Publ. Math. Debrecen
38 / 3–4 (1991), 185–188

On arithmetical functions over Gaussian integers having constant values in some domain

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1. Let G be the set of Gaussian integers, $G^* = G\backslash\{0\}, \mathbf{C} =$ complex field. A function $f : G^* \to \mathbf{C}$ is called completely additive, if $f(\alpha\beta) = f(\alpha) + f(\beta)$ holds for each $\alpha, \beta \in G^*$. The set of completely additve functions is denoted by $\mathcal{A}_G^*$. Some function $F : G^* \to \mathbf{C}$ is called completely multiplicative, if $F(\alpha\beta) = F(\alpha) \cdot F(\beta)$ holds for each pair $\alpha, \beta \in G^*$. The set of completely multiplicative functions is denoted by $\mathcal{M}_G^*$.

Let $S(a, r)(\subseteq \mathbf{C})$ be the closed disc with center a and radius r, i.e. $S(a, r) = \{z : |z - a| \leq r\}$.

Our purpose in this short paper is to give the following analogon of a theorem due to KÁTAI [1] for additive functions taking on constant values in some relatively short intervals.

Theorem 1. *Let $f \in \mathcal{A}_G^*$. Assume that there exists a sequence $z_1, z_2 \dots$ of complex numbers such that $|z_\nu| \to \infty$ and that*

$f(\alpha) = A_\nu =$ constant on $\alpha \in S(z_\nu, (2+\varepsilon)\sqrt{|z_\nu|})$ with some arbitrary positive constant ε. Then $f(\alpha) = 0$, identically.

2. The assertion is an immediate consequence of the following three lemmas.

Lemma 1. *Let $f \in \mathcal{A}_G^*$, $z \in G^*$ with $|z| = M(\geq 2)$. Assume that $f(\alpha) = A$ in $R := \{\alpha \in G^*, M \leq |\alpha| \leq \sqrt{2}M\}$. Then $A = 0, f(\alpha) = 0$ for every $\alpha \in G^*$ with $|\alpha| \leq \sqrt{2}M$.*

PROOF. Let $\lambda = 1 + i \in G$. Then $|\lambda z| = \sqrt{2}M$, i.e. $z \in R, \lambda z \in R$, consequently $f(\lambda) = f(\lambda z) - f(z) = 0$. Let k be such an integer for which $\lambda^k \in R$. It is clear that such a k exists, and $k \in \mathbf{N}$. Then $A = f(\lambda^k) = kf(\lambda)$, which by $f(\lambda) = 0$ implies that $A = 0$. Let now $\alpha \in G^*$ with

$|\alpha| < M$. Then with a suitable $k \in \mathbf{N}$, $\alpha\lambda^k \in R$, consequently $0(= A) = f(\alpha\lambda^k) = f(\alpha) + kf(\lambda) = f(\alpha)$.

The assertion is proved for $\alpha, |\alpha| < M$. But it is clear, if $\alpha \in R$. □

For $r \geq 1$ let $[r]_G$ be defined by

$$[r]_G = \max_{\alpha \in G^*, |\alpha| \leq r} |\alpha|.$$

It is clear that $r - 1 \leq [r]_G \leq r$.

Lemma 2. *Let δ be an arbitrary number in the interval $0 < \delta < 1$. Then there exists a constant $N_0(\delta)$ with the following property : If $f \in \mathcal{A}_G^*$ and $N \in \mathbf{R}$, $N > N_0(\delta)$, $f(\alpha) = A$ for $\alpha \in R_N := \{\alpha \in G^* : N \leq \alpha \leq (1+\delta)N\}$, then $f(\alpha) = 0$ for each $\alpha \in G^*$ in the disc $|\alpha| \leq \left[\frac{\delta N}{2} - 1\right]_G - 1$ (and so in $|\alpha| \leq \frac{\delta N}{2} - 3$).*

PROOF. Let $\varepsilon \in \{\pm 1, \pm i, 0\} = E$. Let $\beta \in G^*$, $|\beta| < N$.
If

$$\frac{N(1+\delta)}{|\beta| + 1} - \frac{N}{|\beta| - 1} \geq 1 \tag{2.1}$$

hold, the there exists $\mu \in G^*$ for which

$$\frac{N}{|\beta| - 1} \leq |\mu| \leq \frac{N(1+\delta)}{|\beta| + 1} \tag{2.2}$$

holds. But then

$$\frac{N}{|\beta + \varepsilon|} \leq |\mu| \leq \frac{N(1+\delta)}{|\beta + \varepsilon|}, \tag{2.3}$$

and so

$$N \leq |\,(\beta + \varepsilon)\mu\,| \leq N(1+\delta),$$

$(\beta + \varepsilon)\mu \in R_N$. Since $\varepsilon = 0 \in E$, therefore for each unit ε we have $A = f(\mu\beta) = f(\mu(\beta + \varepsilon))$, which gives that $f(\beta) = f(\beta + \varepsilon)$.

Now we shall prove that the inequality (2.1) holds, if $|\beta|$ runs in the range $L := \left[\frac{4}{\delta} + 1, \frac{\delta N}{2} - 1\right]$, whenever N is large enough. It is enough to prove that (2.1) holds at the endpoints of L. Let first $|\beta| = \frac{4}{\delta} + 1$. Substituting this into (2.1), the left hand side of (2.1) is $\frac{\delta^2 N}{2(4+2\delta)}$ and this is clearly ≥ 1, if $N \geq 8/\delta^2 + 4/\delta$. Let now $|\beta| = \frac{\delta N}{2} - 1$. Then the left hand side of (2.1) is $\frac{2\delta^2 N - 8 - 8\delta}{\delta^2 N - 4\delta}$, and this is ≥ 1, if $N > \frac{8}{\delta^2} + 4/\delta$.

So we have proved that $f(\beta) = f(\beta+\varepsilon)$, whenever $|\beta| \in L, \beta \in G^*$, and $N > \frac{8}{\delta^2} + \frac{4}{\delta}$. But then $f(\beta) =$ constant in the ring $|\beta| \in L$.

Observe that

$$\left[\frac{\delta N}{2} - 1\right]_G - 1 \geq \sqrt{2}\left(\left[\frac{4}{\delta} + 1\right]_G + 1\right)$$

for each large N. But then we can find such a $z \in G^*$ for which $(|z|, \sqrt{2}|z|) \subseteq L$. Then we can apply Lemma 1 with $|z|$ instead of M. This implies that $f(\alpha) = 0$ whenever $|\alpha| \leq \sqrt{2}|z|$. Since $f(\alpha) =$ constant when $|\alpha| \in L$, therefore $f(\alpha) = 0$ in the whole disc indicated in Lemma 2. □

Lemma 3. *Let $\varepsilon > 0$ be an arbitrary fixed constant. Then there exist positive numbers $N_1(\varepsilon), c(\varepsilon)$ with the following properties. If $f \in \mathcal{A}_G^*$, $a \in \mathbf{C}$ satisfying $|a| > N_1(\varepsilon)$, $r = (2+\varepsilon)\sqrt{|a|}$ and $f(\alpha) =$ constant $=$ A in the disc $\alpha \in S(a,r)$, then $f(\alpha) = 0$ for each $\alpha \in G^*$ in the disc $|\alpha| \leq c(\varepsilon)\sqrt{|a|}$.*

PROOF. For each $w \in \mathbf{C}$ the disc $S\left(w, \frac{\sqrt{2}}{2}\right)$ contains at least one Gaussian integer. Let $\beta \in G^*$, $|\beta| \leq \sqrt{2}r$. Then there exists $\mu \in G^*$ such that

$$\left|\mu - \frac{a}{\beta}\right| \leq \frac{1}{\sqrt{2}} \leq \frac{r}{|\beta|}. \tag{2.4}$$

Let $\varepsilon \in \{1, -1, i, -i\}$, and assume that β is so chosen that

$$\frac{1}{\sqrt{2}} + \frac{|a|}{|\beta||\beta+\varepsilon|} \leq \frac{r}{|\beta+\varepsilon|} \tag{2.5}$$

holds. Then

$$\left|\mu - \frac{a}{\beta+\varepsilon}\right| \leq \left|\mu - \frac{a}{\beta}\right| + \left|\frac{a}{\beta} - \frac{a}{\beta+\varepsilon}\right| \leq \frac{1}{\sqrt{2}} + \frac{a}{|\beta||\beta+\varepsilon|} \leq \frac{r}{|\beta+\varepsilon|},$$

consequently $\beta(\mu+\varepsilon), \beta\mu \in S(a,r)$. But this implies that $f(\beta) = f(\beta+\varepsilon)$ under the condition (2.5).

By simple computation we deduce that (2.5) holds for each $\varepsilon = \{\pm 1, \pm i\}$ if $|\beta|(|\beta|+1) - r|\beta| + |a| \leq 0$ holds. It is clear that this inequality holds in the interval $|\beta| \in (x_1, x_2)$, where $x_1 < x_2$ are the roots of the quadratic polynomial

$$x^2 - (r-1)x + |a| = 0.$$

It is clear that $x_1 = x_1(|a|) \to \infty$ if $|a| \to \infty$. Furthermore,

$$x_1 = \frac{(r-1) - \sqrt{(r-1)^2 - 4|a|}}{2}, \quad x_2 = \frac{(r-1) + \sqrt{(r-1)^2 - 4|a|}}{2}.$$

Observing that

$$\frac{2x_1}{r} = 1 - \sqrt{\left(1 - \frac{1}{r}\right)^2 - \frac{4|a|}{r^2}} - \frac{1}{r} = 1 - \sqrt{1 - \left(\frac{2}{(2+\varepsilon)}\right)^2} + \mathcal{O}\left(\frac{1}{r}\right),$$

and similarly that

$$\frac{2x_2}{r} = 1 + \sqrt{1 - \left(\frac{2}{2+\varepsilon}\right)^2} + \mathcal{O}\left(\frac{1}{r}\right),$$

we get that

$$\frac{x_2}{x_1} \to \frac{1 + \sqrt{1 - \left(\frac{2}{2+\varepsilon}\right)^2}}{1 - \sqrt{1 - \left(\frac{2}{2+\varepsilon}\right)^2}} (=: b) \quad \text{as} \quad |a| \to \infty.$$

Since $b > 1$, therefore choosing δ to be less than $b-1$, we get that $\frac{x_2(|a|)}{x_1(|a|)} > 1 + \delta$, if $|a|$ is large enough. But then the conditions of Lemma 2 are satisfied with $N := x_1(|a|)$. Observing that $x_1(|a|)$ has the same order as $\sqrt{|a|}$, we get our lemma immediately. □

3. Without any important change in the proof we can deduce the following assertion.

Theorem 2. *Let $f \in \mathcal{M}_G^*$, which does not take the zero value. Assume that there exists a sequence $z_1, z_2, \ldots$ of complex numbers such that $|z_\nu| \to \infty$ and that*

$$f(\alpha) = A_\nu = \text{constant on } \alpha \in S(z_\nu, (2+\varepsilon)\sqrt{|z_\nu|}) \quad .$$

with some arbitrary positive constant ε. Then $f(\alpha) = 1$, identically.

References

[1] I. Kátai, On the determination of an additive arithmetical function by its local behaviour, *Colloquium Math.* **20** (1969), 265–267.

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(Received August 18, 1987)