\mathbf{R}}$ and has kernel $\{f:\mathbf{R}\to\mathbf{R}, f(x+a)=f(x),(x\in\mathbf{R})\}$. If $f\in C(\mathbf{R})$ is the sum of the functions f_i, periodic mod a_i,

$$f=f_1+\dots+f_n,\quad \Delta_{a_i}f_i=0, \tag{1}$$

then f satisfies the homogeneous difference equation

$$\Delta_{a_1}\Delta_{a_2}\dots\Delta_{a_n}f=0. \tag{2}$$

Here we investigate the converse, and prove, that if $f\in C(\mathbf{R})$ is bounded and satisfies (2), then there exists a representation (1) with f_i continuous $(i=1,\dots,n)$. This extends a theorem of M. WIERDL, *Mat. Lapok*, **32** (1984), 107–113, which settles the case $n=2$. Using a result of H. WHITNEY, *J. Math. Pures Appl.* **36** (1957), 67–95, and the above theorem, we can characterize the functions f having a representation

$$f=P+f_1+\dots+f_n,\ f_i\in C(\mathbf{R}),\ \Delta_{a_i}f_i=0,\ P\in\mathbf{R}[x],\ \deg P<n. \tag{3}$$

These functions are the trivial solutions of (2) in $C(\mathbf{R})$, and an $f\in C(\mathbf{R})$ can be decomposed according to (3) if and only if (2) and

$$\omega(n,f)<\infty\quad\big(\omega(n,f):=\sup\{\|\underbrace{\Delta_h\dots\Delta_h}_{n\text{-times}}f\|_\infty : h\in\mathbf{R}\}\big) \tag{4}$$

is satisfied.

The same decomposition problem can be investigated in various other function spaces. Moreover, we can prove a generalization of our first theorem to certain topological spaces (in place of $\mathbf{R}$) and commuting transformations (in place of the translation $x\to x+a_i$).

BRYDAK, D., **On the stability of the iterative functional equation.**

The problem of stability of the equation

$$\varphi[f(x)]=g[x,\varphi(x)],\quad x\in\mathcal{J}=[a,b), \tag{1}$$

where f and g are given functions, will be discussed in the case where equation (1) has a one–parameter family of continuous solutions. We shall deal with the iterative stability in the sense of Hyers being on the iterative stability of the stability of the so–called comparison equation

$$\varphi[f(x)]=f(x)\varphi(x),\quad x\in\mathcal{J}$$

such that

$$|\,g(x,y)-g(x,z)\,|\ge f(x)|y-z|,\quad x\in\mathcal{J}.$$

POWAZKA, Z., **Über eine Funktionalungleichung.**

In dieser Arbeit werden Ergebnisse aus der Theorie der Funktionalungleichungen einer Veränderlichen in der Theorie der Funktionalungleichungen mehrerer Veränderlichen angewendet.

Mit anderen Worten beweisen wir Sätze für die stetigen Lösungen der Funktionalungleichung

$$\psi(\alpha x + \beta y + \gamma) \leq \alpha\psi(x) + \beta\psi(y) + \gamma,$$

wo $x, y \in \mathbf{R}$, α, β, γ positive reelle Zahlen sind.

BARBANTI, L., **Application of the Gronwall–Belman inequality for Volterra – Stieltjes integral equations.**

In this work we solve two problems that are posed in the context of numerical linear Volterra–Stieltjes integral equations that is, in the context of the equations

$$\text{(K)} \qquad x(t) - \int_0^t \cdot d_s K(t,s)x(s) = f(t) \qquad 0 \leq t \leq t_0$$

where x and f belong to $G([0,t_0],\mathbf{R})$ the set of all regulated functions (i.e. that has only discontinuities of first kind) from $[0,t_0]$ into $\mathbf{R}$, and $K : [0,t_0]^2 \to L(\mathbf{R})$ is an admissible nucleus (in the sense that the interior (or Dushnik type) integral $\int \cdot$ exist and is regulated, and that there exists a unique resolvent R satisfying $x(t) = f(t) + \int_0^t \cdot d_s R(t,s) f(s)$, and the equalities

$$(R_\star) \quad R(t,s)x - x - K(t,s)x + \int_s^t \cdot d_\sigma K(t,\sigma)(R(\sigma,s)x) = 0, \qquad x \in \mathbf{R}$$

$$(R^\star) \quad R(t,s)x - x + \int_s^t \cdot d_\sigma (R(t,\sigma)(K(\sigma,s)x) = 0, \quad x \in \mathbf{R},\ s \leq t \leq t_0).$$

(For the conditions on K to be admissible, and for the general theory of the interior integral and the (K) equation, see HÖNIG Equations intégrales generalisées et applications – *Public. Math. d'Orsay* **83–01** (1983)).

We say that the pair (K,R) satisfies the G–B–property if K is increasing and R decreasing when we fix the first variable in both.

The Gronwall–Bellman inequality for (K) is the following: "for all $f,g \in G([0,t_0],\mathbf{R})$ if (K,R) satisfies the G–B-property then we have : $g(t) \le f(t) + \int_0^{t_0} \cdot d_s K(t,s)g(s)$ implies $g(t) \le f(t) - \int_0^{t} \cdot d_s R(t,s)f(s)$ " (see HÖNIG, 1.c).

The problems that we are going to solve are due to U. ŽILINSKI when dealing with problems in meteorology (see the technical report by ŽILINSKI, 1981 *Applied Mech. Institute*, Kiev): Given K_1 and K_2 as admissible nuclei and R_1, R_2 as their associated unique resolvents find the conditions under which
1.) There exists a $T(t,s)$, where T is an admissible nucleus with resolvent R, such that (T,R) has the G–B-property and

$$(\star)\qquad \begin{aligned} &\left\| \int_0^t \cdot d_s[K_1(t,s)\circ R_1(s,0) + K_2(t,s)\circ R_2(s,0)]x \right\| \le \\ &\le \left\| \int_0^t \cdot d_s T(t,s)\circ [R_1(s,0) - R_2(s,0)]x \right\| \end{aligned}$$

for all $x \in \mathbf{R}$.
2.) There exist a $T(t,s)$, where T is an admissible nucleous with (T,R) having the G–B-property, and a constant operator G such that

$$(\star\star)\qquad \begin{aligned} &([K_2(t,s)\circ R_2(s,0) - R_2(t,s)\circ K_2(s,0)] - [K_1(t,s)\circ R_1(s,0) - \\ &- R_1(t,s)\circ K_1(s,0)])x = ([T(t,s)+G]\circ[K_1(s,0) - K_2(s,0)])x \end{aligned}$$

for all $x \in \mathbf{R}$.

The solution of these problems is obtained essentially using $(R^\star)$ and $(R_\star)$ and taking into account that in the Gronwall – Bellman – inequality, in this case, the term $f(t)$ becomes zero. The answer is affirmative if and only if the first terms in both $(\star)$ and $(\star\star)$ are zero.

TABOR, J., **On mappings preserving spheres, directions and sense of vectors.**

Let E be a nonzero real normed space, G a group (written additively) and $\varphi : E \times G \to E$ a mapping. Consider the following conditions.

(i) $\varphi(x, 0) = x \quad$ for $x \in E$,

(ii) $\varphi(\varphi(x, t), s) = \varphi(x, t + s) \quad$ for $x \in E,\ t, s \in G$,

(iii) $|\varphi(x, t)| = |\varphi(y, t)| \quad$ for $|x| = |y|,\ t \in G$,

(iv) There exists a function $\alpha : \mathbf{R} \times \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}^+$ such that

$$\varphi(ax, t) = \alpha(x, a, t)\varphi(x, t) \qquad \text{for } x \in E,\ a \in \mathbf{R}^+,\ t \in \mathbf{R}.$$

Theorem. *If the function* $\varphi : E \times G \to E$ *satisfies conditions* (i) – (iii) *then the function*

$$\Theta(|x|, t) = |\varphi(x, t)| \qquad \textit{for } x \in E,\ t \in G \tag{1}$$

satisfies the translation equation and the identity condition. If furthermore φ *satisfies condition* (iv) *then it can be written in the form*

$$\varphi(x, t) = \begin{cases} \Theta(x, t)\lambda\left(\frac{x}{|x|}, t\right) & \text{for } x \in E,\ x \neq 0,\ t \in G, \\ 0 & \text{for } x = 0,\ t \in G \end{cases} \tag{2}$$

where $\lambda : S \times G \to S,\ S = \{x \in E : |x| = 1\}$ *and* λ *satisfies the translation equation and the identity condition.*

Conversely, if $\Theta : \mathbf{R}^+ \times G \to \mathbf{R}^+$ satisfies the translation equation and the identity condition, then the function φ of the form (2) satisfies conditions (i) - (iv) and (1).

FEHÉR, J., **Über die Polynomlösungen einer Funktionalgleichung.**

DHOMBRES, J., **About functional equations meetings from 1350 to 1820 or some historical facts concerning functional equations and inequalities.**

Problems and remarks

1. Problem. (D. BRYDAK). Let F be a one–parameter family of functions defined and continuous in an interval $[a,b)$. Let $D \subset \mathbf{R}^2$ be a region such that through every point of D there passes a unique function $\varphi \in F$. Let us define the first integral R of the family F as follows

$$R(x,y) := \varphi(x), \qquad (x,y) \in D,$$

where φ is such a member of F that $\varphi(x) = y$.

What are the assumptions implying the differentiability of R ?

2. Problem. (D. BRYDAK). Let f be a function defined, strictly increasing and continuous in $\mathcal{J} = [0,a)$, $a > 0$. Moreover, let $f(0) = 0$, $0 < f(x) < x$ for $x \in (0,a)$. Let g be a function defined, nonnegative and continuous in $\mathcal{J}$. Let the equation

$$(1) \qquad \varphi[f(x)] = g(x)\varphi(x), \qquad x \in \mathcal{J}$$

have a continuous solution, positive in $(0,a)$ and depending on an arbitrary function. Let ψ be a nonnegative continuous solution of the inequality

$$(2) \qquad \psi[f(x)] \le g(x)\psi(x), \qquad x \in \mathcal{J}.$$

Does there always exist a solution of φ such that there exists the limit $\lim\limits_{x\to 0+} \psi(x)/\varphi(x)$?

Remark. The function φ does not have to be continuous. It is already known that such a continuous φ does not have to exist for every solution of (2).

3. Remark. (L. PAGANONI). Consider the functional equation

$$(1) \quad \big(f(x+y) - f(x) - f(y)\big)\big(g(x+y) - g(x) - g(y)\big) = 0 \qquad x,y \in \mathbf{R}$$

in the unknown functions $f,g : \mathbf{R} \to \mathbf{R}$. L. GIUDICI, a student of the University of Milano, has found the general solution of (1) under the hypothesis of the continuity of f (no assumption of g). In particular, if both f and g are continuous, then at least one of them has to be additive.

4. Remark. (D. RUSCONI). A function G; $G : \mathbf{R}^2 \to \mathbf{R}$, is of the form

$$G(x,y) = f(y) - f(x) + \varphi(y-x) \qquad x,y \in \mathbf{R}$$

where $f,\varphi : \mathbf{R} \to \mathbf{R}$ if and only if it is a solution of the equation

$$G(x+z, y+z) - G(x,y) = G(y, y+z) - G(x, x+z) \qquad x,y,z \in \mathbf{R}.$$

All continuous functions f and φ, which appear in the previous representation, are given explicitely, when G is a continuous solution of the equation

$$G(x,y)=G\left(x,\frac{x+y}{2}\right)+G\left(\frac{x+y}{2},y\right) \qquad x,y\in\mathbf{R}.$$

5. Remark. (Z. SEBESTYÉN). A Schwarz type inequality for positive operators on a Hilbert space proved to be useful in obtaining extension theorems of KREIN [2] and FRIEDRICHS type [3] as well. Another application of the method of proof gives a simple proof of a remarkable theorem of PARROTT [1] concerning the quotient norm with respect to spaces of Hilbert space operators.

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6. Remark. (Z. GAJDA). Remark on a problem of M. SABLIK. Let F_f denote the Cauchy difference of a function $f:[0,\infty)\to\mathbf{R}$, i.e.

$$F_f(x,y):=f(x+y)-f(x)-f(y), \qquad x,y\in[0,\infty).$$

The following two classes of functions have appeared in the course of investigations concerning conditional Cauchy functional equations carried out recently by Dr. M. SABLIK :

$$\begin{aligned} A &:= \{f:[0,\infty)\to\mathbf{R} : F_f \text{ is differentiable at the origin as a function}\\ &\qquad\qquad\qquad\qquad\quad \text{of two variables}\},\\ B &:= \{f:[0,\infty)\to\mathbf{R} : f=a+g, \text{ where } a \text{ is additive and } g \text{ is}\\ &\qquad\qquad\qquad\qquad\quad \text{differentiable at zero}\}. \end{aligned}$$

It is easily seen that the class B is contained in A. Dr. SABLIK was interested in answering the question whether the two classes coincide. Adopting the terminology introduced by M. LACZKOVICH one may also formulate this question as follows:

Is it true that the class of all functions differentiable at the origin posseses the double difference property?

Some facts seemed to suggest that the solution to this problem should be positive. For instance the well known results of N. G. DE BRUIJN imply that the class of all functions differentiable everywhere has the double difference property. Moreover M. LACZKOVICH has recently proved the double difference property for the class of all functions continuous at zero. The following example shows, however, that the class of functions differenctiable at zero fails to share the same property.

Let us take a function $h : [0, \infty) \to \mathbf{R}$ such that $h(x) = x$ for $x \in [0, 1)$ and h is bounded on the interval $[1, \infty)$. Now, define $f : [0, \infty) \to \mathbf{R}$ by

$$f(x) := \sum_{n=1}^{\infty} \frac{h(2^n x)}{n \cdot 2^n}, \qquad x \in [0, \infty).$$

Then one can check that $F_f'(0, 0) = 0$ but f does not admit a decomposition into a sum of an additive function and a function differentiable at zero.

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