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Nonstable representable K-theory for σ-C^*-algebras

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Abstract. In this paper, Thomsen's nonstable K-theory for C^*-algebras is extended to σ-C^*-algebras (countable inverse limits of C^*-algebras). We show that the homotopy groups of the group of quasi-unitaries in σ-C^*-algebras form a homology theory on the category of all σ-C^*-algebras which becomes topological K-theory when stablized.

0. Introduction

The purpose of this paper is to extend non-stable K-theory for C^*-algebras to a generalized homology theory on the category of σ-C^*-algebras defined in [1] and studied in [10]. We hope that these homotopy groups define invariants that are finer and contain more information than the invariants given by representable K-theory or KK-theory without being completely hopeless to calculate.

This paper is organized as follows. In Section 1, we present some basic definitions and propositions, conventions and notation which will be used throughout the paper, and prove several technical results which are needed for the development of non-stable representable K-theory. In Section 2 we define non-stable representable K-theory for σ-C^*-algebras and prove its more elementary properties. In particular, we show that the homotopy groups of the group of quasi-unitaries in σ-C^*-algebras leads to a long exact sequence and, thus, to a homology theory on the category of all σ-C^*-algebras. In Section 3 we define a σ-C^*-algebra A to be $\mathcal{RK}$-stable

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when, roughly speaking, the non-stable $\mathcal{RK}$-groups of A agree with the Phillip's representable K-theory groups. The main purpose of this section is to show that if a σ-C^*-algebra contains a $\mathcal{RK}$-stable ideal such that the corresponding quotient is also $\mathcal{RK}$-stable, then it must itself be $\mathcal{RK}$-stable.

1. Generalized invertibles and unitaries in σ-C^*-algebras

Suppose that $A \cong \varprojlim A_n$ is a σ-C^*-algebra, we will name the associated maps as follows: $\pi_{m,n} : A_m \to A_n$, for $m \geq n$, are the maps of the system, with $\pi_n = \pi_{n+1,n} : A_{n+1} \to A_n$; also $\kappa_n : A \to A_n$ is the canonical homomorphism; and finally (p_n) is the associated family of seminorms on A, determined by $p_n(a) = \|\kappa_n(a)\|$.

Let $A = \varprojlim A_n$ be a unital σ-C^*-algebra. Then $a \in A$ is invertible if and only if $\kappa_n(a)$ is invertible in A_n for every n. It follows that inversion is continuous on the set $\mathrm{Inv}(A)$ of invertible elements of A. (See [6, Proposition 2.8])

Definition 1.1. Let A be a σ-C^*-algebra and let X be a topological space. Then $C(X, A)$ denotes the algebra of all continuous functions from X to A, with the pointwise operations and the topology of uniform convergence in each seminorm on A on each compact subset of X. For $A = \mathbb{C}$, we just write $C(X)$. We will usually only use compact spaces X.

Lemma 1.2. *Let X be a compact Hausdorff space. Then the functor $A \to C(X, A)$ from σ-C^*-algebras to σ-C^*-algebras sends homomorphisms with dense range to homomorphisms with dense range, and preserves short exact sequences.*

Proof. This is a particular case of [11, Lemma 1.4]. □

Let A be a σ-C^*-algebra (not necessarily unital). Define a composition in A by

$$a \bullet b = a + b - ab, \quad \forall a, b \in A. \tag{1.1}$$

Definition 1.3.

$$gI(A) = \{a \in A \mid \exists b \in A : a \bullet b = b \bullet a = 0\},$$
$$gU(A) = \{a \in A \mid a \bullet a^* = a^* \bullet a = 0\}.$$

Elements of $gI(A)$ and $gU(A)$ will be called quasi-invertibles and quasiunitaries, respectively. We denote by $gI_0(A)$ or $gU_0(A)$ the path component of the identity in $gI(A)$ or $gU(A)$, respectively. With these definition, we may consider these groups. Palmer [15] who seems to have been the first to consider these groups. If A is a unital σ-C^*-algebra, we write $\mathrm{Inv}(A)$ for the group of invertibles in A and $U(A)$ for the unitary group in A.

Lemma 1.4. *Let B be a σ-C^*-algebra with unit 1 and $A \subseteq B$ a closed two sided ideal in B. Then*

$$gI(A) = (1 - \mathrm{Inv}(B)) \cap A, \quad gU(A) = (1 - U(B)) \cap A.$$

Proof. Let $a \in \mathrm{Inv}(B)$, there exists $b \in B$ such that $ab = ba = 1$. Since $(1-a) \bullet (1-b) = (1-b) \bullet (1-a) = 0$, it follows that $(1 - \mathrm{Inv}(B)) \cap A \subseteq gI(A)$. Conversely, if $c \in gI(A)$ then there is a d in A such that $c \bullet d = d \bullet c = 0$. Set $a = 1-c$ and $b = 1-d$. Then $a \in \mathrm{Inv}(B)$ and $ab = 1$. We conclude that

$$gI(A) \subseteq (1 - \mathrm{Inv}(B)) \cap A. \qquad \square$$

Since any σ-C^*-algebra can be embedded as an ideal of a unital σ-C^*-algebra, it follows immediately from Lemma 1.4 that both $gI(A)$ and $gU(A)$ are groups with the composition $\bullet$ given by (1.1).

Lemma 1.5. *Let ϕ: $A \to B$ be a surjective homomorphism of C^*-algebras. Then $\phi(gU_0(A)) = gU_0(B)$.*

Proof. This follows immediately from Theorem 1.9 of [21]. $\square$

Theorem 1.6. *Let $\phi : A \to B$ be a surjective map of σ-C^*-algebras. Then $\phi(gU_0(A)) = gU_0(B)$.*

Proof. Obviously we only need to show that $gU_0(B) \subseteq \phi(gU_0(A))$. Let $I = \mathrm{Ker}(\phi)$. Using Proposition 5.3(2) of [10], we can write the exact sequence

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\phi} B \longrightarrow 0$$

as the inverse limit of exact sequences

$$0 \longrightarrow I_n \longrightarrow A_n \xrightarrow{\phi_n} B_n \longrightarrow 0,$$

with surjective maps $\pi_n : A_{n+1} \to A_n$, $\sigma_n : B_{n+1} \to B_n$, $\kappa_n : A \to A_n$, and $\lambda_n : B \to B_n$. We can also assume that $I_{n+1} \to A_n$, and $I \to I_n$ are surjective.

Now let $v \in gU_0(B)$. Let $t \to b(t)$ be a continuous path in $gU(B)$ with $b(0) = 0$ and $b(1) = v$, and regard b as an element of $C([0,1], B)$. Note that in fact $b \in gU_0(C([0,1], B))$. Define $b_n \in gU_0(C([0,1], B_n))$ by $b_n(t) = \lambda_n(b(t))$. We construct inductively a coherent sequence of continuous paths $a_n \in gU_0(C([0,1], A_n))$ such that $a_n(0) = 0$ and $\phi_n(a_n(t)) = b_n(t)$ for all $t \in [0,1]$ and all n. Begin by using Lemma 1.5 to choose $z \in gU_0(C([0,1], A_1))$ whose image in $C([0,1], B_1)$ is b_1, and set $a_1(t) = z(t)$.

For the induction step, suppose we have found a_n such that

$$a_n(0) = 0, \quad \phi_n(a_n(t)) = b_n(t), \quad \pi_{n-1}(a_n(t)) = a_{n-1}(t) \\ \text{for} \quad t \in [0,1].$$

As above, find x, $y \in gU_0(C([0,1], A_{n+1}))$ such that

$$x(0) = y(0) = 0, \quad \phi_{n+1}(x(t)) = b_{n+1}(t) \quad \text{and} \quad \pi_n(y(t)) = a_n(t).$$

Since $b \in gU_0(C([0,1], B))$, we have $b \bullet b^* = b^* \bullet b = 0$. Similarly $x \bullet x^* = x^* \bullet x = 0$, because

$$\phi_n \circ \pi_n = \sigma_n \circ \phi_{n+1}, \quad \sigma_n \circ \lambda_{n+1} = \lambda_n.$$

Then

$$\begin{aligned}
\phi_n \circ \pi_n(y(t) \bullet x^*(t)) &= \phi_n \circ \pi_n(y(t)) \bullet \phi_n \circ \pi_n(x^*(t)) \\
&= \phi_n(a_n(t)) \bullet \sigma_n \circ \phi_{n+1}(x^*(t)) \\
&= \phi_n(a_n(t)) \bullet \sigma_n(b^*_{n+1}(t)) \\
&= b_n(t) \bullet \sigma_n(b^*_{n+1}(t)) \\
&= \lambda_n(b(t)) \bullet \sigma_n \circ \lambda_{n+1}(b^*(t)) \\
&= \lambda(b(t)) \bullet \lambda_n(b^*(t)) \\
&= \lambda_n(b(t) \bullet b^*(t)) = 0.
\end{aligned}$$

With $c(t) = \pi_n(y(t) \bullet x^*(t))$, we then have $c \in C([0,1], I_n)$. Since $c(0) = 0$, we in fact have $c \in gU_0(C[0,1], I_n)$. So, by Lemma 1.5, there is $z \in gU_0(C([0,1], I_{n+1}))$ whose image in $C([0,1], I_n)$ is c, and as usual we may insist that $z(0) = 0$. For each t, the quasi-unitary $\phi_{n+1}(z(t))$ is a scalar multiple of the zero in B_{n+1}, in fact, be 0 since $\sigma_n \circ \phi_{n+1}(z(t)) = \phi_n \circ \pi_n(z(t)) = 0$. Now define $a_{n+1}(t) = z(t) \bullet x(t)$. It is immediately verified that $a_{n+1}(0) = 0$, $\phi_{n+1}(a_{n+1}(t)) = \phi_{n+1}(z(t) \bullet x(t)) = \phi_{n+1}(x(t)) = b_{n+1}(t)$, and

$$\begin{aligned}\pi_n(a_{n+1}(t)) &= \pi_n(z(t) \bullet x(t)) = \pi_n(z(t)) \bullet \pi_n(x(t)) \\ &= c(t) \bullet \pi_n(x(t)) = \pi_n(y(t) \bullet x^*(t)) \bullet \pi_n(x(t)) \\ &= \pi_n(y(t) \bullet (x^*(t) \bullet x(t))) = \pi_n(y(t) \bullet 0) \\ &= \pi_n(y(t)) = a_n(t),\end{aligned}$$

as desired. This completes the induction step.

To finish the proof, let $a(t) \in gU(A)$ be the element defined by the coherent sequence $(a_n(t))$, and set $u = a(1)$. Then $\phi(u) = v$ and u is a continuous path in $gU(A)$ connecting u to the identity. □

Notation 1.7. We let X be a compact Hausdorff space. If A is a σ-C^*-algebra, we denote by $C(X, gU_0(A))$ the group of all continuous functions from X to $gU_0(A)$. Then the analog of Lemma 1.9 in [11] is:

Corollary 1.8. *Suppose that X be a compact Hausdorff space, A and B are σ-C^*-algebras. Then $A \to B$ surjective implies $C(X, gU_0(A)) \to C(X, gU_0(B))$ surjective.*

Proof. This follows from Theorem 1.6 and an easy modification of the argument in Lemma 1.9 of [11]. □

Finally, in this section we have the following:

Theorem 1.9. *Let*

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\phi} B \longrightarrow 0$$

be a short exact sequence of σ-C^*-algebras. Then the sequences

$$0 \longrightarrow i^{-1}(gU_0(A)) \xrightarrow{i} gU_0(A) \xrightarrow{\phi} gU_0(B) \longrightarrow 0$$

and

$$0 \longrightarrow gU(I) \xrightarrow{i} gU(A) \xrightarrow{\phi} \phi(gU(A)) \longrightarrow 0$$

are both exact sequences of topological groups.

Proof. Using Theorem 1.6, in both cases, it is straightforward to check that the sequences are exact. □

2. Nonstable representable K-theories on σ-C^*-algebras

Let X be any topological space and A is a σ-C^*-algebra. Choose a base point $+ \in X$. We can then consider the group $[X, gU(A)]$ consisting of the homotopy classes of base point preserving continuous maps from X to $gU(A)$. The base point of $gU(A)$ is the neutral element $0 \in gU(A)$. Given a continuous base point preserving map $f : X \to gU(A)$, we let $[f]$ denote its class in $[X, gU(A)]$.

It is clear that the group structure of $gU(A)$ gives $[X, gU(A)]$ the structure of a group in the obvious way, and that $[X, gU(\cdot)]$ is a covariant and homotopy invariant functor from the category of σ-C^*-algebras to the category of groups (not necessarily abelian).

Let SA denote the suspension of a σ-C^*-algebra A, i.e.,

$$SA = \{f : [0,1] \to A : f(0) = f(1) = 0\} \cong C_0(\mathbb{R}) \otimes A.$$

For $n > 1$, we set $S^nA = S(S^{n-1}A)$ — the nth suspension of A (to be $C_0(\mathbb{R}^n) \otimes A$). For any topological group G, we let ΩG denote the space of loops in G based at the identity. ΩG is equipped with the compact-open topology.

Lemma 2.1. *$gU(SA)$ is homeomorphic as a topological group to $\Omega gU(A)$.*

Proof. Let $\pi_t : SA \to A$ be the $*$-homomorphism obtained by evaluation at $t \in [0,1]$. It is then straightforward to see that the map which takes $f \in gU(SA)$ to the loop $\hat{f}(t) = \pi_t(f)$, $t \in [0,1]$, in $gU(A)$ is a homeomorphism of topological group. $\square$

It follows that we have a natural isomorphism $[SX, gU(A)] \simeq [X, gU(SA)]$ (where SX is the reduced suspension of X). In particular, $[X, gU(S\cdot)]$ is abelian. Hence, for any space X, we obtain a homotopy invariant covariant functor from the category of σ-C^*-algebras to the category of abelian groups, namely $[X, gU(S\cdot)]$.

Lemma 2.2. *Let A be a σ-C^*-algebra. Then*

$$[S^n, gU(A)] \simeq \pi_n(gU(A)) \simeq \pi_0(gU(S^nA)),$$

where S^n is the n-sphere, $n = 0, 1, 2, \ldots$.

Proof. This is trivial for $n = 0$ since S^0 consists of two points and $S^0A = A$ by definition. Since $[S^n, gU(A)] \simeq [S^{n-1}, \Omega gU(A)]$, then it follows from Lemma 2.1 by induction. $\square$

Definition 2.3. For any σ-C^*-algebra A and any $n = 0, 1, 2, \ldots$. We set

$$\mathcal{RK}_n(A) = \pi_{n+1}(gU(A)).$$

For each n, $\mathcal{RK}_n$ is a homotopy invariant functor from the category of σ-C^*-algebras to the category of abelian groups.

Lemma 2.4. *Let $A \cong \varprojlim A_n$ be a σ-C^*-algebra and X be any topological space, then*

$$gU(A) = \varprojlim gU(A_n); \tag{2.4.1}$$

$$C(X, \varprojlim gU(A_n)) = \varprojlim C(X, gU(A_n)). \tag{2.4.2}$$

Proof. The proof is omitted. $\square$

The following theorem generalizes Proposition 2.1 of [21].

Theorem 2.5. *Let*

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\phi} B \longrightarrow 0$$

be a short exact sequence of σ-C^*-algebras and X a compact Hausdorff space. Then

$$[X, gU(I)] \xrightarrow{i_*} [X, gU(A)] \xrightarrow{\phi_*} [X, gU(B)]$$

is exact at $[X, gU(A)]$.

Proof. Using Proposition 5.3(2) of [10], we can write the exact sequence

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\phi} B \longrightarrow 0$$

as the inverse limit of exact sequences

$$0 \longrightarrow I_n \xrightarrow{i_n} A_n \xrightarrow{\phi_n} B_n \longrightarrow 0,$$

with surjective maps $\pi_n : A_{n+1} \to A_n$, $\sigma_n : B_{n+1} \to B_n$, $\kappa_n : A \to A_n$, and $\lambda_n : B \to B_n$. We can also assume that $I_{n+1} \to I_n$ and $I \to I_{n+1}$ are surjective.

It is immediate that $\phi_* \circ i_* = 0$, so let $f : X \to gU(A)$ be a (base point preserving and continuous) map and assume that $\phi_*([f]) = 0$, i.e.,

$\phi f : X \to gU(B)$ is homotopic to the constant map. We want to conclude that f is homotopic to a map of the form ig, where $g : X \to gU(I)$. By the assumption, there is a continuous function $F : X \times [0,\ 1] \to gU(B)$ such that

(2.5.a) $$F(x, 1) = \phi f(x), \quad x \in X,$$

(2.5.b) $$F(+, t) = 0, \quad t \in [0, 1],$$

(2.5.c) $$F(x, 0) = 0, \quad x \in X.$$

In particular, F maps into $gU_0(B)$. Note that in fact $F : X \times [0, 1] \to gU_0(B)$. Define $F_n : X \times [0, 1] \to gU_0(B_n)$ by $F_n(x,\ t) = \lambda_n(F(x, t))$. By the definition of F_n, we see that

(2.5.i) $$F_n(x, t) = \lambda_n(F(x, 1)) = \lambda_n \phi f(x), \quad x \in X,$$

(2.5.ii) $$F_n(x, 0) = 0, \quad t \in [0, 1],$$

(2.5.iii) $$F_n(x, 0) = 0, \quad x \in X.$$

So, by the proof of Proposition 2.1 [21], we can find a continuous map $H_n : X \times [0, 1] \to gU_0(A_n)$ for each n such that

(i) $$\phi_n H_n(t) = F_n(x, t), \quad (x, t) \in X \times [0, 1],$$

(ii) $$H_n(x, 1) = \kappa_n f(x), \quad x \in X.$$

After substituting H_n by the map $(x, t) \to H_n(x, t) \bullet H_n(+, t)^*$, we can furthermore assume that

(iii) $$H_n(+, t) = 0, \quad t \in [0, 1].$$

we construct inductively a coherent sequence of continuous maps $\widetilde{H}_n : X \times [0, 1] \to gU_0(A_n)$ such that

(1) $$\phi_n \widetilde{H}_n(x, t) = F_n(x, t), \quad (x, t) \in X \times [0, 1],$$

(2) $$\widetilde{H}_n(x, 1) = \kappa_n f(x), \quad x \in X,$$

(3) $$\widetilde{H}_n(+, t) = 0.$$

Let $\widetilde{H}_1(x,t) = H_1(x,t), (x,t) \in X \times [0,1]$. For the induction step, suppose we have found $\widetilde{H}_n$ such that

$$\phi_n \widetilde{H}_n(x,t) = F_n(x,t), \quad (x,t) \in X \times [0,1], \tag{2.5.1}$$

$$\widetilde{H}_n(x,1) = \kappa_n f(x), \quad x \in X, \tag{2.5.2}$$

$$\widetilde{H}_n(+,t) = 0, \tag{2.5.3}$$

$$\pi_{n-1}(\widetilde{H}_n(x,t)) = \widetilde{H}_{n-1}(x,t). \tag{2.5.4}$$

Since $\pi_n : A_{n+1} \to A_n$ is surjective. Thus, we can find a continuous map $Q : X \times [0,\ 1] \to gU_0(A_{n+1})$ such that $\pi_n(Q(x,t)) = \widetilde{H}_n(x,t)$. Then, because $\phi_n \circ \pi_n = \sigma_n \circ \phi_{n+1}$ and $\sigma_n \circ\ \lambda_{n+1} = \lambda_n$, we have

$$\begin{aligned}
\phi_n \circ \pi_n(Q(x,t) \bullet H_{n+1}(x,t)^*) &= \phi_n \circ \pi_n(Q(x,t)) \bullet \phi_n \circ \pi_n(H_{n+1}(x,t)^*) \\
&= \phi_n(\widetilde{H}_n(x,t)) \bullet \sigma_n \circ \phi_{n+1}(H_{n+1}(x,t)^*) \\
&= F_n(x,t) \bullet \sigma_n(F_{n+1}(x,t)^*) \\
&= \lambda_n(F(x,t)) \bullet \sigma_n \circ \lambda_{n+1}(F(x,t)^*) \\
&= \lambda_n(F(x,t)) \bullet \lambda_n(F(x,t)^*) = 0.
\end{aligned}$$

Put

$$C(x,t) = \pi_n(Q(x,t) \bullet H_{n+1}(x,t)^*),$$

we then have $C : X \times [0,1] \to I_n$. Furthermore, we in fact have $C \in C(X \times [0,1],\ gU_0(I_n))$. So, there is a $z \in C(X \times [0,1],\ gU_0(I_{n+1}))$ whose image in $C(X \times [0,1],\ gU_0(I_n))$ is C. After substituting z by the map $(x,t) \to [z(x,t) \bullet z(x,1)^*] \bullet [z(+,t) \bullet z(+,1)^*]^*$, we can suppose that $z(+,t) = 0$ and $z(x,1) = 0$. For $(x,t) \in X \times [0,1]$, the quasi-unitary $\phi_{n+1}(z(t))$ is a scalar multiple of the zero in B_{n+1}. Now define $\widetilde{H}_{n+1}(x,t) = z(x,t) \bullet H_{n+1}(x,t)$. It is immediately verified that

$$\begin{aligned}
\phi_{n+1}(\widetilde{H}_{n+1}(x,t)) &= \phi_{n+1}(z(x,t) \bullet H_{n+1}(x,t)) \\
&= \phi_{n+1}(z(x,t)) \bullet \phi_{n+1}(H_{n+1}(x,t)) \\
&= \phi_{n+1}(H_{n+1}(x,t)) = F_{n+1}(x,t),
\end{aligned} \tag{2.5.d}$$

$$\widetilde{H}_{n+1}(x,1) = z(x,1) \bullet H_{n+1}(x,1) = \kappa_{n+1} f(x), \tag{2.5.e}$$

$$\widetilde{H}_{n+1}(+,t) = z(+,t) \bullet H_{n+1}(+,t) = 0, \tag{2.5.f}$$

$$\begin{aligned}
\text{(2.5.h)} \qquad \pi_n(\widetilde{H}_{n+1}(x,t)) &= \pi_n(z(x,t) \bullet H_{n+1}(x,t)) \\
&= \pi_n(z(x,t)) \bullet \pi_n(H_{n+1}(x,t)) \\
&= C(x,t) \bullet \pi_n(H_{n+1}(x,t)) \\
&= \pi_n(Q(x,t) \bullet H_{n+1}(x,t)^*)\pi_n(H_{n+1}(x,t)) \\
&= \pi_n(Q(x,t) \bullet (H_{n+1}(x,t)^* \bullet H_{n+1}(x,t))) \\
&= \pi_n(Q(x,t)) = \widetilde{H}_n(x,t),
\end{aligned}$$

as desired. This completes the induction.

To finish the proof, let $H : X \times [0,1] \to gU(A)$ be the continuous map defined by the coherent sequence $(\widetilde{H}_n)$. Obviously, we have $\phi H(x,t) = F(x,t)$, $H(x,1) = f(x)$, and $H(+,t) = 0$ for $x \in X$, $t \in [0,1]$. Thus, H provides a homotopy between f and $H(\,\cdot\,,0)$. But (2.5.c) and the above imply that $H(x,\,\cdot\,)$ lies in $i(gU(I))$ for all $x \in X$. □

Remark 2.6. If X is any compact Hausdorff space and A is a σ-C^*-algebra, as usual let $C(X,A)$ stand for the σ-C^*-algebra of all continuous maps from X to A. Then it is known that $C(X,A)$ is *-isomorphic to $C(X) \otimes A$ [10]. Consider the quasi-unitary group $gU(C(X,A))$ with the pointwise multiplication. It is clear that each quasi-unitary $u{=}u(\cdot)$ of $C(X,A)$ can be regarded as a continuous map from X to $gU(A)$, i.e., $gU(C(X,A)) \cong C(X,gU(A))$. Obviously, the identity component $gU_0(C(X,A))$ of $gU(C(X,A))$ is a normal subgroup of both $C(X,gU_0(A))$ and $C(X,gU(A))$. Since $gU_0(A)$ is a normal subgroup of $gU(A)$, $C(X,gU_0(A))$ is a normal subgroup of $C(X,gU(A))$.

We have the following long exact sequence, of course, it is really just the long exact sequence of homotopy groups and the connecting map ∂ can therefore also be described, alternatively, in the usual way in terms of loops.

Theorem 2.7. *Let*

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\phi} B \longrightarrow 0$$

be a short exact sequence of σ-C^*-algebras. Then the following sequence of groups is exact:

$$\begin{aligned}
&\cdots \longrightarrow \mathcal{RK}_n(I) \xrightarrow{i_*} \mathcal{RK}_n(A) \xrightarrow{\phi_*} \mathcal{RK}_n(B) \xrightarrow{\partial} \mathcal{RK}_{n-1}(I) \longrightarrow \cdots \\
&\cdots \longrightarrow \mathcal{RK}_1(B) \xrightarrow{\partial} \mathcal{RK}_0(I) \xrightarrow{i_*} \mathcal{RK}_0(A) \xrightarrow{\phi_*} \mathcal{RK}_0(B).
\end{aligned}$$

Proof. This follows from Lemma 2.2 and Theorem 2.5. $\square$

For convenience, we describe the connecting homomorphism ∂ (cf. [2, Theorem 21.4.3]). We begin by introducing a bit of notation. Given an operator algebra A, define

$$\begin{aligned} IA =& C[0,1] \otimes A; \\ CA =& \{\xi \in IA : \xi(0) = 0\} \quad \text{the cone over } A; \\ SA =& \{\xi \in CA : \xi(1) = 0\} \cong C_0(\mathbb{R}) \otimes A \quad \text{the suspension of } A. \end{aligned}$$

If $\phi : A \to B$, then the mapping cone of ϕ is defined by

$$C_\phi = \{(\xi, a) \in CB \oplus A : \xi(1) = \phi(a)\}.$$

There is a natural mapping cone sequence

$$0 \longrightarrow SB \xrightarrow{\iota} C_\phi \xrightarrow{\pi} A \longrightarrow 0$$

defined by $\pi(\xi, a) = a$ and $\iota(\xi) = (\xi, 0)$. Let

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\phi} B \longrightarrow 0$$

be a short exact sequence of σ-C^*-algebras. Hence, there is a homomorphism $e : I \to C_\phi$ given by

$$e(x) = (0, i(x)), \quad x \in I.$$

Thus e_* is invertible and $\partial = e_*^{-1} \circ i_*$.

Corollary 2.8. *$\{\mathcal{RK}_n\}$ is a homology theory on the category of all σ-C^*-algebras.*

Proof. The proof is omitted. $\square$

Theorem 2.9. *For any σ-C^*-algebra A, $\mathcal{RK}_n(K \otimes A)$ is naturally isomorphic to $RK_n(A)$, $n = 1, 2, \ldots$.*

Proof. By [9, Definition 1.1], [24, 7.2], Bott periodicity for representable K-theory of σ-C^*-algebras [12, Theorem 3.4] and Remark 2.6, we have

$$\begin{aligned} RK_1(A) &\cong RK_1(C_0(\mathbb{R}^2) \otimes A) \\ &\cong [gU(K \otimes C_0(\mathbb{R}^2) \otimes A)] \end{aligned}$$

$$
\begin{aligned}
&\cong [gU(C_0(\mathbb{R}^2) \otimes K \otimes A)] \\
&\cong [\mathbb{R}^2, gU(K \otimes A)] \\
&\cong [S^2, gU(K \otimes A)] \\
&\cong \pi_2(gU(K \otimes A)) = \mathcal{RK}_1(K \otimes A).
\end{aligned}
$$

Similarly, we see that

$$
\begin{aligned}
RK_2(A) &\cong RK_1(SA) \\
&\cong \mathcal{RK}_1(K \otimes SA) \\
&\cong \mathcal{RK}_1(S(K \otimes A)) \\
&\cong \pi_2(gU(S(K \otimes A))) \\
&\cong \pi_0(gU(S^3(K \otimes A))) \\
&\cong \pi_3(gU(K \otimes A)) \\
&\cong \mathcal{RK}_2(K \otimes A).
\end{aligned}
$$

Consider the definition of the representable K-theory, by suspending and by use of Lemma 2.2, we conclude that the case of higher n $(n = 3, 4, \dots)$ holds by induction. □

Corollary 2.10. *If A is a C^*-algebra then $\mathcal{RK}_n(K \otimes A)$ is naturally isomorphic to $K_n(A)$, $n = 1, 2, 3, \dots$.*

Proof. This follows from Theorem 2.8 and Theorem 3.4 of [12]. □

3. $\mathcal{RK}$-Stability and its applications

As we shall see, the calculation of the groups $\mathcal{RK}_n(A)$ is, in many cases, already contained in the calculation of the usual (stable) RK-groups of A. To make this precise, we make the following definition.

Definition 3.1. Let A be a σ-C^*-algebra, i.e., $A \cong \varprojlim A_n$. We shall say that A is $\mathcal{RK}$-stable when all $A_n (n = 1, 2, 3, \dots)$ are $\mathcal{K}$-stable.

It is clear that $\mathcal{RK}_n(A) \cong \mathcal{RK}_n(K \otimes A)$ for all $n = 0, 1, 2, \dots,$ when A is a $\mathcal{RK}$-stable σ-C^*-algebra. So for such σ-C^*-algebra, $\mathcal{RK}_n(A) \cong RK_0(A)$ for n even and $\mathcal{RK}_n(A) \cong RK_1(A)$ for n odd. In particular, for such σ-C^*-algebras, $\mathcal{RK}_n(A) \cong K_0(A)$ for n even and $\mathcal{RK}_n \cong K_1(A)$ for n odd.

Theorem 3.2. *Let A be a $\mathcal{RK}$-stable σ-C^*-algebra, I is an ideal of A such that the quotient σ-C^*-algebra $B = A/I$ is also $\mathcal{RK}$-stable, then I is also $\mathcal{RK}$-stable.*

Proof. It follows from [21] and the exactness of the $\mathcal{RK}_n$-functors. □

Theorem 3.3. *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a σ-C^*-extension. If B and I are $\mathcal{RK}$-stable, then so is A.

Proof. Write the exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

as an inverse limit of exact sequences

$$0 \longrightarrow I_n \longrightarrow A_n \longrightarrow B_n \longrightarrow 0$$

of C^*-algebras, with all maps in the inverse systems being surjective (see [10, Proposition 5.3(2)]).

Now this theorem immediately from [21]. □

Theorem 3.4. *Let A be a C^*-algebra, then*

$$\mathcal{RK}_n(A) = \mathcal{K}_n(A), \quad n = 0, 1, 2, \dots .$$

Proof. It follows from [21] and Definition 2.3. □

Corollary 3.5. *For every compact Hausdorff space X, we have*

$$\mathcal{RK}_n(C(X)) \cong H^{-n}(X, \mathbb{Z}), \quad n = 0, 1, 2, \dots .$$

Proof. This follows from the above theorem and [21, Theorem 4.1]. □

The following result states that non-stable representable K-theory is countable additive for products.

Theorem 3.6. *Let $\{A_n \colon n \geq 1\}$ be a family of σ-C^*-algebras. Then there is a canonical isomorphism*

$$\mathcal{RK}_i\left(\prod_{n=1}^{\infty} A_n\right) \cong \prod_{n=1}^{\infty} \mathcal{RK}_i(A_n).$$

Proof. We first consider the case in which $i = 0$. Then one easily checks that

$$gU\left(\prod_n A_n\right) \cong \prod_n gU(A_n),$$

and that

$$\left[X, gU\left(\prod_n A_n\right)\right] \cong \prod_n [X, gU(A_n)],$$

here X is a compact Hausdorff space. It now follows easily that

$$\mathcal{RK}_0\left(\prod_n A_n\right) \cong \prod_n \mathcal{RK}_0(A_n).$$

The statement for $\mathcal{RK}_i$ for $i > 0$ now follows by tensoring all the algebras with $C_0(\mathbb{R}^i)$. □

Lemma 3.7 [21]**.** *For each $n = 2, 3, 4, \ldots$, and any C^*-algebra A, the tensor product C^*-algebra $O_n \otimes A$ is $\mathcal{K}$-stable.*

Call a σ-C^*-algebra nuclear if it is the inverse limit of a system of nuclear C^*-algebras with surjective maps, as in the remarks following [10, Proposition 3.3].

Theorem 3.8. *For each $n = 2, 3, \ldots$, and any σ-C^*-algebra A, the tensor product σ-C^*-algebra $O_n \otimes A$ is $\mathcal{RK}$-stable.*

Proof. Write $A = \varprojlim A_n$ with all maps surjective. Then $O_n \otimes A = \varprojlim O_n \otimes A_n$ since O_n is nuclear. Therefore $O_n \otimes A$ is $\mathcal{RK}$-stable by Lemma 3.7. □

Next we want to consider AF-algebras. For this purpose we consider AF-algebra A for which $K_0(A)$ has large denominators, as defined by Nistor in [7]. Let $A_1 \subseteq A_2$ be an inclusion of finite-dimensional C^*-algebras with inclusion matrix $A = (a_{ij})$. Set $\beta(A_1, A_2) = \min\{a_{ij} : a_{ij} \neq 0\}$. Thus $\beta(A_1, A_2)$ is the minimum of the number of edges between points in the Bratteli diagram of the inclusion.

Lemma 3.9 [21]. *Let A be an AF-algebra for which $K_0(A)$ has large denominators and B an arbitrary C^*-algebra. Then $A \otimes B$ is $\mathcal{K}$-stable.*

Theorem 3.10. *Let A be an AF-algebra for which $K_0(A)$ has large denominators and B an arbitrary σ-C^*-algebra. Then $A \otimes B$ is $\mathcal{RK}$-stable.*

Proof. Write $B = \varprojlim B_n$ with all maps surjective. Then $A \otimes B = \varprojlim A \otimes B_n$, since A is nuclear. Therefore $A \otimes B$ is $\mathcal{RK}$-stable. □

According to Proposition 2.3 of [7], if A is an infinite-dimensional simple AF-algebra other than K, then $K_0(A)$ has large denominators. The above theorem therefore has the following corollary.

Corollary 3.11. *Let A be an infinite-dimensional simple AF-algebra. Then $A \otimes B$ is $\mathcal{RK}$-stable for all σ-C^*-algebras B.*

Lemma 3.12 [21]. *Let A be a σ-unital C^*-algebra and B an arbitrary C^*-algebra. Then $Q(A) \otimes B$ is $\mathcal{K}$-stable, where $Q(A)$ denotes the out stable multiplier algebra of A.*

Theorem 3.13. *Let A be a σ-unital σ-C^*-algebra and B an arbitrary σ-C^*-algebra. Then $Q(A) \otimes B$ is $\mathcal{RK}$-stable.*

Proof. Write $A = \varprojlim A_n$ and $B = \varprojlim B_m$. By Proposition 3.2, Theorem 3.14, the proof of Proposition 5.3 and Corollary 5.4 in [10], we have

$$\begin{aligned} Q(A) \otimes B &= [\varprojlim (M(K \otimes A_n)/K \otimes A_n)] \otimes B \\ &= \varprojlim [(M(K \otimes A_n)/K \otimes A_n) \otimes B_m], \end{aligned}$$

then, by Lemma 3.12, $Q(A) \otimes B$ is $\mathcal{RK}$-stable. □

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