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Summing power series with exponential polynomial coefficients

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To the memory of Professor Béla Barna

In the paper [1] the author derives the formula

$$f(x) = \sum_k \Delta^k a(0) x^k / (1-x)^{k+1} \tag{1}$$

for

$$f(x) = \sum_{n=0}^{\infty} a(n) x^n \tag{2}$$

where a is a polynomial and Δ denotes the difference operator (see also [2]). Using the Fourier transform of exponential polynomials introduced by the author in [3] here we show that we can go one step further to sum any power series with exponential polynomial coefficients. We note that the method can easily be extended to multiple power series.

Theorem. *Let f be a power series having the form (2), where the coefficient a is any complex exponential polynomial. Then we have*

$$f(x) = \sum_\lambda \sum_k \Delta^k \widehat{a}(\lambda)(0)(\lambda x)^k / (1-\lambda x)^{k+1} \quad , \tag{3}$$

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and here both sums are actually finite. ($\widehat{a}$ denotes the Fourier transform of a.)

PROOF. The proof is straightforward by using the "inversion formula" (Theorem 2.2 in [3]) for the exponential polynomial a

$$a(n) = \sum_{\lambda} \widehat{a}(\lambda)(n)\lambda^n \tag{4}$$

Substituting into (2) we have

$$f(x) = \sum_{\lambda} \sum_{n=0}^{\infty} \widehat{a}(\lambda)(n)(\lambda x)^n \tag{5}$$

whenever the inner sum is convergent (the sum $\sum_{\lambda}$ is finite).

Now we apply formula (1) for the inner series to obtain (3) which was to be proved.

Example. For the power series

$$f(x) = \sum_{n=0}^{\infty} n x^n \cos(n\alpha)$$

where α is arbitrary, we have

$$\widehat{a}(\lambda)(n) = \begin{cases} \dfrac{n}{2} & \text{for } \lambda = e^{\pm i\alpha} \\ 0 & \text{otherwise} \end{cases},$$

and hence

$$f(x) = \frac{1}{2}\frac{e^{i\alpha}x}{(1-e^{i\alpha}x)^2} + \frac{1}{2}\frac{e^{-i\alpha}x}{(1-e^{-i\alpha}x)^2} = \frac{(x^3+x)\cos\alpha - 2x^2}{(1-2x\cos\alpha + x^2)^2}.$$

References

[1] J. KLIPPERT, Summing power series with polynomial coefficients, *Amer. Math. Monthly,* **90** (1983), 284–285.

[2] R. NICOLESCU, Letters to the Editor, *Amer. Math. Monthly,* **92** (1985), 375–376.

[3] L. SZÉKELYHIDI, The Fourier transform of exponential polynomials, *Publ. Math. Debrecen,* **33** (1986), 13–20.

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