Publ. Math. Debrecen
38 / 3–4 (1991), 225–229

Skew pure projective abelian groups

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In [2], we have investigated skew projective abelian groups. Recently J.D. REID and BERLINGHOFF have studied the structure of quasi–pure projective abelian groups. In this paper we study the structure of skew pure projective (s.p.p.) torsion abelian groups. We prove the following theorem assuming the generalized continuum hypothesis.

Theorem. *A torsion abelian group $G = \sum G_p$ is skew pure projective if and only if each G_p is a direct sum of cyclic groups, or a direct sum of a divisible group and a bounded p–group (primary p–group).*

All groups considered are abelian.

Section 1. We start with the following:

Definition 1.1. A group G is called skew pure projective if every diagram of the following type

$$\begin{array}{c} 0 \to S \to G \xrightarrow{\eta} G/S \to 0 \\ \qquad\qquad\quad \downarrow \alpha \\ \qquad\qquad\quad G/S \end{array}$$

with S pure in G can be completed to

$$\begin{array}{c} 0 \to S \to G \xrightarrow{\eta} G/S \to 0 \\ \qquad \downarrow \theta \qquad \downarrow \alpha \\ 0 \to S \to G \xrightarrow[\eta]{} G/S \to 0 \end{array}$$

where η is the natural map and α and θ are endomorphisms of G/S and G respectively such that $\alpha\eta = \eta\theta$.

Remark 1.2. Every direct summand of an s.p.p. group G is s.p.p.

The proof of the above remark is obvious.

Remark 1.3. A torsion group G is s.p.p. if and only if each of its primary components is.

PROOF. We notice that if $G = \sum G_p$ and S is a pure subgroup of G, and $S = \sum S_p$ then $G/S \simeq \sum G_p/S_p$ and since the G_p's are fully invariant, the endomorphisms of G/S are specified by those of G_p/S_p.

Section 2. In view of remark 1.3, in this section we consider only p–primary s.p.p. groups and we study their structure.

Lemma 2.1. *Let $G = A \oplus B$ be s.p.p. If A is a pure epimorphic image of B, then $A \simeq$ to a direct summand of B.*

PROOF. Let $\beta : A \simeq B/S$ where S is a pure subgroup (p.s.g.) of B. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] & A \oplus B \arrow[r, "1_A+\delta=\nu"] \arrow[d, "\theta"] & A \oplus B/S \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & S \arrow[r] & A \oplus B \arrow[r, "1_A+\delta=\nu"] & A \oplus B/S \arrow[r] & 0
\end{tikzcd}$$

Let α be the endomorphism of $A \oplus B/S$ where $\alpha|B/S = \beta^{-1}$ and $\alpha|_A = \beta$.

Now by hypothesis there exists a $\theta : A \oplus B \to A \oplus B$ such that $\nu\theta = \alpha\nu$. Clearly $\theta(A) \subseteq B$. Also if $a \neq 0$ and $\theta(a) = 0$, then $\nu\theta(a) = 0$ whence $\alpha\nu(a) = \alpha(a) = 0$ which contradicts that $\alpha \,|\, A$ is an isomorphism. Thus θ is monic on A. Then we have $A \xrightarrow{\theta} B \xrightarrow{\alpha^{-1}\nu} A$ so that $\alpha^{-1}\nu\theta = \alpha^{-1}\alpha\nu = 1_A$. Hence A is isomorphic to a direct summand of B.

Lemma 2.2. *Let $G = D \oplus R$, where D is divisible and R is reduced, be s.p.p. If $D \neq 0$, then R must be bounded.*

PROOF. Consider $G = Z(p^\infty) \oplus R \oplus D_1$. Then clearly $G_1 = R \oplus Z(p^\infty)$ is s.p.p. Now if R is unbounded R contains a pure subgroup S such that $R/S \simeq Z(p^\infty)$. Hence by Lemma 2.1 $Z(p^\infty) \simeq$ a direct summand of R which is reduced, a contradiction.

Proposition 2.3. *Let C be s.p.p. and let R be pure in C. Then C/R is s.p.p. if and only if the following holds:*

($\star$) *For every pure subgroup $L \supseteq R$ and for every endomorphism θ of C such that $\theta(L) \subseteq L$, there exists an endomorphism σ of C such that $\sigma(R) \subseteq R$ and $(\theta - \sigma)(C) \subseteq L$.*

PROOF. Assume C/R is s.p.p., L a pure subgroup $\supseteq R$. Let $\eta : C \to C/R$; $\overline{\eta} : C/R \to C/L$. Let θ be an endomorphism of C such that

$\theta(L) \subseteq L$. Let $\overline{\theta}$ be the endomorphism on C/L induced by θ. Now by hypothesis there exists $\overline{\sigma} : C/R \to C/R$ such that $\overline{\eta}\,\overline{\sigma} = \overline{\theta}\overline{\eta}$.

Since C is s.p.p. there exists a $\sigma : C \to C$ such that $\eta\sigma = \overline{\sigma}\eta$. Also $\eta\sigma(R) = \overline{\sigma}\eta(R) = 0$. Hence $\sigma(R) \subseteq R$. Further consider $\overline{\eta}\eta(\theta - \sigma)(c) = \overline{\eta}(\eta\theta(c)) - \overline{\eta}(\eta\sigma(c)) = \overline{\eta}(\theta(c) + R) - \overline{\eta}(\overline{\sigma}\eta(c)) = [\theta(c) + L] - \overline{\theta}\overline{\eta}(c + R) = [\theta(c) + L] - \overline{\theta}(c + L) = [\theta(c) + L] - [\theta(c) + L] = 0$ of C/L. Hence $(\theta - \sigma)(c) \in L$ for every c i.e. $(\theta - \sigma)(C) \subseteq L$.

Conversely, let C be an s.p.p. satisfying ($\star$). Then we claim C/R is s.p.p. where R is pure in C. Identifying a given pure quotient of C/R as C/L where L is pure in C containing R, let $\overline{\theta}$ be an endomorphism of C/L, and let $\eta : C \to C/R$ and $\overline{\eta} : C/R \to C/L$ be the natural maps. Now since C is s.p.p. there exists a $\theta = C \to C$ such that $\overline{\theta}\overline{\eta}\eta = \overline{\eta}\eta\theta$. Also it is clear that $\theta(L) \subseteq L$ since $(\overline{\eta}\eta)\theta(L) = \overline{\theta}\overline{\eta}\eta(L) = 0$ of C/L. Hence by ($\star$) there exists a $\sigma : C \to C$ such that $\sigma(R) \subseteq (R)$ and $(\theta - \sigma)(C) \subseteq L$.

Let $\overline{\sigma}$ be the induced map on C/R. Now consider $\overline{\eta}\,\overline{\sigma}(c+R) = \overline{\eta}(\sigma(c) + R) = \sigma(c) + L$, and $\overline{\theta}\overline{\eta}(c + R) = \overline{\theta}(c + L) = \theta(c) + L$. Also $(\theta - \sigma)(c) \in L$ for every $c \in C$. Hence $\overline{\eta}\,\overline{\sigma} = \overline{\theta}\overline{\eta}$. Thus C/R is s.p.p.

Lemma 2.4. *Let G be a reduced s.p.p. group. If B is a basic subgroup of G then $|B| = |G|$.*

PROOF. [Reid] Let B be basic in G and assume $|B| < |G|$. Put $D = G/B$ so that $|D| = |G|$. Since G is s.p.p. and B is pure in G we have for every endomorphism θ of D, and endomorphism α of G such that $\nu\alpha = \theta\nu$, ν the natural map of G onto D. Let $E(G, B) = \{\sigma : \sigma$ is an endomorphism of G such that $\sigma(B) \subseteq B\}$. Then we have an isomorphism $E(D) \simeq \frac{E(G,B)}{\mathrm{Hom}(G,B)}$. Since G is reduced, restriction gives a monomorphism of $E(G, B)$ into $E(B)$ so that $|E(G, B)| \leq |E(B)|$. But $|E(D)| = 2^{|D|}$ and $|E(B)| = 2^{|B|}$. Hence $2^{|D|} = |E(D)| \leq |E(G, B)| \leq |E(B)| \leq 2^{|B|}$. But by assumption $2^{|B|} < 2^{|D|}$, a contradiction to G.C.H.. This proves the lemma.

Lemma 2.5. *If G is s.p.p. and B is basic in G, then we can assume $r(G) = \mathrm{fin}r(G) = r(B) = \mathrm{fin}r(B)$.*

PROOF. First we write $G = G' \oplus G''$ where G' is bounded and $r(G'') = \mathrm{fin}r(G'')$. [4 §.35]. Then G'' is s.p.p. and G is a direct sum of cyclic groups if G'' is. So we may assume $r(G) = \mathrm{fin}r(G)$ to begin with. Similarly for B basic in G, we write $B = B' \oplus B''$ with B' bounded and $r(B'') = \mathrm{fin}r(B'')$. Then $G = B' \oplus H$ for some H and we have $r(G) \geq r(H) \geq \mathrm{fin}r(H) = \mathrm{fin}r(G) = r(G)$. Hence $r(H) = \mathrm{fin}r(H)$. Now consider a basic subgroup B_1 of H. Then $B' \oplus B_1$ is basic in G and so $B' \oplus B_1 \simeq B' \oplus B''$. Hence $B'' \simeq$ a basic subgroup of H. Thus we have replaced G by an s.p.p. group H satisfying $r(H) = \mathrm{fin}r(H)$ and $r(B) = \mathrm{fin}r(B)$ for B basic in H. Further by Lemma 2.4, $|B| = |H|$ and $r(B) = |B| = r(H)$. Hence the Lemma is proved.

Lemma 2.6. *Let C be a direct sum of cyclic groups with L basic in C. Let B_1 and B_2 be basic in L, such that $C/B_1 \simeq L/B_2$. Then there exists an isomorphism $\phi : C \simeq L$ such that $\phi(B_1) = B_2$.*

PROOF. Since C is a direct sum of cyclic groups and L is basic in C, there is an isomorphism $\psi : C \simeq L$. Put $B_1' = \psi(B_1)$. Then $L/B_2 \simeq C/B_1 \simeq L/B_1'$. So by Hill's theorem, there is an automorphism λ of L such that $\lambda(B_1') = B_2$. Consider $\phi = \lambda\psi$. Then ϕ does the work.

Theorem 2.7. *A reduced p–group G is s.p.p. if and only if it is a direct sum of cyclic groups.*

PROOF. Let G be a reduced a s.p.p. p–group. By Lemma 2.5 we may assume $\operatorname{fin}r(G) = \operatorname{fin}r(B)$ for every basic subgroup B of G. Taking B to be a lower basic subgroup of G and B' to be a lower basic subgroup of B, since $r(G/B) = \operatorname{fin}r(G)$ and $r(B/B') = \operatorname{fin}r(B)$ and G/B and B/B' are divisible, we have $G/B \simeq B/B'$. Now there exists a pure exact sequence $0 \to R \to C \xrightarrow{\eta} G \to 0$ where C is a direct sum of cyclic groups. Since $|G| = \operatorname{fin}r(G), G$ has a pair B_1, B_2 of disjoint lower basic subgroups.

Now we can choose B_1', B_2' basic in B_1, B_2 such that $G/B_1 \simeq B_1/B_1'$ and $G/B_2 \simeq B_2/B_2'$. Let $L_1 = \eta^{-1}(B_1)$, $L_1' = \eta^{-1}(B_1')$ and $L_2 = \eta^{-1}(B_2)$, $L_2' = \eta^{-1}(B_2')$. Clearly L_1, L_2 are basic in C and L_1', L_2' are basic in L_1, L_2 and $\dfrac{C}{L_i} \simeq \dfrac{L_i}{L_i'}$, $i = 1, 2$. Then by Lemma 2.6, there are isomorphisms $\phi_i : C \simeq L_i$ such that $\phi_i(L_i) = L_i'$. Since $R \subset L_i' \subset L_i$, $i = 1, 2$, by Proposition 2.3 there exist endomorphisms σ_i, $i = 1, 2$ of C such that $\sigma_i(R) \subseteq R$ and $\delta_i(C) = (\phi_i - \sigma_i)(C) \subseteq L_i$.

Let θ_i be the endomorphism of G induced by σ_i. Let K_i be the kernel of θ_i. Then K_i is the image under η of $\{x \in C$ such that $\sigma_i(x) \in R\}$. Now if $\sigma_i(x) \in R$, then $(\phi_i - \sigma_i)(x) \in L_i'$. Hence $\phi_i(x) \in L_i'$ and ϕ_i being an automorphism $x \in L_i$. Hence $\eta(x) \in B_i$. Hence $K_i \subseteq B_i$.

Now consider the two endomorphisms θ_1, θ_2 of G into the disjoint basic subgroups B_1 and B_2 with kernel $\theta_i \subseteq B_i$. Define $\theta_1 \times \theta_2 : G \to B_1 \otimes B_2$ by $(\theta_1 \times \theta_2)(x) = \theta_1(x) + \theta_2(x)$. Now the kernel of $\theta_1 \times \theta_2 = \{x|\theta_1(x) + \theta_2(x) = 0\}$. Also $\theta_1(x) = -\theta_2(x) = 0$ since the θ_i's map G into B_i. And as the kernel $\theta_i = K_i \subseteq B_i$ we have $x \in B_i, i = 1, 2$. So $x = 0$. Thus G is isomorphic to a subgroup of the direct sum of cyclic groups $B_1 \oplus B_2$ and so is itself a direct sum of cyclic groups.

The converse is well known.

Theorem. *(under G.C.H.) *A torsion abelian group $G = \sum G_p$ is*

*The author is grateful to the referee for suggesting the correct form of this theorem and significant changes.

skew pure projective if and only if each G_p is a direct sum of cyclic groups, or a direct sum of a divisible group and a bounded p–group.

PROOF. This easily follows from Remark 1.3, Lemma 2.2 and Theorem 2.7.

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(Received February 2, 1988)