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Another version of a common fixed point theorem

By ROY O. DAVIES (Leicester)

Abstract. The existence of a unique common fixed point for two weakly commuting self–mappings in a Banach space, of which one is linear and non–expansive, is established under a contraction hypothesis which is shown to be weaker than that in a similar theorem of DIVICCARO, FISHER, and SESSA.

The main result of the present paper is the following.

Theorem 1. *Let T and I be two self-mappings of a non-empty closed convex subset C of a Banach space X, satisfying the inequality*

$$(1)\qquad \begin{aligned}\|Tx-Ty\| \le \alpha\cdot\|Ix-Iy\| + \beta\cdot\max[\,\|Tx-Ix\|,\,\|Ty-Iy\|\,]+\\ +\gamma\cdot\max[\,\|Ix-Iy\|,\,\|Tx-Ix\|,\,\|Ty-Iy\|\,]\end{aligned}$$

for all x, y in C, where $\alpha,\beta,\gamma > 0$ and $\alpha+\beta+\gamma = 1$. Further, let I weakly commute with T, that is, $\|TIx - ITx\| \le \|Tx - Ix\|$ for all x in C, and let I be linear and non-expansive in C. If $I(C)$ contains $T(C)$, then the equations $x = Ty = Iy$ have a unique solution for $x \in C$, and x is a common fixed point of T and I, at which T is continuous.

The above theorem is the same as that in the paper [1] by DIVICCARO, FISHER, and SESSA – to which the reader is referred for a more general discussion – except that we have given a slightly more precise statement about the common fixed point, and (more importantly) instead of (1) their 'contraction' condition was

$$\text{(I)}\quad \|Tx-Ty\|^p \le a\cdot\|Ix-Iy\|^p + (1-a)\cdot\max[\,\|Tx-Ix\|^p, \|Ty-Iy\|^p],$$

where $0 < a < 1/2^{p-1}$ and $p \ge 1$. The following will also be proved here.

Theorem 2. *Condition (I) with $0 < a < 1$ and $p \ge 1$ implies (1) for a certain triple $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma = 1$.*

Thus our Theorem 1 not only implies the theorem of [1], but also implies that the condition $0 < a < 1/2^{p-1}$ in [1] can be relaxed to $0 < a < 1$.

I am grateful to BRIAN FISHER for useful comments.

PROOF OF THEOREM 1. This closely follows the proof in [1], with one significant extra feature, but for completeness we give most of the details, since the use of (1) instead of (I) makes a number of changes necessary.

Let x_0 be an arbitrary point of C. Applying the fact that $I(C) \supseteq T(C)$, inductively define points $x_1, x_2, \dots$ in C by choosing as x_{r+1} any point of C such that

$$(2) \qquad Ix_{r+1} = Tx_r \qquad (r = 0, 1, 2, \dots).$$

Write $c_r = \|Ix_r - Ix_{r+1}\|$ $(r = 0, 1, 2, \dots)$ and observe that from (1) with $x = x_r,\ y = x_{r+1}$, and applying (2) where appropriate – we shall do this in future without special mention – we have for $r \geq 0$

$$(3) \qquad c_{r+1} \leq \alpha \cdot c_r + \beta \cdot \max\{c_r, c_{r+1}\} + \gamma \cdot \max\{c_r, c_r, c_{r+1}\}.$$

From (3) it follows that if $c_r \leq c_{r+1}$ then $c_{r+1} \leq \alpha \cdot c_r + (\beta + \gamma) \cdot c_{r+1}$ and therefore $c_{r+1} \leq c_r$ and so $c_{r+1} = c_r$. Consequently

$$(4) \qquad c_0 \geq c_1 \geq c_2 \geq \dots .$$

We now introduce the extra feature referred to earlier, which if incorporated in the proof in [1] would have made it immediately possible to relax the condition on a to $0 < a < 1$. Write $d_r = \|Ix_r - Ix_{r+2}\|$ $(r = 0, 1, 2, \dots)$. Observe that from (1) with $x = x_r,\ y = x_{r+2}$, and using (4), for $r \geq 0$ we have

$$(5) \qquad \begin{aligned} d_{r+1} &\leq \alpha \cdot d_r + \beta \cdot \max\{c_r, c_{r+2}\} + \gamma \cdot \max\{d_r, c_r, c_{r+2}\} \leq \\ &\leq \alpha \cdot d_r + \beta \cdot c_0 + \gamma \cdot \max\{d_r, c_0\} \leq \max\{d_r, c_0\}. \end{aligned}$$

It follows that $\max\{d_{r+1}, c_0\} \leq \max\{d_r, c_0\}$; thus $(\max\{d_r, c_0\})$ is a decreasing sequence, and consequently d_r is bounded, that is, $\limsup d_r$ is a finite number d. Taking the upper limit on both sides of (5), we conclude that $d \leq \alpha \cdot d + \beta \cdot c_0 + \gamma \cdot \max\{d, c_0\}$, from which it follows that $d \leq c_0$. Thus we have proved that

$$(6) \qquad \limsup d_r \leq c_0.$$

Now let $\varepsilon > 0$ be chosen so small that

$$(7) \qquad (\alpha + \gamma)\varepsilon/4 \leq \alpha c_0/8\ ;$$

in view of (6) we can choose r so large that

$$(8) \qquad d_r \leq c_0 + \varepsilon.$$

Define a point z by $z = \frac{1}{2}x_{r+1} + \frac{1}{2}x_{r+2}$. (The point z used in [1] corresponded to $r = 1$, and instead of (6) a weaker, but more complicated, estimate was used.) Since C is convex, z is in C, and since I is linear, $Iz = \frac{1}{2}Ix_{r+1} + \frac{1}{2}Ix_{r+2}$. From (4) and (8) we have

$$\text{(9)} \qquad \|Iz - Ix_r\| = \left\|\frac{1}{2}(Ix_{r+1} - Ix_r) + \frac{1}{2}(Ix_{r+2} - Ix_r)\right\| \le c_0 + \frac{1}{2}\varepsilon.$$

From (4) again we have

$$\text{(10)} \qquad \|Iz - Ix_{r+1}\| = \left\|\frac{1}{2}(Ix_{r+2} - Ix_{r+1})\right\| \le \frac{1}{2}c_0.$$

Write $\lambda = \|Tz - Iz\|$. By the triangle inequality

$$\lambda = \left\|\frac{1}{2}(Tz - Ix_{r+1}) + \frac{1}{2}(Tz - Ix_{r+2})\right\| \le \frac{1}{2}\|Tz - Tx_r\| + \frac{1}{2}\|Tz - Tx_{r+1}\|.$$

Apply (1) to each term on the right, and use (4), (9), and (10): this gives

$$\begin{aligned}
\lambda \le & \frac{1}{2}[\alpha \cdot \|Iz - Ix_r\| + \beta \cdot \max\{\lambda, c_r\} + \gamma \cdot \max\{\|Iz - Ix_r\|, \lambda, c_r\}] + \\
& + \frac{1}{2}[\alpha \cdot \|Iz - Ix_{r+1}\| + \\
& + \beta \cdot \max\{\lambda, c_{r+1}\} + \gamma \cdot \max\{\|Iz - Ix_{r+1}\|, \lambda, c_{r+1}\}] \le \\
\le & \frac{1}{2}\left[\alpha\left(c_0 + \frac{1}{2}\varepsilon\right) + \beta \cdot \max\{\lambda, c_0\} + \gamma \cdot \max\left\{\lambda, c_0 + \frac{1}{2}\varepsilon\right\}\right] + \\
& + \frac{1}{2}\left[\alpha \cdot \frac{1}{2}c_0 + \beta \cdot \max\{\lambda, c_0\} + \gamma \cdot \max\{\lambda, c_0\}\right] \le \\
\le & 3\alpha c_0/4 + (\alpha + \gamma)\varepsilon/4 + (\beta + \gamma) \cdot \max\{\lambda, c_0\},
\end{aligned}$$

and hence by (7)

$$\text{(11)} \qquad \lambda \le 7\alpha c_0/8 + (\beta + \gamma) \cdot \max\{\lambda, c_0\}.$$

It follows from (11) that $\lambda \le A \cdot c_0$ where $A = 7\alpha/8 + \beta + \gamma$. Thus we have shown that there exists a constant $A < 1$ such that for every x_0 in C we can find z in C with $\|Tz - Iz\| \le A \cdot \|Tx_0 - Ix_0\|$; we now argue similarly to [1], with minor modifications.

It follows first of all that $\inf\{\|Tx - Ix\| : x \in C\} = 0$. Therefore the sets $K_n = \{x \in C : \|Tx - Ix\| \leq 1/n\}$ $(n = 1, 2, \dots)$ are non-empty, and moreover $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$. Therefore the closures $\overline{IK_n}$ are non-empty subsets of C satisfying $\overline{IK_1} \supseteq \overline{IK_2} \supseteq \overline{IK_3} \supseteq \dots$. For $x, y \in K_n$, by the triangle inequality we have

$$\|Ix - Iy\| \leq \|Ix - Tx\| + \|Tx - Ty\| + \|Ty - Iy\| \leq \|Tx - Ty\| + \frac{2}{n},$$

and therefore by (1)

$$\|Tx - Ty\| \leq \alpha\left(\|Tx - Ty\| + \frac{2}{n}\right) + \beta \cdot \frac{1}{n} + \gamma\left(\|Tx - Ty\| + \frac{2}{n}\right) < \\ < (\alpha + \gamma) \cdot \|Tx - Ty\| + \frac{2}{n},$$

whence

$$\|Tx - Ty\| \leq 2/\beta n, \quad \|Ix - Iy\| \leq (2 + 2/\beta)/n. \tag{12}$$

Hence $(\overline{IK_n})$ is a decreasing sequence of non-empty closed subsets of C with diameters tending to zero, and so their intersection is a one-point set $w \in C$:

$$\bigcap_{n=1}^{\infty} \overline{IK_n} = \{w\}. \tag{13}$$

If $x \in K_{2n}$, then because T, I are weakly commuting and I is non-expansive (this is the first point at which we use these hypotheses)

$$\|T(Ix) - I(Ix)\| \leq \|TIx - ITx\| + \|ITx - IIx\| \leq 2\|Tx - Ix\| \leq \frac{1}{n} \; ;$$

we have shown that

$$IK_{2n} \subseteq K_n \qquad (n = 1, 2, \dots). \tag{14}$$

Choose a positive integer M so large that $\alpha M > 1$; we shall prove that

$$\overline{K_{Mn}} \subseteq K_n \qquad (n = 1, 2, \dots). \tag{15}$$

Let x_0 be any element of $\overline{K_{Mn}}$; then $x_0 = \lim x_r$ where each $x_r \in K_{Mn}$, and by continuity $Ix_r \to Ix_0$ as $r \to \infty$. Write $A = \limsup \|Tx_r - Tx_0\|$. By the triangle inequality

$$\|Tx_0 - Ix_0\| \leq \|Tx_0 - Tx_r\| + \|Tx_r - Ix_r\| + \|Ix_r - Ix_0\|$$

and therefore (taking the upper limit on the right)

$$\|Tx_0 - Ix_0\| \leq A + \frac{1}{Mn}. \tag{16}$$

By (1) we have

$$\|Tx_r - Tx_0\| \leq \alpha \cdot \|Ix_r - Ix_0\| + \beta \cdot \max\{\|Tx_r - Ix_r\|, \|Tx_0 - Ix_0\|\}+ \\ + \gamma \cdot \max\{\|Ix_r - Ix_0\|, \|Tx_r - Ix_r\|, \|Tx_0 - Ix_0\|\}$$

and hence by (16), and taking upper limits,

$$A \leq (\beta + \gamma) \cdot \max\left\{\frac{1}{Mn}, \|Tx_0 - Ix_0\|\right\} \leq (\beta + \gamma) \cdot (A + \frac{1}{Mn}),$$

from which it follows that $A \leq (\beta + \gamma)/\alpha Mn$, whence by (16)

$$\|Tx_0 - Ix_0\| \leq 1/\alpha Mn \leq 1/n.$$

This establishes (15).

From (13), (14), (15) we see that

$$w \in \bigcap_{n=1}^{\infty} \overline{IK_{2Mn}} \subseteq \bigcap_{n=1}^{\infty} \overline{K_{Mn}} \subseteq \bigcap_{n=1}^{\infty} K_n. \tag{17}$$

Therefore $Tw = Iw$. But it also follows from (17) and (13) that

$$Iw \in \bigcap_{n=1}^{\infty} IK_n \subseteq \bigcap_{n=1}^{\infty} \overline{IK_n} = \{w\};$$

this implies that $w = Iw$, so w is a common fixed point of T and I. The solution $x = w$ of the equations $x = Ty = Iy$ is unique since if $x = w'$ is any solution then $w' \in IK_n$ for all n and hence $w' = w$.

Finally, let $x_n \to w$ as $n \to \infty$. Since $Tw = Iw$ and I is non-expansive,

$$\|Tx_n - Ix_n\| \leq \|Tx_n - Tw\| + \|Tw - Ix_n\| \leq \|Tx_n - Tw\| + \|w - x_n\|.$$

Applying (1), we have

$$\begin{aligned}\|Tx_n - Tw\| &\leq \alpha\|Ix_n - Iw\| + \beta \cdot \max\{\|Tx_n - Ix_n\|, \|Tw - Iw\|\}+ \\ &\quad + \gamma \cdot \max\{\|Ix_n - Iw\|, \|Tx_n - Ix_n\|, \|Tw - Iw\|\} \leq \\ &\leq \alpha \cdot \|x_n - w\| + (\beta + \gamma)(\|Tx_n - Tw\| + \|x_n - w\|) \leq \\ &= \|x_n - w\| + (\beta + \gamma)\|Tx_n - Tw\|,\end{aligned}$$

and therefore $\|Tx_n - Tw\| \leq (1/\alpha)\|x_n - w\|$, proving the continuity of T at w.

Remark 1. Since $x = w$ is the unique solution of the equations $x = Ty = Iy$ it is automatically the unique common fixed point of T and I. It is not necessarily the unique fixed point of I or T, as is shown by examples in which $C = X = \mathbf{R}$ with the Euclidean norm and $Ix = x,\ Tx = \alpha x$ or $Ix = -x,\ Tx = x$ respectively.

Remark 2. Theorem 1 becomes false if either of the first two terms on the right hand side of (1) is omitted, that is, if either of the conditions $\alpha > 0,\ \beta > 0$ is weakened to $\alpha \geq 0,\ \beta \geq 0$ respectively. This can be shown by simple examples in which $C = X = \mathbf{R}$ with the Euclidean norm, $Ix = x$, and $Tx = \delta x + 1$ for some small $\delta > 0$; here T and I have no common fixed point. If the third term in (1) is omitted we obtain a stronger condition than (1).

PROOF OF THEOREM 2. It is clearly sufficient to show that if $0 < a < 1$ and $p \geq 1$ then there exist constants $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma = 1$ such that for $W, X, Y, Z \geq 0$

$$W^p \leq a \cdot X^p + (1-a) \cdot \max(Y^p, Z^p) \tag{17}$$

implies

$$W \leq \alpha \cdot X + \beta \cdot \max(Y, Z) + \gamma \cdot \max(X, Y, Z).$$

Considering the maximum possible value of W permitted by (17), we see that this is the same as proving the inequality

$$\begin{aligned} &a \cdot X^p + (1-a) \cdot \max(Y^p, Z^p) \leq \\ &\leq \left[\alpha \cdot X + \beta \cdot \max(Y, Z) + \gamma \cdot \max(X, Y, Z)\right]^p. \end{aligned} \tag{18}$$

By symmetry we may assume that $Z \leq Y$. Then (18) becomes

$$a \cdot X^p + (1-a) Y^p \leq \left[\alpha \cdot X + \beta \cdot Y + \gamma \cdot \max(X, Y)\right]^p. \tag{19}$$

We first find the conditions on α, β, γ for (19) to hold whenever $0 \leq X \leq Y$, in which case (19) can be rewritten as

$$f(X, Y) \equiv \left[\alpha \cdot X + (1-\alpha) Y\right]^p - \left[a \cdot X^p + (1-a) Y^p\right] \geq 0. \tag{20}$$

For this inequality to be true for $X = 0$ it is necessary that

$$(1-a) \leq (1-\alpha)^p, \quad \text{that is,} \quad \alpha \leq 1 - (1-a)^{1/p}. \tag{21}$$

Moreover for $Y = X$ the inequality (20) becomes an equality, while

$$\begin{aligned} \partial f(X, Y)/\partial Y &= (1-\alpha) p[\alpha \cdot X + (1-\alpha) Y]^{p-1} - (1-a) p Y^{p-1} \geq \\ &\geq p Y^{p-1}\{(1-\alpha)^p - (1-a)\}, \end{aligned}$$

which is non-negative provided that (21) is satisfied. Thus (21) is necessary and sufficient for (19) to hold whenever $0 \le X \le Y$.

It is now clear by symmetry that (19) will hold for $0 \le Y \le X$ if and only if

$$\beta \le 1 - a^{1/p}. \tag{22}$$

Finally we can now conclude that provided (21) and (22) are satisfied and α, β are also chosen so small that $\gamma = 1 - (\alpha + \beta)$ is positive, all our conditions are met. The proof is complete.

References

[1] M. L. Diviccaro, B. Fisher and S. Sessa, A common fixed point theorem of Greguš type, *Publicationes Math. (Debrecen)* **34** (1987), 83–89.

ROY O. DAVIES
DEPARTMENT OF MATHEMATICS
THE UNIVERSITY, # LEICESTER
LE1 7RH
ENGLAND

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