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On the solutions of the functional equation $f(xf(y)^\ell + yf(x)^k) = tf(x)f(y)$

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Let **N, Z, Q, R** denote sets of all positive integers, integers, rational numbers, and real numbers, respectively. The functional equation

$$f(x\,f(y)^\ell + y\,f(x)^k) = t\,f(x)f(y), \tag{1}$$

where $t \neq 0$ is a fixed real number, k and ℓ are fixed positive integers, and the unknown function f maps $\mathbf{R}$ into itself, has been studied by many authors in various cases (cf. [1] – [11]). N. BRILLOUËT [2] (cf. also [3], [4]) found all continuous solutions $f : \mathbf{R} \to \mathbf{R}$ of (1) in the case $k = \ell = 1$ and $t > 0$, M. SABLIK and P. URBAN [9] – [11] solved (1) for $t = 1$ in the class of continuous functions $f : \mathbf{R} \to \mathbf{R}$, and W. BENZ [1] determined the cardinality of the set of discontinuous solutions of (1) for many $t \in \mathbf{R}$.

We are going to find all continuous solutions of (1) in the case where $t \in \mathbf{R}\backslash\{0\}$ and $k \neq \ell$ are positive integers. We always suppose, without loss of generality, that $\ell < k$. The result presented here is a generalization of the one from [10].

Let us start with the following

Lemma 1. *If a function $f : \mathbf{R} \to \mathbf{R}$ satisfies functional equation (1), then the following conditions hold:*

(i) *if there exists an $x \in \mathbf{R}$ such that $f(x) = 0$, then $f(0) = 0$,*

(ii) *if an $x_0 \in \mathbf{R}\backslash f^{-1}(\{0\})$, then the functions*

$$\mathbf{R} \ni y \to x_0 f(y)^\ell + y\,f(x_0)^k, \tag{2}$$

$$\mathbf{R} \ni y \to y\,f(x_0)^\ell + x_0 f(y)^k \tag{3}$$

are one – to – one.

PROOF. ad (i) Let f (z) = 0 and $x = y = z$. Then by (1) $f(0) = 0$.
ad (ii) We are going to prove that function (2) is one – to – one. The proof for function (3) is analogous.

Let us suppose that $f(x_0) \neq 0$ and function (2) is not one –to – one. Then there exist $y, z \in \mathbf{R}$ such that $y \neq z$ and

$$x_0 f(y)^\ell + y\, f(x_0)^k = x_0 f(z)^\ell + z\, f(x_0)^k.$$

Since by (1) $t\, f(x_0) f(y) = t\, f(x_0) f(z)$, so $f(y) = f(z)$ and $y\, f(x_0)^k = z\, f(x_0)^k$. Thus $y = z$. It is a contradiction.

This completes the proof.

Lemma 2. *If a continuous function $f : \mathbf{R} \to \mathbf{R}$ satisfies (1) and there exists an $x \in \mathbf{R}$ such that $f(x) \neq 0$, then for each $r > 0$ there exists an $x_r \in \mathbf{R}$ such that $f(x_r) \neq 0$ and*

$$\left| \frac{f(x_r)^\ell}{x_r} \right| < r.$$

PROOF. Let us denote $F := \{x : f(x) \neq 0\}$, $s := \sup F$, and $i := \inf F$. Since f is continuous and there exists an $x \in \mathbf{R}$ such that $f(x) \neq 0$, so $F \neq \emptyset$ and $s \neq 0$ or $i \neq 0$.

The following two cases are possible:

1^0 F is bounded,

2^0 F is not bounded.

Suppose that 1^0 holds. If $s \neq 0$, then there exists a sequence $\{x_n\}_{n \in \mathbf{N}} \subset F \setminus \{0\}$ such that $\lim_{n\to\infty} x_n = s$. If $s = 0$, then $i \neq 0$ and there exists a sequence $\{y_n\}_{n\in\mathbf{N}} \subset F\setminus\{0\}$ such that $\lim_{n\to\infty} y_n = i$.

Let $s \neq 0$. Then by 1^0 $s < +\infty$ and $\lim_{n\to\infty} f(x_n) = 0$. Thus

$$\lim_{n\to\infty} \frac{f(x_n)^\ell}{x_n} = 0.$$

When $s = 0$, then in a similiar way, we obtain that

$$\lim_{n\to\infty} \frac{f(y_n)^\ell}{y_n} = 0.$$

Now suppose that 2^0 holds. Then there exists a sequence $\{x_n\}_{n\in\mathbf{N}} \subset F\setminus\{0\}$ such that $\lim_{n\to\infty} |x_n| = +\infty$. If there exists a subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$ of $\{x_n\}_{n\in\mathbf{N}}$ such that

$$\lim_{n\to\infty} \frac{f(x_{n_k})^\ell}{x_{n_k}} = 0, \tag{4}$$

then the assertion holds.

On the contrary, if each subsequence of $\{x_n\}_{n\in\mathbf{N}}$ does not satisfy (4), then there exist a $d > 0$ and an $n_0 \in \mathbf{N}$ such that

$$\left|\frac{f(x_n)^\ell}{x_n}\right| > d \qquad \text{for all } n > n_0.$$

Hence

$$|f(x_n)^\ell| > d|x_n| \qquad \text{for all } n > n_0 \tag{5}$$

and

$$|x_n f(x_n)^{-\ell}| < \frac{1}{d} \qquad \text{for all } n > n_0. \tag{6}$$

Since $\ell < k$ and $\lim_{n\to\infty} |x_n| = +\infty$, by (5) we have that

$$\lim_{n\to\infty} |f(x_n)^{k-\ell}| = +\infty. \tag{7}$$

Let

$$W(x,y) := x\,f(y)^\ell + y\,f(x)^k \qquad \text{for all } x,y \in \mathbf{R}.$$

Then

$$\frac{f(W(x,y))^\ell}{W(x,y)} = \frac{t^\ell f(x)^\ell f(y)^\ell}{x\,f(y)^\ell + y f(x)^k} = \frac{t^\ell f(y)^\ell}{x\,f(x)^{-\ell} f(y)^\ell + y\,f(x)^{k-\ell}}$$

for all $x, y \in F\backslash\{0\}$.

Thus by (6) and (7)

$$\lim_{n\to\infty} \frac{f(W(x_n,y))^\ell}{W(x_n,y)} = 0 \qquad \text{for all } y \in F\backslash\{0\}.$$

This implies the assertion.

Lemma 3. *If a continuous function $f : \mathbf{R} \to \mathbf{R}$ satisfies (1) and there exist $a, b \in \mathbf{R}, a < b$ such that $f(a) = f(b) \neq 0$, then $f(x) = f(a)$ for all $x \in [a,b]$.*

PROOF. Let us denote

$$V(x,y,z) := \frac{f(x)^\ell}{x}(y-z) + f(y)^k - f(z)^k \qquad \text{for all } x,y,z \in \mathbf{R}, x \neq 0$$

and suppose that there exists a $c \in (a,b)$ such that $f(c) \neq f(a)$. Then

$$\operatorname{sgn}(f(c)^k - f(a)^k) \neq \operatorname{sgn}(f(b)^k - f(c)^k), \tag{8}$$

where

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Since

$$|f(c)^k - f(a)^k| = |f(b)^k - f(c)^k| \neq 0,$$

by Lemma 2 there exists an $x_0 \in \mathbf{R}$ such that

$$\left|\frac{f(x_0)^\ell}{x_0}(c-a)\right| < |f(c)^k - f(a)^k|$$

and

$$\left|\frac{f(x_0)^\ell}{x_0}(b-c)\right| < |f(b)^k - f(c)^k|.$$

Hence

$$\operatorname{sgn} V(x_0, c, a) = \operatorname{sgn}(f(c)^k - f(a)^k) \tag{9}$$

and

$$\operatorname{sgn} V(x_0, b, c) = \operatorname{sgn}(f(b)^k - f(c)^k). \tag{10}$$

It is easy to observe that by (8), (9), and (10)

$$\operatorname{sgn} V(x_0, b, a) \neq \operatorname{sgn} V(x_0, c, a) \tag{11}$$

or

$$\operatorname{sgn} V(x_0, b, a) \neq \operatorname{sgn} V(x_0, b, c). \tag{12}$$

Since f is continuous, the following functions:

$$\mathbf{R} \ni y \to V(x_0, y, a),$$
$$\mathbf{R} \ni y \to V(x_0, b, y)$$

are continuous. Thus, if (11) holds, then there exists an $e_1 \in [c, b]$ such that

$$V(x_0, e_1, a) = 0 \tag{13}$$

and if (12) holds, then there exists an $e_2 \in [a, c]$ such that

$$V(x_0, b, e_2) = 0. \tag{14}$$

It is easy to verify that (13) implies that

$$e_1 f(x_0)^\ell + x_0 f(e_1)^k = a\, f(x_0)^\ell + x_0 f(a)^k$$

and (14) implies that

$$e_2 f(x_0)^\ell + x_0 f(e_2)^k = b\, f(x_0)^\ell + x_0 f(b)^k.$$

In view of Lemma 1 (ii) it is impossible.

Lemma 4. *If a continuous function $f : \mathbf{R} \to \mathbf{R}$ satisfies (1) and there exist $a, b \in \mathbf{R}, a < b$ such that $f(x) = f(a) = f(b) \neq 0$ for all $x \in (a, b)$, then $f(x) = f(a)$ for all $x \in \mathbf{R}$.*

PROOF. Let us suppose that there exists an $x \in \mathbf{R}$ such that $f(x) \neq f(a)$. Then $\operatorname{card} f(\mathbf{R}) > \operatorname{card} \mathbf{N}$ and by axiom of choise there exists a set $Y \subset \mathbf{R}$ such that

$$\operatorname{card} Y > \operatorname{card} \mathbf{N}, \tag{15}$$

$$f(x) \neq f(y), \qquad \text{for all } x, y, \in Y, x \neq y, \tag{16}$$

and

$$0 \notin f(Y). \tag{17}$$

Fix a $y \in \mathbf{R}$. We define a function $W^y : \mathbf{R} \to \mathbf{R}$ as follows

$$W^y(x) := y\, f(x)^\ell + x\, f(y)^k, \qquad \text{for all } x \in \mathbf{R}.$$

It follows from (1) that $f(W^y(x)) = t\, f(x) f(y)$ for all $x, y \in \mathbf{R}$. Thus

$$f(z) = t\, f(a) f(y), \qquad \text{for all } z \in W^y([a, b]),\ y \in \mathbf{R}$$

and according to (16) and (17)

$$W^x([a, b]) \cap W^y([a, b]) = \emptyset, \qquad \text{for all } x, y \in Y,\ x \neq y. \tag{18}$$

It is easy to notice that for each $y \in Y$ the function W^y is continuous and by Lemma 1 (ii) is one – to – one. Hence $W^y([a, b])$ is a nontrivial interval for all $y \in Y$.

Let $A := \{W^y([a, b]) : y \in Y\}$. Then (18) implies that A is a family of disjoint intervals. By (15) $\operatorname{card} A > \operatorname{card} \mathbf{N}$. It is impossible.

Finally we prove the following

Theorem 1. *The functions $f_1 \equiv 0$ and $f_2 \equiv \frac{1}{t}$ are the only continuous solutions of (1) in the class of functions $f : \mathbf{R} \to \mathbf{R}$.*

PROOF. Let a continuous function $f : \mathbf{R} \to \mathbf{R}$ satisfy (1) and let $f \not\equiv 0$. We define $F := \{x : f(x) \neq 0\}$. Thus

$$F \neq \emptyset. \tag{19}$$

Two cases are possible
(a) $f|_F$ is not one – to – one,
(b) $f|_F$ is one – to –one,
where the function $f|_F : F \to \mathbf{R}$ is defined as follows

$$f|_F(x) := f(x) \qquad \text{for all } x \in F.$$

When (a) holds, then there exist $a, b \in F, a < b$ such that $f(a) = f(b) \neq 0$. Thus by Lemma 3 we obtain that $f|_{[a,b]} \equiv f(a) \neq 0$ and consequently on account of Lemma 4 and (1) $f \equiv \frac{1}{t}$.

On the contrary, if (b) holds, then by (1) the following condition is true:

$$x\, f(y)^k + y\, f(x)^\ell = y\, f(x)^k + x\, f(y)^\ell, \qquad \text{for all } x, y \in F.$$

Thus

$$f(x)^\ell - f(x)^k = x\frac{f(y)^\ell - f(y)^k}{y} \qquad \text{for all } x, y \in F,\ y \neq 0.$$

Since f is continuous and $f \not\equiv const$, there exists a $y_0 \in F \setminus \{0\}$ such that $f(y_0)^\ell \neq f(y_0)^k$. Denote

$$p := \frac{f(y_0)^\ell - f(y_0)^k}{y_0}.$$

Then

$$\text{(20)} \qquad f(x)^\ell - f(x)^k = px, \qquad \text{for all } x \in F.$$

Hence the function $g : \mathbf{R} \to \mathbf{R}$ defined as follows

$$\text{(21)} \qquad g(x) = \frac{1}{p}(x^\ell - x^k), \qquad \text{for all } x \in \mathbf{R}$$

satisfies the following condition

$$\text{(22)} \qquad g(f(F)) = F.$$

Now suppose that

$$\text{(23)} \qquad F = \mathbf{R}.$$

Then by the fact that f is continuous we obtain that $f(F)$ is connected. Thus $f(F) \subset (0, +\infty)$ or $f(F) \subset (-\infty, 0)$. Since by (21) $g((0, +\infty)) \neq \mathbf{R}$ and $g((-\infty, 0)) \neq \mathbf{R}$, by (22) we obtain that (23) does not hold. Hence by Lemma 1 (i)

$$f(0) = 0. \tag{24}$$

It follows from (21) that $g(1) = g(0) = 0$. Thus by (24) and (22)

$$\{0, 1\} \cap f(F) = \emptyset. \tag{25}$$

Observe that since f is continuous and (b) holds, we obtain that $f(F) \cup \{0\} = f(\mathbf{R})$ is connected and $f^{-1}(\{0\})$ is connected. Thus by (19) F is unbounded and consequently according to (20) $f(F)$ is unbounded. Therefore in view of (25)

$$f(F) = (-\infty, 0). \tag{26}$$

Hence and from (21) and (22) it follows that

$$\operatorname{card} \mathbf{R} \setminus F > 2. \tag{27}$$

In the case where k and ℓ are even numbers, we have that $g(-1) = 0$. Thus by (22), (24), and (26) at least one of the following cases holds:

(j) k is an odd number,

(jj) ℓ is an odd number.

Suppose that (j) holds and denote $A_k := \{f(x)^k : x \in F\}$. Then on account of (26)

$$A_k = (-\infty, 0). \tag{28}$$

Since according to (1)

$$f(x\,f(y)^k) = 0, \qquad \text{for all } y \in F,\ x \in f^{-1}(\{0\}),$$

so

$$x\,A_k := \{x\,z : z \in A_k\} \subset f^{-1}(\{0\}), \qquad \text{for all } x \in f^{-1}(\{0\}) \tag{29}$$

and

$$z f^{-1}(\{0\}) \subset f^{-1}(\{0\}), \qquad \text{for all } z \in A_k. \tag{30}$$

Fix any $x_0 \in f^{-1}(\{0\})\setminus\{0\}$ (cf. (27)). If $x_0 < 0$, then by (28) and (29)

$$x_0 A_k = (0, +\infty) \subset f^{-1}(\{0\})$$

and consequently in virtue of (28) and (30)

$$(-\infty, 0) \subset f^{-1}(\{0\}).$$

Thus according to (24)

$$f^{-1}(\{0\}) = \mathbf{R}.$$

On the contrary, if $x_0 > 0$, then by (28) and (29)

$$x_0 A_k = (-\infty, 0) \subset f^{-1}(\{0\})$$

and consequently in view of (28) and (30)

$$(0, +\infty) \subset f^{-1}(\{0\}).$$

Thus on account of (24)

$$f^{-1}(\{0\}) = \mathbf{R}.$$

Hence $f \equiv 0$. It leads to a contradiction.
In the case (jj) we obtain a contradiction in the same way.
This completes the proof.

We can generalize the above theorem in a similar way as in [10].

Theorem 2. *Let X be a linear space over $\mathbf{R}$ and let a function $f : X \to \mathbf{R}$ satisfy* (1). *Then, if for each $x \in X\setminus\{0\}$ the function $g_x : \mathbf{R} \to \mathbf{R}$ given by*

$$g_x(s) = f(s\,x) \qquad \text{for all } s \in \mathbf{R}$$

is continuous, then $f \equiv 0$ or $f \equiv \frac{1}{t}$.

Proof. It is easy to verify that g_x satisfies (1) for $x \in X\setminus\{0\}$. Thus by Theorem 1 $g_x \equiv const$ for $x \in X\setminus\{0\}$.

Since $g_x(0) = f(0)$ for $x \in X\setminus\{0\}$, so $g_x \equiv f(0)$ for $x \in X\setminus\{0\}$ and therefore $f \equiv const$. This completes the proof.

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