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Bounding functions concerning character degrees of solvable groups

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To the memory of Professor Béla Barna

Throughout the paper G always denotes a solvable group of finite order. We shall use the notations of [6]. We shall investigate some properties of the functions $\alpha(r)$ and $\delta(r)$ of [3] and [5]:

$$\alpha(r) = \max\{d.l.\,(G/\operatorname{Ker}\chi) \mid \chi \in \operatorname{Irr}(G),\quad \chi(1) \le f_r\}$$

where

$$c.d.(G) = \{f_1 < f_2 \cdots < f_n\}$$

and

$$\delta(1) = 1 \;\;,\;\; \delta(r) = \alpha(r) - \alpha(r-1) \quad \text{for } r = 2, \dots, n.$$

We know from [5] that $\delta(i) \le 3$ for $i = 1, \dots, n$ and according to [3] $\alpha(i) \le 2i$ for $i = 1, \dots n$. Our aim is to find properties which enable us to say somewhat more about these functions.

Let us introduce the following

Hypothesis 1. Let $\psi \in \operatorname{Irr}(G)$ and $M \triangleleft G$. If $M \le \operatorname{Ker}\chi$ for all $\chi \in \operatorname{Irr}(G)$ such that $\chi(1) < \psi(1)$ then $M' \le \operatorname{Ker}\psi$.

Hypothesis 1'. The same as Hypothesis 1 but M must be a member of the derived series of G.

Remark 1. According to [1], all solvable groups of odd order satisfy Hypoyhesis 1. We can see easily that M–groups also satisfy it since if $\psi \in \operatorname{Irr}(G)$ then $\psi = \lambda^G$ for some $\lambda \in \operatorname{Irr}(H)$ linear character of $H \le G$. If ψ is not linear then every nonprincipal irreducible constituent of 1_H^G has degree less than $\psi(1)$. So $M \le \operatorname{Ker} 1_H^G \le H$. But then $M' \le H' \le \operatorname{Ker}\lambda$ and so also $M' \le \operatorname{Ker}\psi$. It is trivial that Hypothesis 1 yields Hypothesis 1'.

First we prove the following simple

Lemma 1. *If the group G satisfies Hypothesis 1' then $\delta(i) \leq 1$ for $i = 1, \ldots, n$ and consequently $\alpha(i) \leq i$ for $i = 1, \ldots, n$. Conversely, if $\delta(i) \leq 1$ for $i = 1, \ldots, n$ then Hypothesis 1' is satisfied in G.*

PROOF. We use induction. $\delta(1) = 1$ by definition. Let us suppose that Hypothesis 1' is satisfied in G and from this we have already proved that $\delta(i) \leq 1$ for $i = 1, \ldots, r-1$. By the definition of the function α , $G^{\alpha(r-1)} \leq \operatorname{Ker}\chi$ for all $\chi \in \operatorname{Irr}(G)$ satisfing $\chi(1) \leq f_{r-1}$. From Hypothesis 1' we have that $G^{\alpha(r-1)+1} \leq \operatorname{Ker}\psi$ for all $\psi \in \operatorname{Irr}(G)$ satisfying $\psi(1) \leq f_r$. So $\alpha(r) \leq \alpha(r-1) + 1$ that is $\delta(r) \leq 1$. As $\alpha(r) = \sum_1^r \delta(i) \leq r$, the first part of the statement is proved.

Conversely, if $\delta(i) \leq 1$ for $i = 2, \ldots, n$ then $\alpha(i) \leq \alpha(i-1) + 1$ so $G^{\alpha(i-1)+1} \leq \operatorname{Ker}\chi$ for all $\chi \in \operatorname{Irr}(G)$ satisfying $\chi(1) \leq f_i$ and from this Hypothesis 1' easily follows.

Corollary 1. *Groups of odd order and M-groups satisfy $\delta(i) \leq 1$ and $\alpha(i) \leq i$ for $i = 1, \ldots, n$.*

PROOF. Follows from Remark 1 and Lemma 1.

Now we weaken our hypotheses:

Hypothesis 2. Let $M \triangleleft G$ and let us suppose that $M \leq \operatorname{Ker}\chi$ for all $\chi \in \operatorname{Irr}(G)$ satisfying $\chi(1) \leq f_{r-1}$. If there exists a $\psi \in \operatorname{Irr}(G)$ for which $M' \not\leq \operatorname{Ker}\psi$ and $\psi(1) = f_r$ then $M'' \leq \operatorname{Ker}\vartheta$ for all $\vartheta \in \operatorname{Irr}(G)$ satisfying $\vartheta(1) \leq f_{r+1}$. Here $r \leq n$ and we define f_{n+1} to be equal to f_n.

Hypothesis 2'. The same as Hypothesis 2 but M must be a member of the derived series of G.

Before dealing with Hypothesis 2 or Hypothesis 2' we introduce a further weakening of them, namely

*Hypothesis 2 *.* Let $M \triangleleft G$ and $\chi \in \operatorname{Irr}(G)$. Let us suppose that $M \leq \operatorname{Ker}\varphi$ for all $\varphi \in \operatorname{Irr}(G)$ such that $\varphi(1) < \chi(1)$; then $M'' \leq \operatorname{Ker}\chi$.

*Hypothesis 2 *'.* The same as Hypothesis 2* but M must be a member of the derived series of G.

Remark 2. It is easy to see that Hypothesis 1 yields Hypothesis 2 and Hypothesis 2 yields Hypothesis 2*. The same is true for Hypothesis i' for $\mathrm{i} = 1, 2, 2^*$.

As we can see e.g. in the case of SL(2,3) Hypothesis 2 does not yield Hypothesis 1'. Here $\alpha(1) = 1,\ \alpha(2) = 3,\ \alpha(3) = 3$ and $\delta(1) = 1,\ \delta(2) = 2$ and $\delta(3) = 0$.

Now we have a similar statement as in Lemma 1:

Lemma 2. *Hypothesis 2*' is equivalent to* $\delta(i) \leq 2$ *for* $i = 1, \dots, n$.

PROOF. Easy to prove.

For Hypothesis 2' we can prove more:

Lemma 3. *If G satisfies Hypothesis 2' then* $\delta(i) \leq 2$ *for* $i = 1, \dots, n$ *and if for some* $r < n$ $\delta(r) = 2$ *then* $\delta(r+1) = 0$.

PROOF. The first statement follows from Lemma 2. If $\delta(r) = 2$ for some $r < n$ then by Hypothesis 2' $G^{\alpha(r-1)+2} \leq \operatorname{Ker} \vartheta$ for every $\vartheta \in \operatorname{Irr}(G)$ satisfying $\vartheta(1) \leq f_{r+1}$. So we have that $\alpha(r+1) \leq \alpha(r-1) + 2$ that is $\delta(r+1) + \delta(r) \leq 2$. So $\delta(r+1) = 0$.

In a similar way as in the proof of Corollary 9 of [3] we get the following

Theorem 1. *If G satisfies Hypothesis 2' then* $\alpha(r) \leq r + 1$ *for all* $r \leq n$. *If for an* r $\delta(r) \leq 1$ *is also true then* $\alpha(r) \leq r$.

PROOF. By Lemma 3 we have that $\delta(i) \leq 2$ for $i = 1, \dots, n$. If $\delta(i) \leq 1$ for all $i = 1, \dots, n$ then we are done by Lemma 1. If $\delta(i) = 2$ for some $i < n$ then by Lemma 3 we have that $\delta(i+1) = 0$. Set

$$S = \{i \mid i < r, \ \ \delta(i) = 2\},$$
$$T = \{i \mid i \leq r, \ \ i - 1 \in S\}.$$

Then $T \cap S = \emptyset$ and $|S| = |T|$.
So $\alpha(r) = \sum_1^r \delta(i) \leq 1 + \delta(r) + 2|S| + 0 \cdot |T| + 1 \cdot (r - |T \cup S \cup \{1, r\}|) =$

$$= \begin{cases} r & \text{if } \ \delta(r) = 0 \text{ and } r \in T \text{ or } \delta(r) = 1 \\ r - 1 & \text{if } \ \delta(r) = 0 \text{ and } r \notin T \\ r + 1 & \text{if } \ \delta(r) = 2. \end{cases}$$

Corollary 2. *If a group G satisfies Hypothesis 2' then* $d.l.(G) \leq |c.d.(G)| + 1$. *If in addition* $\delta(n) \leq 1$ *then* $d.l.(G) \leq |c.d.(G)|$.

Corollary 3. *If for a group G* $d.l.(G) \leq 3$ *then* $\alpha(i) \leq i + 1$ *for* $i = 1, \dots, n$. *This bound is strict. These groups satisfy* $\delta(n) \leq 1$ *and* $d.l.(G) \leq |c.d.(G)|$.

PROOF. As for these groups Hypothesis 2' is easily seen to be satisfied, the first statement follows from Theorem 1. $\alpha(i) = i + 1$ for $i = 2$ in $SL(2, 3)$. It is easy to see that $\delta(n) \leq 1$ so the last assertion also follows from Theorem 1.

Remark 3. Although we cannot prove in general that Hypothesis 2' yields $d.l.(G) \leq |c.d.(G)|$, we mention that it is true e.g. in the case of Frobenius groups. Frobenius groups satisfy Hypothesis 2' if and only if their complement H satisfies it and $\delta_H(|c.d.(H)|) \leq 1$.

Let us introduce for $\psi \in \mathrm{Irr}(G)$

Hypothesis 2 (ψ). Let $\psi(1) = f_r$. Let $M \triangleleft G$ such that $M \leq \mathrm{Ker}\,\chi$ for all $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) \leq f_{r-1}$. If $M' \not\leq \mathrm{Ker}\,\psi$ then $M'' \leq \mathrm{Ker}\,\vartheta$ for all $\vartheta \in \mathrm{Irr}(G)$ such that $\vartheta(1) \leq f_{r+1}$. Here f_{n+1} is defined to be equal to f_n.

Using Lemma 7 of [3] we can prove the following

Lemma 4. *If G is a finite solvable group such that $3, 5 \nmid |G|$ and $\psi \in \mathrm{Irr}(G)$ is faithful and primitive then Hypothesis 2 (ψ) is satisfied in G.*

PROOF. Let $M \triangleleft G$ such that $M \triangleleft G$ such that $M \leq \mathrm{Ker}\,\chi$ for all $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) < \psi(1)$. Let us assume that $M'' \neq 1$. Then $[M, M'] \neq 1$ either.

By Lemma 7 of [3] there exists an $\omega \in \mathrm{Irr}(G)$ such that $\psi(1) < \omega(1) \leq 3/2\,\psi(1)$ and $M \not\leq \mathrm{Ker}\,\omega$. Also $\psi(1) = 2$ or $\psi(1) = 4$. So either $2 < \omega(1) \leq 3/2{\cdot}2 = 3$, which yields that $\omega(1) = 3$, or $4 < \omega(1) \leq 3/2{\cdot}4 = 6$, which yields $\omega(1) = 5$ or 6. As $\omega(1) \mid |G|$, these are all impossible. So $M'' = 1$ and we are done.

Remark 4. Hypothesis 2*' does not hold for all solvable groups as we can see in the example of $GL(2,3)$. Here we have $\alpha(1) = 1,\ \alpha(2) = 4$, $\alpha(3) = 4,\ \alpha(4) = 4$ and $\delta(1) = 1,\ \delta(2) = 3,\ \delta(3) = 0,\ \delta(4) = 0$. So for $i = 2$ we have $\alpha(i) = i + 2$ and $\delta(i) = 3$.

Now we prove a sufficient condition for Hypothesis 2*. This can be considered as an extension of part b/ of Theorem 6 in [5].

Theorem 2. *Let G be a finite solvable group such that $3, 5 \nmid |G|$ then Hypothesis 2 * is satisfied in G.*

PROOF. Let G be a counterexample of minimal order. Let $M \triangleleft G$, $\chi \in \mathrm{Irr}(G)$; we suppose that $M'' \not\leq \mathrm{Ker}\,\chi$ and $M \leq \mathrm{Ker}\,\varphi$ for all $\varphi \in \mathrm{Irr}(G)$ with $\varphi(1) < \chi(1)$. By Lemma 4, χ cannot be faithful and primitive.

If $\mathrm{Ker}\,\chi > 1$ then by the inductive hypothesis applied to χ, $M\,\mathrm{Ker}\,\chi/\mathrm{Ker}\,\chi$ and $G/\mathrm{Ker}\,\chi$ $M'' \leq \mathrm{Ker}\,\chi$, which is a contradiction.

If χ is not primitive then $\chi = \alpha^G$ for an $\alpha \in \mathrm{Irr}(H)$ for some $H < G$. As $1_H^G \notin \mathrm{Irr}(G)$, $M \leq \mathrm{Ker}\,1_H^G \leq H$. If $\nu \in \mathrm{Irr}(H)$ such that $\nu(1) < \alpha(1)$ then as $\nu^G(1) < \chi(1)$ we have $M \leq \mathrm{Ker}\,\nu^G \leq \mathrm{Ker}\,\nu$ and by the inductive hipothesis applied to M, H and α we get $M'' \leq \mathrm{Ker}\,\alpha$ so $M'' \leq \mathrm{Ker}\,\chi$, and thus we have a final contradiction.

In [4] we have proved that Hypothesis 1 is inherited by direct products. Similarly, one can prove easily that this is also true for Hypothesis 2*, namely we have the following

Lemma 5. *Let $G = G_1 \times G_2$ where G_i $i = 1, 2$ satisfy Hypothesis 2*. Then G satisfies it as well.*

Now we have an analogous statement as Theorem 1.5 in [4]:

Theorem 3. *Let G be a finite solvable group such that for every $x \in G^\star$, $C_G(x)$ is the direct product of a $\{3,5\}$-group and of a $\{3,5\}'$ group. Then G satisfies Hypothesis 2 *.*

PROOF. Using the above Remark 1 and Lemma 5 with Theorem B and Lemma 2 of [2] the proof is similar to that of Theorem 1.5 in [4].

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