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Torsion and curvature in smooth loops

K. H. HOFMANN (Darmstadt) and KARL STRAMBACH (Erlangen)

Abstract. A Lie group is flat with respect to its natural left invariant affine connection, and its torsion relates to the Lie bracket via $T(X,Y) = -[X,Y]$. A smooth loop is, roughly, a Lie group without associativity. Its tangent algebra at the origin has, in addition to the binary bracket, a ternary operation which is a measure of deviation from associativity locally. There are several ways of giving a smooth loop affine connections. There is one for which the torsion behaves like in groups while the curvature relates to the ternary operation of the tangent algebra via $R(X,Y)(Z)_e = -\langle X_e, Y_e, Z_e\rangle + \langle Y_e, X_e, Z_e\rangle$.

Introduction

Each Lie group carries a canonical left invariant connection giving rise to torsion and curvature tensors. The torsion T associates with two left invariant vector fields X and Y their Lie bracket $T(X,Y) = -[X,Y]$ (up to sign), and the curvature R vanishes identically. Lie groups are flat in this sense. These matters are familiar and emerge already in any local theory dealing with differential geometric aspects of Lie theory. (Sample references: HELGASON 1978, KLINGENBERG 1984, GRAEUB 1961.)

Conversely, if one associates with a smooth manifold M endowed with a connection an exponential function $\mathrm{Exp} : T(M) \to M$ and considers a normal neighborhood U of any point $e \in M$ (see HELGASON 1978, p. 32 ff.), then one obtains a smooth binary partial multiplication

$$\mu : U \times U \to M, \quad \mu(x,y) = \mathrm{Exp}_x(\tau(x,e)(\mathrm{Exp}^{-1} y))$$
with the parallel transport $\tau(x,e) : T_e(M) \to T_x(M)$ along the unique geodesic connecting e and x in U.

The multiplication satisfies $\mu(e,x) = \mu(x,e) = x$ and allows unique solutions x and y of the equations $\mu(a,x) = b = \mu(y,a)$ for all a and b sufficiently close to e; such solutions are usually denoted $x = a \setminus b$ and

$y = b/a$ and they depend smoothly on a and b. (See e.g. KIKKAWA 1975 or SABININ and MIKHEEV 1985.) Thus we have produced all ingredients of a local Lie group with the sole exception of associativity of multiplication. Such "nonassociative" (local) Lie groups are called *(local) Lie loops,* or *smooth, respectively, analytical (local) loops.*

Lie loops have attracted the attention of authors in topological algebra (A.I. MALCEV 1955, SAGLE 1961 ff., HUDSON 1963 ff., KUZMIN 1968 ff.), in the foundations of geometry (SALZMANN 1957, FREUDENTHAL 1960; see also the survey by HOFMANN and STRAMBACH 1990) and in differential geometry (LOOS 1966, AKIVIS 1969 ff., KIKKAWA 1975 ff., SABININ 1981 ff.; see also the survey by GOLDBERG 1990 and the lecture notes by SABININ and MIKHEEV 1985). In particular, passing through various levels of generality, one has finally learned to associate with a (local) Lie loop G a tangent algebra $L(G) = T_e(G)$ which carries a bilinear anticommutative multiplication $(x, y) \mapsto [\![x, y]\!]$, called *commutator bracket*, and a trilinear operation $(x, y, z) \mapsto \langle x, y, z\rangle$, called *associator bracket* vanishing identically in the case of a (local) Lie group. The two operations are linked by the so called *Akivis identity*

$$(A) \qquad \Sigma\{\operatorname{sgn}(\sigma)\cdot\langle x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\rangle : \sigma \in S_3\} = [\![[\![x, y]\!], z]\!] + [\![[\![y, z]\!], x]\!] + [\![[\![z, y]\!], x]\!].$$

As in the Lie group case, the commutator bracket is a measure for noncommutativity near the identity; but on the top of that, the associator bracket is now a measure for nonassociativity near the identity. We call an algebra with a binary and a ternary multiplication satisfying (A) an *Akivis algebra*, and, in particular, the tangent algebra of a (local) Lie loop its Akivis algebra. One notes that (A) reduces to the Jacobi identity in the Lie group case. (See AKIVIS 1976, HOFMANN and STRAMBACH 1987, 1990.)

The objective of this paper is to nominate reasonable canonical connections on a (local) Lie loop and to compute the associated torsions and curvatures with the particular aim of linking these with the algebraic data of the Akivis algebra of the loop.

For this purpose we observe a general procedure by which certain connection may be constructed on a smooth manifold M. We shall say that we have *a transitive family* $m(x, y) : M \to M$ *of diffeomorphisms of* M, if each function $m(x, y)$ is a diffeomorphism such that $(x, y, z) \mapsto m(x, y)(z)$ is smooth and the relations $m(x, x) = 1_M$ and $m(x, y)(y) = x$ hold for all $x, y \in M$. (It would suffice that $m(x, y)$ is defined on an open neighborhood of y.) For each such family we obtain a smooth family of linear maps $p(x, y) : T_y(M) \to T_x(M)$ via $p(x, y) = d(m(x, y))_y$ satisfying $p(x, x) = 1_{T_x(M)}$ for all x. Such a family we shall also call a *linear transport family on* M. This family is at the heart of the matter, because we can define a connection on M via

$\nabla_X(Y) = \left.\frac{d}{dt}\right|_{t=0} p(\gamma(t),x)^{-1}Y_{\gamma(t)}$ with a smooth curve γ which uniquely solves the initial value problem $\gamma'(t) = X_{\gamma(t)}, \quad \gamma(0) = x$, locally.

Let us say that this connection is *associated with the transitive family* m *of diffeomorphisms.*

Now we consider a (local) Lie loop M. Then we have at the very least two natural transitive families of diffeomorphisms related to left translations λ_x (given by $\lambda_x(y) = x \circ y$ with the loop multiplication $\circ$)

I) *The first left canonical family:* $m(x,y)(z) = x \circ (y \setminus z)$,

II) *The second left canonical family:* $m(x,y)(z) = (x/y) \circ z$.

We note $m(x,y) = \lambda_x(\lambda_y)^{-1}$ in the case of the first family. This implies the identity

$$m(x,y)m(y,z) = m(x,z) \quad \text{and} \quad p(x,y)p(y,z) = p(x,z).$$

We shall call such families *associative.* In a (local) Lie group, both families coincide, but not so in nonassociative loops. Accordingly, we have a *first left canonical connection* and a *second left canonical connection.*

We shall show that *the curvature for a connection associated with an associative transitive family of diffeomorphisms vanishes identically.* (Theorem 2.7). As a consequence, *the first left canonical curvature vanishes identically on a (local) Lie loop.* In other words, with respect to the first left canonical connection, Lie loops are flat.

The main result of our paper, however is the following:

Theorem A. *On each (local) Lie loop the torsion and curvature associated with the second left canonical connection and the Akivis algebra of the loop are related as follows:*

(*i*) $T(X,Y)_e = [\![Y_e, X_e]\!]$,
(*ii*) $R(X,Y)(Z)_e = -\langle X_e, Y_e, Z_e\rangle + \langle Y_e, X_e, Z_e\rangle$

for all smooth vector fields X, Y, *and* Z.

This calls for some remarks.

Firstly, the result concerning the torsion corresponds to the Lie group situation. As it seems to be common, it is comparatively easy to establish.

Secondly, in the Lie group case, identity (ii) is a special aspect of the more general fact, that the curvature vanishes identically. Seen in this light, curvature and nonassociativity are linked in the context of the second left canonical connection. This deserves to be emphasized, since the published record indicates that it is not obvious how a connection is to be defined in such a way that this phenomenon becomes visible.

Thirdly, one might ask the following question. Via the torsion, the commutator bracket of the Akivis algebra is determined geometrically. Is the same true for the associator bracket, i.e., can the associator bracket be retrieved from the torsion and curvature in general? Since the curvature $R(X,Y)(Z)$ is skew symmetric in the arguments X and Y while simple examples show that the associator bracket may be invariant under all permutations of its arguments, the answer is no. This calls for a small algebraic elaboration which we shall give in the text. Thus an identity like (ii) is about the best that one can rightfully expect.

At this point, some historical background information is called for. The first author to connect the algebraic operations of the tangent algebra of a local Lie loop with differential geometric invariants was Akivis (AKIVIS and SHELEKHOV 1971a). Since it is important to recognize the relation between his procedure and ours, we shall explain his approach in some detail, Akivis is primarily interested in the differential geometry of webs which was initiated by BLASCHKE in the thirties. This allows him to utilize the close relationship between loop theory and the geometry of 3–webs. In fact each Lie loop G determines a 3–web of three families of smooth foliations on $G \times G$, consisting of the *horizontal* leaves $G \times \{a\}$, $a \in G$, the *vertical* leaves $\{a\} \times G$, $a \in G$, and the *transversal* leaves $\{(x,y) : x \circ y = a\}$, $a \in G$, respectively. (In the geometric literature, these leaves are referred to as the lines of the web.) Conversely, each smooth 3–web gives rise to a class of Lie loops which are equivalent in a sense which is very precisely understood. On the 3–web $G \times G$ we have a connection, which was introduced by CHERN in his dissertation under BLASCHKE as early as 1935. This connection is called the *Chern connection of the 3–web.* (See KIKKAWA 1985.) It gives rise to a torsion $T^{\wedge}$ and a curvature $R^{\wedge}$ on the product manifold $G \times G$. We shall use the abbreviations

$$C(X,Y) = T^{\wedge}((X,0),(0,Y)), \text{ and} \tag{1}$$

$$A(X,Y,Z) = R^{\wedge}((X,0),(0,Y))(Z,0), \tag{2}$$

where we have represented each vector field on $G \times G$ in the form $(X^{(h)}, X^{(v)})$ with its horizontal component $X^{(h)}$ and its vertical component $X^{(v)}$. In terms of the tensor fields C and A we can express the torsion and curvature as follows:

$$\begin{aligned} &T^{\wedge}((X^{(h)},X^{(v)}),(Y^{(h)},Y^{(v)})) \\ &= C(X^{(h)},Y^{(h)}) + C(X^{(v)},Y^{(v)}), \end{aligned} \tag{3}$$

$$\begin{aligned} &R^{\wedge}((X^{(h)},X^{(v)}),(Y^{(h)},Y^{(v)}))(Z^{(h)},Z^{(v)}) \\ &= A(X^{(h)},Y^{(v)})(Z^{(h)}) - A(Y^{(h)},X^{(v)})(Z^{(h)}) \\ &+ A(X^{(h)},Y^{(v)})(Z^{(v)}) - A(Y^{(h)},X^{(v)})(Z^{(v)}). \end{aligned} \tag{4}$$

(See AKIVIS and SHELEKHOV 1971a), KIKKAWA 1985). (There is, as usual, a local version of this theory; it is in the local form that AKIVIS and his school present this frame.) Akivis' main result in this direction is

Theorem B. (AKIVIS and SHELEKHOV 1971a). *If the tensor fields C and A are derived from the torsion and curvature of the Chern connection on the 3–web associated with a (local) Lie loop as indicated in (1) and (2) above, then*

$$(i) \qquad C(X,Y)_e = [\![Y_e, X_e]\!],$$
$$(ii) \qquad A(X,Y,Z)_e = -\langle X_e, Y_e, Z_e\rangle.$$

The Chern connection induces a connection on all horizontal and all vertical leaves. In particular, if the connection induced on $G \times \{a\}$ is transported to G we obtain a connection on G. It was observed by NAGY 1985 that *this connection is the first left canonical connection.* The vertical leaves induce the first right canonical connection. It thus follows that the curvature induced in horizontal and vertical leaves is identically zero : They are flat with respect to the Chern connection. This connection was reconsidered by KIKKAWA 1985, and independently of the web–aspect, the first left canonical connection was investigated on special Lie loops by SABININ and MIKHEEV 1985.

In fact, quite in contrast with the second left canonical connection which we treat in this paper and which is capable of detecting curvature in the presence of nonassociativity, the first left canonical connection is made to order for the treatment of local geodesic Lie loops such as we have mentioned in the beginning of this introduction. Such local geodesic loops always have certain additional algebraic properties. If, in addition, the manifold we consider is symmetric, then the local geodesic Lie loop satisfies the identity $a(ab) = a^2b$ (locally). This identity is called the left Bol identity. (See SABININ and MIKHEEV, 1985, p. 13, 14.) Conversely, if a Lie loop satisfying the left Bol identity is given, then the first left canonical connection (which, as we have remarked, is always flat!) allows a geodesic Lie loop multiplication to be defined, but in the end it turns out to agree with the given Lie loop multiplication. (See SABININ and MIKHEEV 1985, p. 59, Theorem 5. See also NAGY 1986 and, for certain special cases, KIKKAWA 1975, p. 168, Theorem 5.7.)

The tensor fields C and A given by (1) and (2) can be identified internally in the loop. For this purpose we recall that for a smooth function $f : X \times Y \to Z$ we have partial derivatives such as $(D_1f)(x,y) : T_x(X) \to T_{f(x,y)}(Z)$ and mixed second partial derivatives $(D_1D_2f)(x,y) : T_x(X) \otimes T_y(Y) \to T_{f(x,y)}(Z)$. (Since we are exlusively interested in the local theory, it suffices to visualize these concepts for smooth functions defined on an open subset of $\mathbf{R}^n$ taking values in $\mathbf{R}^m$.) Now for a (local) Lie loop G we let $\mu : G \times G \to G$ denote the multiplication $\mu(x,y) = x \circ y$ and $\kappa(x,y) : G \to G$ the family of diffeomorphisms given by $\kappa(x,y) = \lambda_{x\circ y}^{-1}\lambda_x\lambda_y$. Set $\overline{\kappa}(x,y,z) = \kappa(x,y)(z)$.

Theorem C. *In a local Lie loop we have*

$$(i) \qquad C(X,Y)_e = -(D_1D_2\mu)(e,e)(X_e \otimes Y_e - Y_e \otimes X_e)$$
$$(ii) \qquad A(X,Y,Z)_e = -(D_1D_2D_3\overline{\kappa})(e,e,e)(X_e \otimes Y_e \otimes Z_e)$$

for all smooth vector fields X, Y, and Z.

This result was proved by KIKKAWA 1985 for a special class of loops (namely, loops with the left inverse property). It is noteworthy that in the same paper, KIKKAWA considers yet another type of transitive family of diffeomorphisms $m(x,y)$ which is given by

$$\text{III)} \qquad m(x,y)(z) = y \circ ((y \setminus x) \circ (y \setminus z)).$$

For a special class of Lie loops arising in the context of homogeneous spaces, he shows that for the connection associated with this family the torsion and curvature yield essentially the same formulae as in we have in Theorem A. We shall show in the last section of this paper that our methods yield this result without any special hypotheses on the loop.

Our paper is organized as follows. Since our applications address the local theory of Lie loops only, we can restrict our attention completely to manifolds which are open subsets of $\mathbf{R}^n$, and thus we can stay within the purview of the calculus of several variables. In order to keep the presentation self contained and elementary, we record the necessary background material. The calculations themselves tend to get involved. We have made a concerted effort to use coordinate free notation. In existing literature, the excessive use of coordinate calculations has obscured the matter more than it has elucidated it.

We first study connections with transitive families of diffeomorpisms in general, and we calculate the torsion and curvature in this situation (Theorem 2.6). Then we apply this theorem to the case of the second left canonical family of diffeomorphisms on a local Lie loop. In our last section we provide supplementary information such as a proof of Theorem C above and a discussion of the torsion and curvature associated with the Kikkawa family III) above.

In our treatment of local Lie loops we stay within the domain of local analytical loops. But this is merely a matter of convenience. The essential fact is that we use power series computations up through terms of order three. Thus our conclusions remain intact for local differentiable loops of class C^k with $k \geq 3$.

The conceptual purpose of the paper is to elucidate the interrelation of the infinitesimal algebra (the Akivis algebra) of a local analytical loop and the geometric concepts of torsion and curvature. This requires a substantial amount of calculations such as arise in any context where accuracy up to terms of degree 3 is necessary. We have opted for including the calculations because we believe them to be nontrivial and because we have found the probability for errors to be too high in order to leave them entirely to the reader.

1. Background information from calculus

We consider an open subset B of a finite dimensional real vector space L, and we shall have occasion to regard other finite dimensional vector spaces L_1, L_2 etc. The vector spaces $\mathrm{Hom}(L_1, \mathrm{Hom}(L_2, L_3))$ and $\mathrm{Hom}(L_1 \otimes L_2, L_3)$ shall be identified via the rule

$$(1) \qquad F(u)(v) = F(u \otimes v) \quad \text{with} \quad u \in L_1 \quad \text{and} \quad v \in L_2.$$

If $f : B \to L_1$ is a differentiable mapping, then the derivative f' is a mapping
$f' : B \to \mathrm{Hom}(L, L_1)$, i.e. $f'(x) : L \to L_1$ is linear; if f is twice differentiable, then f'' is a mapping $f'' : B \to \mathrm{Hom}(L, \mathrm{Hom}(L, L_1)) = \mathrm{Hom}(L \otimes L, L_1)$ according to (1) so that $f''(x)(u \otimes v)$ is a meaningful expression denoting a vector in L_1.

Any function $X : B \to L$ is called a *vector field on* B; instead of $X(x)$ we shall write X_x. For our purposes we will not be encumbered by assuming that all function, including vector fields are smooth (i.e., are of class C^∞). Every vector field X on B operates on the vector space $C^\infty(B, \mathbf{R})$ of $C^\infty-$ functions via $X(f)(x) = f'(x)(X_x)$, thus yielding a $C^\infty-$ function $X(f) : B \to \mathbf{R}$. The product rule $(fg)'(x)(u) = f'(x)(u)g(x) + f(x)g'(x)(u)$ immediately confirms that X operates as a derivation on $C^\infty(B, \mathbf{R})$:

$$(2) \qquad X(fg)(x) = (Xf)(x)g(x) + f(x)(Xg)(x).$$

(It is convenient to allow the abbreviation $(Xf)(x)$ for $X(f)(x)$.)

On several occasions we shall refer to the following Lemma :

Lemma 1.1. *Let* $f : B \to \mathrm{Hom}(L_1, L_2)$ *and*
$g : B \to \mathrm{Hom}(L_2, L_3)$ *be smooth function and define*
$g \# f : B \to \mathrm{Hom}(L_1, L_3)$ via $(g \# f)(x) = g(x) \circ f(x)$.
The derivative $(g \# f)' : B \to \mathrm{Hom}(L, \mathrm{Hom}(L_1, L_3))$ *is then computed as follows:*

$$(g \# f)'(x)(u)(v) = g'(x)(u)(f(x)(v)) + g(x)f'(x)(u)(v), u \in L, \ v \in L.$$

PROOF. We define the composition $C : \mathrm{Hom}(L_2, L_3) \times \mathrm{Hom}(L_1, L_2) \to \mathrm{Hom}(L_1, L_3)$ by $C(q,p) = q \circ p$ and the diagonal $F : B \to B \times B$ by $F(x) = (x, x)$. Then $C'(q,p)(r,s) = C(r,p) + C(q,s)$, since C is bilinear, and since $f \# g = C \circ (f \times g) \circ F$ we compute by the chain rule that $(g \# f)'(x) = C'(g(x), f(x)) \circ (g'(x) \times f'(x)) \circ F$, or $(g \# f)'(x)(u)(v) = C(g'(x)(u), f(x))(v) + C(g(x), f'(x)(u))(v)$, which transforms into our assertion.

If X is a vector field on B, we may identify it with a function $f : B \to \mathrm{Hom}(\mathbf{R}, L)$ via $X_x = f(x)(1)$. An application of Lemma 1.1 then yields the following corollary :

Lemma 1.2. *Let X be a smooth vector field on B and $g : B \to \mathrm{Hom}(L, L_1)$ a smooth function, and define $g(X)(x) = g(x)(X_x)$. Then $g(X) : B \to L_1$ and*

$$g(X)'(x)(u) = g'(x)(u)(X_x) + g(x)(X'_x(u)). \ \square$$

This Lemma applies to the smooth function $X(f)$ for a smooth vector field X and a smooth function f. We recall $X(f)(x) = f'(x)(X_x) = f'(X)(x)$ in the notation of Lemma 1.2. Hence Lemma 1.2 yields

$$X(f)'(x)(u) = f''(x)(u)(X_x) + f'(x)(X'_x(u)). \tag{3}$$

As a consequence, if Y is a second smooth vector field on B, as an operator on smooth functions we can form $YX(f)$ and by the definition $Y(g) = g'(x)(Y_x)$ obtain from (3) the relation

$$YX(f)(x) = f''(x)(Y_x)(X_x) + f'(x)X'_x(Y_x). \tag{4}$$

Since for any twice continuously differentiable function f we have

$$f''(x)(u \otimes v) = f''(x)(v \otimes u), \tag{5}$$

we derive from (4) and (5) with the definition $[X, Y] = XY - YX$ the equation

$$[X, Y](f)(x) = f'(x) \circ (Y'_x(X_x) - X'_x(Y_x)). \tag{6}$$

We defined the Lie bracket $[X, Y]$ in terms of operators on $C^\infty(B, \mathbf{R})$; but relation (6) allows us to interpret it as a vector field as follows:

$$[X, Y]_x = Y'_x(X_x) - X'_x(Y_x). \tag{7}$$

Lemma 1.3. *Let $F : B \to G\ell(L) \subseteq \mathrm{Hom}(L, L)$ be a smooth function and X a smooth vector field on B. We define a vector field Y on B by $Y_x = F(x)^{-1}(X_x)$. Then*

$$Y'_x(u) = -F(x)^{-1}F'(x)(u)(F(x)^{-1}X_x) + F(x)^{-1}(X'_x(u)),\ u \in L. \tag{8}$$

Proof. We set $g(x) = F(x)^{-1}$ and thus define a smooth function $g : B \to \mathrm{Hom}(L, L)$. With this notation we have $Y = g(X)$ in the terminology of Lemma 1.2. Hence Lemma 1.2 yields

$$Y'_x(u) = g(X)'(x)(u) = g'(x)(u)(X_x) + g(x)(X'_x(u)).$$

Now we write $g = I \circ F$ with $I : G\ell(L) \to \mathrm{Hom}(L, L)$ given by $I(f) = f^{-1}$. By the chain rule, $g'(x) = I'(F(x)) \circ F'(x)$. Now $I'(f)(u) = -f^{-1}uf^{-1}$ for $u \in \mathrm{Hom}(L, L)$; this is an exercise resulting from the expansion $(f+u)^{-1} - f^{-1} = ((1 + f^{-1}u)^{-1} - 1)f^{-1} = -f^{-1}uf^{-1} + \ldots$
Hence $g'(x)(u)(v) = I'(F(x))(F'(x)(u))(v) = -F(x)^{-1}\,(F'(x)(u))\,F(x)^{-1}$ (v). Substitution of this expression gives the assertion. □

We conclude our preliminary section by recalling the definition of the concept of an affine connection on B and of the associated ideas of a torsion and a curvature field.

Definition 1.4. An *affine connection* on B is a function ∇ which assigns to a pair (X, Y) of smooth vector fields a vector field $\nabla_X Y$ satisfying the following two conditions:

$$\begin{aligned} (i)\quad & \nabla_{fX+gY} = f\nabla_X + g\nabla_Y \ \text{ for } \ f, g \in C^\infty(B, \mathbf{R}) \\ (ii)\quad & \nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2 \ \text{ and } \ \nabla_X(fY) \\ & = X(f)Y + f\nabla_X Y \ \text{ for } \ f \in C^\infty(B, \mathbf{R}). \end{aligned}$$

We notice that we wrote fX for the vector field $(fX)_x = f(x)X_x$.

Definition 1.5. If ∇ is an affine connection on B, then the *torsion field is defined* by

$$(i) \qquad T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

and the *curvature field is defined* by

$$(ii) \qquad R(X, Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

2. The affine connection associated with a transitive family of diffeomorphisms

Definition 2.1. Let B be an open subset of the finite dimensional real vector space L. By a *transitive family of (local) diffeomorphisms of B* we understand a family of functions $m(x, y) : B \to L$, $x, y \in B$ which satisfy the following conditions:

- (i) The function $\overline{m} : B \times B \times B \to L$, $\overline{m}(x, y, z) = m(x, y)(z)$ is smooth.
- (ii) Each $m(x, y)$ is a diffeomorphism onto its image, an open subset of L.
- (iii) $m(x, x)$ is the inclusion map of B in L, i.e., $m(x, x)(z) = z$.
- (iv) $m(x, y)(y) = x$ for all $x, y \in B$.

By a *linear transport family* we understand a family of linear maps $p(x, y) : L \to L$, $x, y \in B$ which satisfy the following conditions:

(v) $p : B \times B \to \mathrm{Hom}(L, L)$ is smooth.
(vi) $p(x, x) = \mathbf{1}_L$ for all $x \in B$.

We shall denote with $(D_k f)(x_1, \ldots, x_n)$ the partial derivative with respect to x_k.

Proposition 2.2. *If m is a transitive family of local diffeomorphisms on B, then the definition $p(x,y) = m(x,y)'(y) = (D_3\overline{m})(x,y,y)$ yields a linear transport family.*

PROOF. Every $p(x,y)$ is in $\mathrm{Hom}(L,L)$ and p is smooth by 2.1. (i). Since $m(x,x)$ is the identity on B, then $m(x,x)'(y) = \mathbf{1}_L$ for all $x, y \in B$; in particular we have (vi). □

The family p defined by $p(x,y) = m(x,y)'(y)$ will be called the *linear transport family associated with the family m.*

In order to give an orientation of the ideas involved, let us assume that we have a Lie group structure on B; the simplest case is $B = L$ and $xy = x + y$. Then the equation $m(x,y)(z) = xy^{-1}z$ defines a transitive family of diffeomorphisms of B. This family has the additional property that $m(x,y)m(y,z) = m(x,z)$ holds for all $x, y, z \in B$ and that $m(y,x) = m(x,y)^{-1}$. Nothing of this kind is assumed in Definition 2.1 and this is crucial for the later developments; most of it would become trivial in this situation.

If we denote with $\lambda_x : B \to B$ the left translation given by $\lambda_x(y) = xy$, then $m(x,y)(z) = \lambda_{xy^{-1}}(z)$, whence $m(x,y)' = \lambda'_{xy^{-1}}$ and thus $p(x,y) = \lambda'_{xy^{-1}}(y)$ in this example. Since we have $m(x,y) = \lambda_x\lambda_y^{-1}$, the chain rule secures $p(x,y) = \lambda'_x(e)\lambda'_y(e)^{-1}$ with the identity e of B in this example.

In the remainder of this section we consider a linear transport family p on B as given. It may or may not arise as the linear transport family associated with a transitive family of local diffeomorphisms. We shall attach to the family p an affine connection on B which will then allow us to introduce a torsion field and a curvature field on B.

We recall that for a smooth vector field X on B and each $x \in B$ the differential equation $\gamma'(t) = X_{\gamma(t)}$, has a unique maximal solution with the initial value $\gamma(0) = x$. We call it the *X –trajectory through x.*

Definition 2.3. Let X and Y be smooth vector fields on B and γ the X –trajectory through x. We set $\psi(t) = p(\gamma(t), x)^{-1}Y_{\gamma(t)}$, notice $\psi(0) = Y_x$ by 2.1. (iii) and define

$$(\nabla_X Y)_x = \psi'(0).$$

For a function p in two variables x and y we denote with D_1p the partial derivative with respect to the first variable x and with D_2p the partial derivative with respect to the second variable y.

Proposition 2.4.

$$(\nabla_X Y)_x = -(D_1p)(x,x)(X_x)(Y_x) + Y'_x(X_x)$$
$$= -(D_1p)(x,x)(X_x \otimes Y_x) + Y'_x(X_x).$$

PROOF. We fix x and set $F(z) = p(z,x)$ and $Z_z = F(z)^{-1}(Y_z)$. Then $\psi(t) = Z_{\gamma(t)}$, and by the chain rule we have $\psi'(0) = Z'_{\gamma(0)}(\gamma'(0))$ $= Z'_x(X_x) = F(x)^{-1}F'(x)(X_x)(F(x)^{-1}Y_x) + F(x)^{-1}(Y'_x(X_x))$ by Lemma 1.3. Now we notice that $F(x) = p(x,x) = \mathbf{1}_B$ by 2.1. (vi) and that $F'(x) = (D_1p)(x,x)$. This proves the assertion. □

Proposition 2.5. *The assignment ∇ of 2.2 and 2.3 is an affine connection on B.*

PROOF. We have to verify conditions (i) and (ii) of 1.4. But by the linearity of $(D_1p)(x,x)$ and Y'_x, condition (i) is immediate from 2.4, as is the additivity of $\nabla_X Y$ in Y. But if f is a smooth function on B, then $\nabla_X(fY)_x = -(D_1p)(x,x)(X_x)(f(x)Y_x) + (fY)'_x(X_x)$. The first summand is simply $-f(x)\cdot(D_1p)(x,x)(X_x \otimes Y_x)$. By Lemma 1.2, the second summand is $f'(x)(X_x).Y_x + f(x)\cdot(Y'_x(X_x)) = (X(f)Y)_x + f(x).Y'_x(X_x)$, since $f'(x)(X_x) = X(f)(x)$. We thus obtain $\nabla_X(fY)_x = f(x)\cdot(-(D_1p)(x,x)$ $(X_x \otimes Y_x) + Y'_x(X_x)) + (X(f)Y)_x = f(x)\cdot(\nabla_X Y)_x + (X(f)Y)_x$ which is the remainder of condition (ii) in 1.4. □

We shall say that the affine connection ∇ is *associated with the given linear transport family* p and that the corresponding torsion and curvature fields are also associated with this family.

Theorem 2.6. *The torsion and curvature fields associated with a linear transport family $p(x,y)$ on B are described as follows :*

(i) $T(X,Y)_x = (D_1p)(x,x)(Y_x \otimes X_x - X_x \otimes Y_x)$.
(ii) $R(X,Y)(Z)_x = (D_1p)(x,x)(X_x \otimes (D_1p)(x,x)(Y_x \otimes Z_x)$
$\quad - Y_x \otimes (D_1p)(x,x)(X_x \otimes Z_x))$
$\quad - (D_2D_1p)(x,x)(X_x \otimes Y_x \otimes Z_x - Y_x \otimes X_x \otimes Z_x)$.

Complement. *If the linear transport family $p(x,y)$ is associated with a transitive family of local diffeomorphisms $m(x,y)$ and if we set $\overline{m}(x,y,z)$ $= m(x,y)(z)$, then in the Theorem we may substitute $D_1p(x,x) =$ $(D_1D_3\overline{m})(x,x,x)$. However, in general $(D_2D_1p)(x,x) \neq$ $(D_2D_1D_3\overline{m})(x,x,x) = (D_1D_2D_3\overline{m})(x,x,x)$.*

PROOF. As a first step we shall abbreviate $(D_1p)(x,x)$ by $P(x)$. Then we have

$$(\nabla_X Y)_x = -P(x)(X_x \otimes Y_x) + Y'_x(X_x). \tag{1}$$

Now we calculate $T(X,Y)_x = (\nabla_X Y)_x - (\nabla_Y X)_x - [X,Y]_x =$ $-P(x)(X_x \otimes Y_x - Y_x \otimes X_x) + Y'_x(X_x) - X'_x(Y_x) - [X,Y]_x$, which in view of equation (7) in Section 1 is exactly assertion (i) above.

The calculation of the curvature is more involved. First we calculate $(\nabla_Y \nabla_X Z)_x = (\nabla_Y U)_x$ with $U_x = -P(x)(X_x \otimes Z_x) + Z'_x(X_x)$ by (1) above. Now $(\nabla_Y U)_x = -P(x)(Y_x \otimes U_x) + U'_x(Y_x)$ by the same token. We need $U'_x(u)$ for which we have to calculate the derivative of $x \mapsto U_{1x}$ $= -P(x)(X_x \otimes Y_x) = -P(x)V_x$ with $V_x = X_x \otimes Z_x$. Then by Lemma 1.2 we have

$$
\begin{aligned}
(2) \qquad U'_{1x}(u) &= - P'(x)(u)(V_x) - P(x)(V'_x(u)) \\
&= - P'(x)(u)(X_x \otimes Z_x) - P(x)(X'_x(u) \otimes Z_x \\
&\quad + X_x \otimes Z'_x(u)),
\end{aligned}
$$

in view of the product rule. Next we have to calculate the derivative of $x \mapsto U_{2x} = Z'_x(X_x)$ for which we again invoke Lemma 1.2:

$$
(3) \qquad U'_{2x}(u) = Z''_x(u)(X_x) + Z'_x(X'_x(u)).
$$

This allows us to write down, using (2) and (3):

$$
\begin{aligned}
(4) \quad U'_x(Y_x) =& U'_{1x}(Y_x) + U'_{2x}(Y_x) \\
=& - P'(x)(Y_x)(X_x \otimes Z_x) - P(x)(X'_x(Y_x) \otimes Z_x) \\
& - P(x)(X_x \otimes Z'_x(Y_x)) + Z''_x(Y_x)(X_x) + Z'_x(X'_x(Y_x))
\end{aligned}
$$

In view of

$$
\begin{aligned}
(5) \qquad -P(x)(Y_x \otimes U_x) =& - P(x)(Y_x \otimes -P(x)(X_x \otimes Z_x)) \\
& - P(x)(Y_x \otimes Z'_x(X_x))
\end{aligned}
$$

we calculate $(\nabla_Y U)_x$ collecting (5) and (4) :

$$
\begin{aligned}
& (\nabla_Y \nabla_X Z)_x = P(x)(Y_x \otimes P(x)(X_x \otimes Z_x)) && \text{(row 1)} \\
& - P(x)(X_x \otimes Z'_x(Y_x) + Y_x \otimes Z'_x(X_x)) && \text{(row 2)} \\
(6) \qquad & - P(x)(X'_x(Y_x) \otimes Z_x) && \text{(row 3)} \\
& - P'(x)(Y_x)(X_x \otimes Z_x) && \text{(row 4)} \\
& + Z'_x(X'_x(Y_x)) && \text{(row 5)} \\
& + Z''_x(Y_x)(X_x) && \text{(row 6).}
\end{aligned}
$$

As a next step we have to exchange X and Y in (6) and subtract (6) from the result to obtain $(\nabla_X \nabla_Y Z)_x - (\nabla_Y \nabla_X Z)_x$. Since row 2 is symmetric in X and Y, it will drop out in this process. Also row 6 is symmetric in X and Y by $Z''_x(Y_x)(X_x) = Z''_x(Y_x \otimes X_x)$ in view of relation (5) in Section 1. Hence row 6, too, will drop out in the process. We must record the rest:

$$\begin{aligned}
&(\nabla_X \nabla_Y Z)_x - (\nabla_Y \nabla_X Z)_x = \\
&= P(x)(X_x \otimes P(x)(Y_x \otimes Z_x) - Y_x \otimes P(x)(X_x \otimes Z_x))(\text{row } 1) \\
(7)\quad &- P(x)([X,Y]_x \otimes Z_x) \qquad (\text{row } 2) \\
&- P'(x)(X_x)(Y_x \otimes Z_x) + P'(x)(Y_x)(X_x \otimes Z_x) \qquad (\text{row } 3) \\
&+ Z'_x([X,Y]_x) \qquad (\text{row } 4),
\end{aligned}$$

where, in rows 2 and 4, we have used $Y'_x(X_x) - X'_x(Y_x) = [X,Y]_x$ according to (7) in Section 1.

As a next step we have to calculate $(\nabla_{[X,Y]}Z)_x$. In view of (1) above this is done quickly :

$$(8) \qquad \begin{aligned}(\nabla_{[X,Y]}Z)_x = &- P(x)([X,Y]_x \otimes Z_x) \quad (\text{row } 1) \\ &+ Z'_x([X,Y]_x) \qquad (\text{row } 2).\end{aligned}$$

In order to find $R(X,Y)(Z)_x$, by 1.5.ii, we have to subtract (8) from (7), and we notice that row 2 of (7) and row 1 of (8) cancel, and that row 4 of (7) and row 2 of (8) cancel. Thus $R(X,Y)(Z)_x$ is the sum of rows 1 and 3 in (7). In order to recognize that this result will prove assertion (ii) we have to recall, firstly, that $P(x) = (D_1 p)(x,x)$; this will take care of row 1 in (7) and the first summand in (ii). Secondly, we have to evaluate $P'(x)(u)$: From $P(x) = (D_1 p)(x,x)$ we derive

$$(9) \qquad P'(x)(u) = (D_1^2 p)(x,x)(u) + (D_2 D_1 p)(x,x)(u),$$

and in row 3 of (7) we have to evaluate $P'(x)(u)(v)(w) = P'(x)(u \otimes v \otimes w)$ at $(-X_x \otimes Y_x + Y_x \otimes X_x) \otimes Z_x$. If we momentarily fix x and set $q(y) = p(y,x)$, then $(D_1^2 p)(y,x) = q''(y)$, and by condition (5) in Section 1 we have $q''(y)(u \otimes v - v \otimes u) = 0$ for all $y \in B$ and $u, v \in L$. Hence $(D_1^2 p)(x,x)(-X_x \otimes Y_x + Y_x \otimes X_x) = 0$ and hence row 3 in (7) reduces to $(D_2 D_1 p)(x,x)(-X_x \otimes Y_x \otimes Z_x + Y_x \otimes X_x \otimes Z_x)$ which gives exactly the second summand of (ii). The proof is complete, since the Complement is straightforward from the definitions. □

A linear transport family p on B will be called *associative* if $p(x,y)p(y,z) = p(x,z)$ and if $p(x,y) = p(y,x)$ holds. We then fix a point e in B and define a smooth family of linear maps $t(x) : L \to L$ by setting $t(x) = p(x,e)$. Then we have $p(x,y) = p(x,e)p(e,y) = t(x)t(y)^{-1}$ and

$t(e) = p(e,e) = \mathbf{1}_L$. Conversely, if a smooth family $t(x)$ of linear maps is given for $x \in B$ and $t(e) = \mathbf{1}_L$, then p defined by $p(x,y) = t(x)t(y)^{-1}$ is a linear associative transport family.

We calculate the partial derivatives D_1p and D_2D_1p for such a family. By the chain rule

$$(D_1p)(x,y)(u) = t'(x)(t(y)^{-1}u). \tag{9}$$

If we define a vector field Y by $Y_x = t(x)^{-1}u$, then by Lemma 1.3 we have

$$Y'_x(v) = -t(x)^{-1}t'(x)(v)(t(x)^{-1}u). \tag{10}$$

This gives

$$(D_2D_1p)(x,y)(u \otimes v \otimes w) = -t'(x)(t(y)^{-1}t'(y)(u)(t(y)^{-1}v)(w). \tag{11}$$

Now we calculate the curvature $R(X,Y)(Z)_x =$
$= t'(x)(t(x)^{-1}X_x)(t'(x)(t(x)^{-1}Y_x)(Z_x)-$
$-t'(x)(t(x)^{-1}Y_x)(t'(x)(t(x)^{-1}X_x)(Z_x)+$
$+t'(x)(t(x)^{-1}t'(x)(X_x)(t(x)^{-1}Y_x)(Z_x)-$
$-t'(x)(t(x)^{-1}t'(x)(Y_x)(t(x)^{-1}X_x)(Z_x) = 0.$

We therefore obtain the following

Theorem 2.7. *The curvature tensor field associated with an associative linear transport family vanishes identically. Its torsion field is given by*

$$T(X,Y)_x = t'(x)(t(x)^{-1}Y_x)(X_x) - t'(x)(t(x)^{-1}X_x)(Y_x),$$

where $p(x,y) = t(x)t(y)^{-1}$. □

In this sense, the curvature is a measure of the non–associativity of the linear transport field.

3. The affine connections associated with a local analytical loop

In this Section we have to refer to some background material which has been discussed in greater detail in HOFMANN and STRAMBACH 1987 and 1990. In these sources we gave references to other literature, notably from the schools of AKIVIS and SAGLE. We recapitulate the definition:

Definition 3.1. A local analytical loop is an open neighborhood B of 0 in a finite dimensional real vector space L together with formal power series in two variables on L which converges on $B \times B$ and thus defines an analytical function

$$(x,y) \mapsto x \circ y = x + y + q(x,y) + r(x,x,y) + s(x,y,y) +$$

homogeneous summands of degree 4 or more:

$$B \times B \to L$$

with a bilinear map $q : L \times L \to L$ and two trilinear maps $r, s : L \times L \times L \to L$.

In a local analytical loop we have $0 \circ x = x \circ 0 = x$ for all $x \in B$, and for all $a, b \in B$ which are close enough to 0 the equations

$$a \circ y = b \quad \text{and} \quad x \circ a = b \tag{1}$$

have unique solutions $y = a \setminus b$ and $x = b/a$ which depend analytically on a and b.

We may and will assume in the following that B is chosen so small, that the analytical functions $(x, y) \mapsto x \setminus y,\ \ x/y$ are well defined and analytic on $B \times B$.

We need an explicit formula for x/y in the sense that we have to know the expansion of x/y up to degree 3.

Lemma 3.1.

$$\begin{aligned}
a)\quad & x/y = x - y - q(x,y) + q(y,y) + \\
& + q(q(x,y),y) - q(y,y),y) - \\
& - r(x,x,y) + r(x,y,y) + r(y,x,y) - r(y,y,y) - \\
& - s(x,y,y) + s(y,y,y) + \\
& + \textit{summands of higher homogeneous degree.}
\end{aligned}$$

$$\begin{aligned}
b)\quad & x \setminus y = -x + y - q(x,y) + q(x,x) + \\
& + q(x, q(x,y)) - q(x, q(x,x)) - \\
& - s(x,y,y) + s(x,x,y) + s(x,y,x) - s(x,x,x) - \\
& - r(x,x,y) + r(x,x,x) + \\
& + \textit{summands of higher homogeneous degree.}
\end{aligned}$$

These formulae are valid for all sufficiently small x and y.

PROOF. a) We denote the right hand side of the asserted equation with R. For a proof we now calculate $R \circ y$ up to terms of homogeneous degree three. The straightforward calculation then shows that $R \circ y = y +$ summands of homogeneous degree four or beyond. But this proves a).

b) is proved analogously. □

Lemma 3.2. *For all sufficiently small x, y, z we have*

$$\begin{aligned}
(x/y) \circ z = & x - y + z - q(x,y) + q(y,y) + q(x,z) - q(y,z) + \\
& + q(q(x,y),y) - q(q(y,y),y) - q(q(x,y),z) + q(q(y,y),z) - \\
& - r(x,x,y) + r(x,y,y) + r(y,x,y) - r(y,y,y) + \\
& + r(x,x,z) - r(x,y,z) - r(y,x,z) + r(y,y,z) - \\
& - s(x,y,y) + s(y,y,y) + s(x,z,z) - s(y,z,z) + \\
& + \textit{summands of higher homogeneous degree.}
\end{aligned}$$

PROOF. We substitute the expression for $u = x/y$ according to formula 3.1.a into the series for $u \circ z$ according to 3.1 and obtain the result by straightforward computation. □

Now we assume that B is chosen so small, that for all x, y, z the expression $\overline{m}(x, y, z) = (x/y) \circ z$ is well–defined and thus gives us a transitive family $m(x, y) : B \to L$ of diffeomorphisms given by $m(x, y)(z) = (x/y) \circ z$. In the Introduction, we have called this family the second left canonical family of the local analytical loop. According to Section 2, there is an affine connection ∇ associated with this family which is expressed in terms of the associated linear transport family

$$p(x, y) = (D_3\overline{m})(x, y, y) = m(x, y)'(y),$$

and its partial derivative

$$(D_1 p)(x, y) = (D_1 D_3 \overline{m})(x, y, y).$$

Finally, for the computation of the associated curvature according to Theorem 2.6, we also need a computation of $(D_2 D_1 p)(x, x)$ which we shall give in lemma 3.5 below.

Lemma 3.3. $(D_3\overline{m})(x, y, z)(w) = w + q(x, w) - q(y, w)-$

$$\begin{aligned}
&-q(q(x, y), w) + q(q(y, y), w)+\\
&+r(x, x, w) - r(x, y, w) - r(y, x, w) + r(y, y, w)+\\
&+s(x, w, z) + s(x, z, w) - s(y, w, z) - s(y, z, w)+
\end{aligned}$$

+summands whose homogeneous degree in x, y, z is 3 or more.

PROOF. This follows directly from Lemma 3.2: In order to form the partial derivative with respect to z and to evaluate it at w all terms not containing z vanish; in the terms which are linear in z we substitute w for z, and in the terms which are quadratic in z we proceed according to the Leibniz Rule:

If $F : L_1 \times \ldots \times L_n \to L$ is a multilinear map, then

$$F'(x_1, \ldots, x_n)(u_1 \otimes \ldots \otimes u_n) = F(u_1, x_2, \ldots, x_n)+ +F(x_1, u_2, x_3, \ldots, x_n) + \ldots + F(x_1, \ldots, x_{n-1}, u_n).$$

Lemma 3.4. $(D_1 D_3\overline{m})(x, y, z)(v \otimes w) =$

$$\begin{aligned}
&=q(v, w) - q(q(v, y), w)+\\
&+r(v, x, w) + r(x, v, w) - r(v, y, w) - r(y, v, w)+\\
&+s(v, w, z) + s(v, z, w)+
\end{aligned}$$

+summands whose homogeneous degree in x, y, z is 2 or more.

PROOF. Proceed as in Lemma 3.3.

Lemma 3.5. $(D_2D_1p)(x,y)(u\otimes v\otimes w) =$

$$\begin{aligned}&= -q(q(v,u),w) - r(v,u,w) - r(u,v,w) + \\ &\quad + s(v,w,u) + s(v,u,w) + \\ &\quad + \textit{summands whose homogeneous degree} \\ &\qquad \textit{in } x \textit{ and } y \textit{ is } 1 \textit{ or more.}\end{aligned}$$

PROOF. From Lemma 3.4 we obtain $(D_1p)(x,y) =$ $= (D_1D_3\overline{m})\,(x,y,y) = q(u,v) + q(q(v,y),w) + r(v,x,w) + r(x,v,w)$ $-r(v,y,w) - r(y,v,w) + s(v,w,y) + s(v,y,w) + \dots$

Now we have to compute the derivative of this expression with respect to y and to evaluate at u. The procedure is as in Lemma 3.3.

It is very instructive at this point to observe the warning in the Complement to Theorem 2.6. A calculation of $(D_2D_1D_3\overline{m})(x,y,z)(u\otimes v\otimes w)$ shows that this expression equals $q(q(v,u),w) - r(v,u,w) - r(u,u,w)+$ summands of degree 1 or more in x,y,z. In this way we lose the s–terms, and the result differs from $(D_2D_1p)(x,y)(u\otimes v\otimes w)$.

But now we are ready to compute the torsion and curvature according to Theorem 2.6. In fact from 2.6.i and the value for $(D_1p)(x,x) = (D_1D_3\overline{m})(x,x,x)$ from 3.4 we obtain

$$(i_x\ \text{-})\qquad \begin{aligned}T(X,Y)_x =\,& q(Y_x,X_x) - q(X_x,Y_x) \\ & -q(q(Y_x,x),X_x) + q(q(X_x,x),Y_x) + \\ & +s(Y_x,X_x,x) - s(X_x,Y_x,x) + \\ & +s(Y_x,x,X_x) - s(X_x,x,Y_x) + \\ & +\text{summands which are} \\ & \quad \text{at least quadratic in } x.\end{aligned}$$

In particular, for $x = 0$ we obtain

$$(i_0)\qquad T(X,Y)_0 = q(Y_0,X_0) - q(X_0,Y_0).$$

In principle, Theorem 2.6.ii and 3.4 combined with 3.5 allow us to compute the curvature $R(X,Y)(Z)_x$ through the linear terms. We are, however, interested in the evaluation at 0 and thus obtain

$$(ii_0\ +)\qquad \begin{aligned}R(X,Y)(Z)_0 =\,& q(X_0,q(Y_0,Z_0)) - q(Y_0,q(X_0,Z_0)) \\ & + q(q(Y_0,X_0),Z_0) - q(q(X_0,Y_0),Z_0) - \\ & - s(Y_0,Z_0,X_0) + s(X_0,Z_0,Y_0) - \\ & - s(Y_0,X_0,Z_0) + s(X_0,Y_0,Z_0).\end{aligned}$$

(Notice that the sum $r(u,v,w)+r(v,u,w)$ is symmetric in u and v and thus drops out upon antisymmetrizing in these two variables!)

At this point we pause for a moment with the differential geometry of our local analytical loop and recall its Akivis algebra which was mentioned in the Introduction. If we have a local analytical loop in the sense of Definition 3.1, we may identify L with the tangent space of this loop at the identity 0. Then L carries an Akivis algebra structure with respect to the

commutator bracket $[\![x,y]\!] = \lim t^{-2}((tu \circ tv)/(tv \circ tu))$ for $t \to 0$, and the

associator bracket $\langle x,y,z\rangle = \lim t^{-3}(((tu \circ tv)\circ tw)/(tu \circ (tv \circ tw)))$ for $t \to 0$.

For more details we refer to Hofmann and Strambach 1986. There one also finds the following result (Theorem IX.6.6.), due to Akivis and Shelekhov 1971a):

Lemma 3.6.

(*i*) $[\![u,v]\!] = q(u,v) - q(v,u)$

(*ii*) $\langle u,v,w\rangle = q(q(u,v),w) - q(u,q(v,w)) + r(u,v,w) + r(v,u,w) - s(u,v,w) - s(u,w,v)$,

where q,r and s are as in 3.1. □

The following is then an immediate consequence:

Lemma 3.7. *In the Akivis algebra of a local analytical loop we have the identity*

$$(iii)\qquad -\langle u,v,w\rangle + \langle v,u,w\rangle = q(u,q(v,w)) - q(v,q(u,w)) + q(q(v,u),w) - q(q(u,v),w) - s(v,w,u) + s(u,w,v) - s(v,u,w) + s(u,v,w). \quad \square$$

We recognize immediately that the right had side of this formula agrees with the expression we found for the curvature $R(X,Y)(X)_0$ in 0 if we set $u = X_0, v = Y_0$ and $w = Z_0$. We have now arrived at Theorem A in the introduction. We summarize this in the following statements:

Definition 3.8. Let a local analytical loop $(B,L,\circ)$ be given as in Lemma 3.1 and assume that B is small enough so that the analytical functions $(x,y)\mapsto x\circ y,\ x/y,\ x\setminus y : B\times B \to L$ and $(x,y,z)\mapsto (x/y)\circ z : B\times B\times B \to L$ are well defined. Then the affine connection given by $(\nabla_X)(Y)_x = -(D_1p)(x,x)(X_x\otimes Y_x) + Y'_x(X_x) = Y'_x(X_x) + q(X_x,Y_x) - q(q(X_x,x),Y_x) + s(X_x,Y_x,x) + s(X_x,x,Y_x) +$ terms of higher order in x will be called the ***second left canonical connection*** of the analytical loop. The torsion and curvature computed relative to this connection will be called the ***second left canonical torsion*** and the ***second left canonical curvature.***

Theorem 3.9. *The Akivis algebra and the second left canonical torsion and curvature of a local analytical loop are linked by the following equations:*

$$(i)\quad T(X,Y)_0 = -[\![X_0, Y_0]\!],$$
$$(ii)\quad R(X,Y)(Z)_0 = -\langle X_0, Y_0, Z_0\rangle + \langle Y_0, X_0, Z_0\rangle. \ \square$$

As we have indicated in the introduction, we call *first left canonical connection* the one which is associated with the associative transitive family of diffeomorphisms $m(x,y)$ given by $\overline{m}(x,y,z) = x \circ (y \backslash z)$. By Theorem 2.7, its curvature vanishes identically, and its torsion in 0 is readily calculated to agree with that of the second left canonical connection.

Of course, we can develop the whole theory based on right translations. The following remarks show, how that is reduced to the present one.

If $x \circ y = x + y + q(x,y) + r(x,x,y) + s(x,y,y) + \dots$ describes the multiplication of a local analytical loop, we can define a new local analytical loop

$$x \circ^\star y = x + y + q^\star(x,y) + r^\star(x,x,y) + s^\star(x,y,y) + \dots$$

by setting $q^\star(x,y) = q(y,x), r^\star(x,y,z) = s(z,y,x)$, and $s^\star(x,y,z) = r(z,y,x)$. Then $x \circ^\star y = y \circ x$. If ϱ_x is the local right translation of the original loop given by $\varrho_x(y) = y \circ x = x \circ^\star y$, then $\varrho_x = \lambda_x^\star$, and the right canonical affine connection on B for the given local analytical loop is the left canonical affine connection of the local analytical loop defined by $\circ^\star$. Its commutator bracket $[\![u,v]\!]^\star$ equals $-[\![u,v]\!]$, and its associator bracket is $\langle u,v,w\rangle^\star = q^\star(q^\star(u,v),w) - q^\star(u, q^\star(v,w)) + r^\star(u,v,w) + r^\star(v,u,w)$ $-s^\star(u,v,w) - s^\star(u,w,v) = q(w,q(v,w)) - q(q(w,v),u) + s(w,v,u)$ $+s(w,u,v) - r(w,v,u) - r(v,w,u) = -\langle w,v,u\rangle$.
Consequently $-\langle u,v,w\rangle^\star + \langle v,u,w\rangle^\star = \langle w,v,u\rangle - \langle w,u,v\rangle$.
These remarks give us the following corollary of Theorem 3.9.

Corollary 3.10. *The Akivis algebra and the second right canonical torsion and curvature of a local analytical loop are linked by the following equations:*

$$(i)\quad T^\star(X,Y)_0 = [\![X_0, Y_0]\!],$$
$$(ii)\quad R^\star(X,Y)(Z) = \langle Z_0, Y_0, X_0\rangle - \langle Z_0, X_0, Y_0\rangle. \ \square$$

The following is an immediate consequence of Theorem 3.9:

Corollary 3.11. *If the associator bracket of the Akivis algebra of a local analytical loop satisfies*

$$\text{(B)} \qquad \langle u,u,v\rangle = 0 \quad \textit{for all} \quad u,v,$$

then the second left canonical curvature in is computed as follows:

$$R(X,Y)(Z)_0 = -2\langle X_0, Y_0, Z_0\rangle. \ \square$$

Condition (B) which says that the associator bracket is skew symmetric in its first two arguments is satisfied in the Akivis algebras of all analytical local left Bol loops (see SABININ and MIKHEEV 1985) and all homogeneous loops (KIKKAWA 1975, HOFMANN and STRAMBACH 1986a). In these cases, the curvature and the torsion determine the Akivis algebra completely.

At this point it is a natural impulse to ask whether this is always the case. The answer is no. A closer look shows that this is an algebraic question which we discuss in the following remarks.

Suppose that L is an Akivis algebra; we consider the vector space $E = \mathrm{Hom}(L \otimes L \otimes L, K)$ over the ground field K (which is $\mathbf{R}$ in the case of our application). We have a distinguished element $A \in E$ given by $A(u \otimes v \otimes w) = \langle u, v, w \rangle$. The group S_3 operates on E via $(\sigma^{-1} f)(x_1 \otimes x_2 \otimes x_3) = f(x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)})$ and thus makes E into an R–module on the left, where $R = K[S_3]$ is the group algebra of S_3 over K. We denote with $\tau \in S_3$ the transposition (12) and with γ the cyclic permutation (123). Theorem 3.7 tells us that if we know the left canonical curvature of a local loop whose Akivis algebra is L, then we know $(\mathbf{1} - \tau)A$ since $((\mathbf{1} - \tau)A)(u \otimes v \otimes w) = A(u \otimes v \otimes w) - A(v \otimes u \otimes w) = \langle u, v, w \rangle - \langle v, u, w \rangle$. Since we are allowed to perform all permutations on the arguments, we know in fact the submodule $R(\mathbf{1} - \tau).A$ generated by $(\mathbf{1} - \tau) \cdot A$ in the R–module E. If τ' denotes the transposition (23), from Corollary 3.10 and the same argument we know the R–module $R(\mathbf{1} - \tau') \cdot A$; and $\tau' = \gamma\tau$.

Lemma 3.12. *If $e = \frac{1}{6}\Sigma\{\sigma : \sigma \in S_3\}$ denotes the symmetrisation operator then we have the following conclusions:*

- (*i*) *e is a central idempotent, and $R = Re \oplus R(1 - e)$.*
- (*ii*) *$\dim Re = 1$ and $R(1 - e) = R(\mathbf{1} - \tau) + R(\mathbf{1} - \gamma\tau)$.*

PROOF. (*i*) is immediate from the definitions (see also HOFMANN and STRAMBACH 1986).
(*ii*) $\dim Re = 1$ is equivalent to $\dim R(1 - e) = 5$ and was shown in HOFMANN and STRAMBACH 1986.
Since $(\mathbf{1} - \tau)e = (\mathbf{1} - \gamma\tau)e = 0$, we have $R(\mathbf{1} - \tau) + R(\mathbf{1} - \gamma\tau) \subseteq R(1 - e)$. But the five elements $\mathbf{1} - \tau$, $\gamma - \gamma\tau$, $\gamma^2 - \gamma^2\tau$, $\mathbf{1} - \gamma\tau$, and $\gamma - \gamma^2\tau$ are linearly independent, whence $\dim (R(\mathbf{1} - \tau) + R(\mathbf{1} - \gamma\tau)) \geq 5$, and (*ii*) follows.

This gives us the following result:

Proposition 3.13. *If L is an Akivis algebra over K, $E = \mathrm{Hom}(L \otimes L \otimes L, K)$ as an $R = K[S_3]$ left module for the natural action of S_3 on E, and if $A(u \otimes v \otimes w) = \langle u, v, w \rangle$ then the R–module generated by the two forms $u \otimes v \otimes w \mapsto \langle u, v, w \rangle - \langle v, u, w \rangle$, $\langle u, v, w \rangle - \langle u, w, v \rangle$ is cyclic and is, in fact $R \cdot (1 - e)A$ with the symmetrisation operator e of 3.12. Furthermore, $R \cdot eA = K \cdot eA$ is one–dimensional.*

Finally the following two statements are equivalent:

$$(1) \qquad A \in R(1-e)A.$$
$$(2) \qquad eA = 0.$$

PROOF. The first two assertions follow from Lemma 3.12, and we have to show that (1) and (2) are equivalent. From Lemma 3.12 we know that $R\cdot A = KeA + R(1-e)A$. But this sum is direct, since $keA = r(1-e)A$ implies $keA = e(keA) = r\cdot e(1-e)A = 0$. Hence $A \in R(1-e)A$ is equivalent to $eA = 0$. □

We have seen before that for the case that L is the Akivis algebra of a local analytical loop, from the curvature and torsion we obtain the submodule $R(1-e)A$. Proposition 3.3 says that from this information we can retrieve A if and only if $eA = 0$. Thus, in order to show that this is not generally possible, we have to exhibit an example of a nonzero A with $eA \neq 0$. There are in fact 1–dimensional examples of this type:

Example 3.14. Let $L = \mathbf{R}$ and consider the local analytical loop $\circ : B \times B \to \mathbf{R}$ defined on a sufficiently small neighborhood of 0 by the (globally defined) multiplication $x \circ y = x + y + x^2 y$. Then $A(u \otimes v \otimes w) = \langle u, v, w\rangle = 2uvw$, whence $0 \neq A = eA$ since A is invariant under all permutations. □

4. Supplements

We have observed that torsion and curvature for the second left canonical connection do not completely determine the Akivis algebra of a local analytical group. In fact, the algebraic arguments we noted in 3.13 show that no curvature tensor can determine the associator bracket in general. The first part of this section provides a general proof of Theorem C of the Introduction in the following form:

Theorem 4.1. *Let* $\mu : B \times B \to L,\ \mu(x,y) = x \circ y$ *be a local analytical loop and set* $\kappa(x,y) = \lambda_{x\circ y}{}^{-1}\lambda_x\lambda_y, \overline{\kappa}(x,y,z) = \kappa(x,y)(z) =$
$= (x \circ y) \setminus (x \circ (y \circ z))$. *Then*

(*i*) $(D_1 D_2 \mu)(0,0)(u \otimes v - v \otimes u) = [\![u, v]\!]$
(*ii*) $(D_1 D_2 D_3 \overline{\kappa})(0,0,0)(u \otimes v \otimes w) = -\langle u, v, w\rangle$.

PROOF. (*i*) From Definition 3.1 we calculate directly that
$(D_1 D_2 \mu)(x,y)(u \otimes v) = q(u,v) + r(u,x,v) + r(x,u,v) + s(u,v,y) + s(u,y,v)$
+ summands of homogeneous degree 2 or more in x and y.
Hence $(D_1 D_2 \mu)(0,0) = q(u,v)$ and (*i*) follows.

Before we complete the proof by showing (*ii*) we derive a result which is of interest in itself and from which (*ii*) will follow readily.

Proposition 4.2. *For all sufficiently small elements in a local analytical loop according to 3.1 we have*
$(x\circ y)\setminus(x\circ(y\circ z)) = z - \langle x,y,z\rangle$
+ summands of homogeneous degree 4 or more.

PROOF. The proof is by straightforward calculation. In the following indications we compute modulo summands of homogeneous degree 4 or more.

a) $x\circ(y\circ z) = x+y+z+q(x,y)+q(x,z)+q(y,z)+q(x,q(y,z))+ r(x,x,y)+r(x,x,z)+r(y,y,z)+s(x,y,y)+s(x,y,z) +s(x,z,y)+s(x,z,z)+s(y,z,z).$

b) We set $u = x\circ y$ according to 3.1 and $v = x\circ(y\circ z)$ according to a) above and substitute these expressions into the power series for $u\setminus v$ according to 3.1.*b*, recording only summands up to homogeneous degree 3.

$$\begin{aligned}
&u\setminus v = -u+v-q(u,v)+q(u,u)+q(u,q(u,v))-\\
&-q(u,q(u,u))-s(u,u,u)-s(u,v,v)+s(u,u,v)+s(u,v,u)+\\
&+r(u,u,u)-r(u,u,v)=\\
&=-\,(x+y+q(x,y)+r(x,x,y)+\\
&+s(x,y,y))+x+y+z+q(x,y)+q(x,z)+q(y,z)+\\
&+q(x,q(y,z))+r(x,x,z)+r(y,y,z)+s(x,y,y)+s(x,y,z)+\\
&+s(x,z,y)+s(x,z,z)+s(y,z,z)-\\
&-q(x{+}y{+}q(x,y),x{+}y{+}z{+}q(x,y){+}q(x,z){+}q(y,z)+\\
&+q(x{+}y{+}q(x,y),\ x{+}y{+}q(x,y))+\\
&+q(x{+}y{+}q(x,y),\ q(x{+}y,x{+}y{+}z))-\\
&-q(x{+}y{+}q(x,y),\ q(x{+}y,x{+}y))-\\
&-s(x{+}y,x{+}y,x{+}y)-s(x{+}y,x{+}y{+}z,x{+}y{+}z)+\\
&+s(x{+}y,x{+}y,x{+}y{+}z)+s(x{+}y,x{+}y{+}z,x{+}y)+\\
&+r(x{+}y,\ x{+}y,\ x{+}y)-r(x{+}y,\ x{+}y,\ x{+}y{+}z)
\end{aligned}$$

An evaluation of this sum results in enough cancellation to yield the asserted result. □

Now we can readily complete the Proof of part (*ii*) of Theorem 4.1. Indeed if we set $\overline{\kappa}(x,y,z) = \kappa(x,y)(z) = (x\circ y)\setminus(x\circ(y\circ z))$, then Proposition 4.2 implies $(D_1D_2D_3\overline{\kappa})(x,y,z)\,(u\otimes v\otimes w) = -\langle u,v,w\rangle+\dots$ whence $(D_1D_2D_3\overline{\kappa})(0,0,0)\,(u\otimes v\otimes w) = -\langle u,v,w\rangle$. This completes the proof of 4.1. *ii*.

In the remainder of this section we investigate a third transitive family of local diffeomorphisms which was suggested by Kikkawa 1985 in the special case of loops with the so–called left inverse property. This family is defined by

$$III) \quad m(x,y)(z) = y \circ ((y \setminus x) \circ (y \setminus z)),$$

which is to be seen in competition with the first left canonical family

$$I) \quad m(x,y)(z) = x \circ (y \setminus z),$$

and the second left canonical family

$$II) \quad m(x,y)(z) = (x/y) \circ z.$$

The general process we have described in this paper associates with the Kikkawa family a connection for which we can calculate the torsion and the curvature. This we shall outline in the following.

Proposition 4.3. *For all sufficiently small elements in a local analytical loop according to 3.1 we have, except for summands of homogeneous degree 4 or more :*

$$\begin{aligned} y \circ ((y \setminus x) \circ (y \setminus z)) =& x - y + z + q(x-y, z-y) - \\ & - q(x-y, q(y, z-y)) - \langle y, x-y, z-y \rangle + \\ & + r(x,x,z-y) - r(y,y,z-y) + \\ & + s(x-y, z-y, z-y) + \dots \end{aligned}$$

PROOF. The proof is by straightforward calculation, although it is advisable to organize one's bookkeeping. By Lemma 3.1. b we obtain explicit expressions for $y \setminus x$ and $y \setminus z$, e.g.

$$\begin{aligned} y \setminus x = x - y - q(y, x-y) + q(y, q(y,x) - q(y,y)) - s(y, x-y, x-y) \\ - r(y,y,x-y) + \dots . \end{aligned}$$

With these expressions we go into formula a) in the proof of 4.2 and find

$$\begin{aligned} y \circ ((y \setminus x) \circ (y \setminus z)) = x - y + z + q(x-y, z-y) \\ - q(x-y, q(y, z-y)) + R + \dots \end{aligned}$$

where

$$\begin{aligned} R =& -q(q(y, x-y), z-y) + q(y, q(x-y, z-y)) + \\ & + r(x-y, x-y, z-y) + s(x-y, z-y, z-y) + \\ & + s(y, x-y, z-y) + s(y, z-y, x-y) = \\ =& (-q(q(y, x-y), z-y) + q(y, q(x-y, z-y)) - \\ & - r(y, x-y, z-y) - r(x-y, y, z-y) + \\ & + s(y, x-y, z-y) + s(y, z-y, x-y)) + r(x,x,z-y) - \\ & - r(y,y,z-y) + s(x-y, z-y, z-y) \end{aligned}$$

The assertion now follows from the formula for the associator bracket in 3.6. □

From the transitive family $m(x,y)$ we derive the associated linear transport family $p(x,y)$ according to the formula $p(x,y)(w) = (D_3\overline{m})(x,y,y)(w)$ with $\overline{m}(x,y,z) = y \circ ((y \setminus x) \circ (y \setminus z))$. We obtain in this fashion

$$p(x,y)(w) = w + q(x-y,w) - q(x-y,q(y,w)) - \langle y, x-y, w\rangle \\ + r(x,x,w) - r(y,y,w)$$

(noticing that the s–terms drop out).

This allows us to calculate

$$(D_1p)(x,y)(v\otimes w) = q(v,w) - q(v,q(y,w)) - \langle y,v,w\rangle \\ + r(x,v,w) + r(v,x,w),$$

and, as a consequence,

$$(D_2D_1p)(x,y)(u\otimes v\otimes w) = -\,q(v,q(u,w)) - \langle u,v,w\rangle \\ + r(u,v,w) + r(v,u,w).$$

Theorem 4.4. *Let $\circ : B\times B\to L$ be a local analytical loop and T and R the torsion and curvature derived from the connection associated with the Kikkawa family III) of local diffeomorphisms. Then*

(*i*) $T(X,Y)_0 = -[\![X_0,Y_0]\!]$,
(*ii*) $R(X,Y)(Z)_0 = \langle X_0,Y_0,Y_0\rangle - \langle Y_0,X_0,Z_0\rangle$.

PROOF. By Theorem 2.6 we have
$T(X,Y)_0 = (D_1p)(0,0)(Y_0\otimes X_0 - X_0\otimes Y_0) =$
$= q(Y_0,X_0) - q(X_0,Y_0) = [\![Y_0,X_0]\!]$ by 3.6.

Likewise, from Theorem 2.6 we obtain

$$\begin{aligned} R(X,Y)(Z)_0 &= (D_1p)(0,0)(X_0\otimes(D_1p)(0,0)(Y_0\otimes Z_0) - \\ &- Y_0\otimes(D_1p)(0,0)(X_0\otimes Z_0)) - \\ &- (D_2D_1p)(0,0)(X_0\otimes Y_0\otimes Z_0 - Y_0\otimes X_0\otimes Z_0) = \\ &= q(X_0,q(Y_0,Z_0)) - q(Y_0,q(X_0,Z_0)) + \langle X_0,Y_0,Z_0\rangle - \\ &- \langle Y_0,X_0,Z_0\rangle + q(Y_0,q(X_0,Z_0)) - q(X_0,q(Y_0,Z_0)). \ \square \end{aligned}$$

We observe that the curvatures calculated in Theorem 3.9 and in Theorem 4.4 differ by a minus sign.

The preceding results show the dependence between the Akivis algebra of a local analytical loop and torsion and curvature may be based on various transitive families of local diffeomorphisms, but the first and the second left canonical family remain the most natural ones.

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K. H. HOFMANN
FACHBEREICH MATHEMATIK TECHNISCHE HOCHSCHULE DARMSTADT
SCHLOSSGARTENSTR.
7 D–6100 DARMSTADT
FEDERAL REPUBLIC OF GERMANY
ELECTRONIC ADDRESS: XMATDA41@DDATHD21.BITNET

KARL STRAMBACH
MATHEMATISCHES INSTITUT UNIVERSITÄT ERLANGEN–NÜRNBERG
BISMARCKSTRAßE 1 1/2
D–8520 ERLANGEN
FEDERAL REPUBLIC OF GERMANY
ELECTRONIC ADDRESS: @ CNVE. RRZE. UNI–ERLANGEN. DBP. DE

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