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On speed of mean convergence of Lagrange and Hermite interpolation based on the roots of Laguerre polynomials

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Dedicated to Professor Béla Gyires on his 80th birthday

The mean convergence of interpolating processes was investigated by many authors after the pioneer work of ERDŐS, TURÁN and BALÁZS [2,4]. See for references on this topic, e.g. [10,12]. We mention first the classical result of ERDŐS and TURÁN [1,2] stating that for a general class of weights, the Lagrange interpolation converges in L^2 –norm on finite interval. For the case of infinite interval, the analogous theorem was proved by BALÁZS and TURÁN [4,5]. In [13] the authors gave an estimate for the speed of L^2 –norm convergence of the Lagrange interpolation with Laguerre abscissas; it was based on a Jackson type theorem corresponding to the Laguerre weight. For the Hermite interpolation on finite interval a general result is given in [1, p. 419].

In [15] JOÓ and SZABADOS proved an L^1 –norm convergence for two interpolating processes based on roots of Laguerre polynomials $L_n^\alpha(x)$ $(-1 < \alpha \le 0)$. Using the ideas of the present paper, one can extend this result for any $\alpha > -1$.

Here we concern ourselves only to the case of Laguerre nodes, because this contains the ideas for Jacobi and Hermite nodes too. We use the ideas of [2–7].

Denote by $L_n^{(\alpha)}(x) := x^{-\alpha}e^x\frac{1}{n!}\left[e^x x^{n+\alpha}\right]^{(n)}$ the Laguerre polynomials for $\alpha > -1$ and by $0 < x_1 < \cdots < x_n$ the zeros of $L_n^{(\alpha)}(x)$ (in fact $x_k = x(k,n,\alpha)$; we simplify the notation).
Define further

$$\ell_k(x) := \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)\prime}(x_k)(x - x_k)} \qquad (k = 1,\dots,n)$$

the fundamental polynomials of the Lagrange interpolation.

Let $L_n(f,x) := \sum_{k=1}^{n} f(x_k)\ell_k(x)$ be the Lagrange interpolation of f. For any $f \in C^1(0,\infty)$ the Hermite interpolating polynomial of f is defined as follows:

$$H_n(f,x) := \sum_{k=1}^{n} \left[f(x_k) \frac{x_k(x_k-\alpha)+x(\alpha+1-x_k)}{x_k} + \right.$$
$$\left. + f'(x_k)(x-x_k) \right] \ell_k^2(x).$$

As in [15], introduce the following space:

$$C(\lambda) := \left\{ f \in C\,[0,\infty) \ : \ \lim_{x\to\infty} f(x)e^{-\lambda x} = 0 \right\},$$

further let

$$E_n(f) := E_n^{\lambda}(f) := \inf_{P_n \in \Pi_n} \max_{x \geq 0} |f(x) - p_n(x)| e^{-\lambda x}.$$

We shall prove the following

Theorem. *Let $\alpha > -1$, $0 < \lambda < 1/2$, then we have the estimates*

(1)
$$\left(\int_0^{\infty} x^{\alpha} e^{-x} |f(x) - L_n(f,x)|^2 dx \right)^{1/2} \leq c \cdot E_{n-1}^{\lambda}(f) \quad (f \in C\,[0,\infty))$$

(2)
$$\int_0^{\infty} x^{\alpha} e^{-x} |f(x) - H_n(f,x)| dx \ \leq c \cdot E_{2n-1}^{\lambda}(f') \ (f \in C^1\,[0,\infty))$$

$(n = 1, 2, \dots)$ where c is an absolute constant (independent of f and of n). (Π_n denotes the set of algebraic polynomials of degree at most n).

Lemma. *Let $\alpha > -1$ and $0 < \mu < 1/2$, then for any $f \in C\,(0,\infty)$:*

$$(3) \quad \left(\int_0^{\infty} x^{\alpha} e^{-x} \cdot L_n(f,x)^2 dx \right)^{1/2} \leq c \cdot \left(\max_{1 \leq k \leq n} |f(x_k)|^2 e^{-\mu x_k} \right)^{1/2}$$

and for any $f \in C^1(0,\infty)$

$$\int_0^\infty x^\alpha e^{-x}|H_n(f,x)|dx \le c\cdot \max_{1\le k\le n}|f(x_k)|e^{-\mu x_k}+ \tag{4}$$
$$+\bar{c}\cdot \max_{1\le k\le n}|f'(x_k)|e^{-\mu x_k}.$$

PROOF of the Lemma.
Consider a function $g \in C^{2n}(0,\infty)$ with property $g^{(2n)} > 0$ on $(0,\infty)$. Then either $g \ge H_n(g)$ on $(0,\infty)$ or $g \le H_n(g)$ on $(0,\infty)$. Indeed, if $g(y_1) > H_n(g,y_1)$ and $g(y_2) < H_n(g,y_2)$ then the function $g - H_n(g)$ has at least $2n+1$ zeros (counted with multiplicity) and then $(g-H_n(g))^{(2n)} = g^{(2n)}$ has also a zero – a contradiction. Choose specially $g(x) = e^{\mu x}$, we get

$$\sum_{k=1}^n \left\{ e^{\mu x_k}\frac{x_k(x_k-\alpha)+x(\alpha+1-x_k)}{x_k} + \mu e^{\mu x_k}(x-x_k)\right\}\ell_k^2(x) \le$$
$$\le e^{\mu x} \quad (x\ge 0).$$

Multiplying by $x^\alpha e^{-x}$ and integrating from 0 to ∞ we obtain:

$$\int_0^\infty x^\alpha e^{-x}\sum_{k=1}^n e^{\mu x_k}\frac{x_k(x_k-\alpha)+x(\alpha+1-x_k)}{x_k}\ell_k^2(x)dx \le \tag{5}$$
$$\le \int_0^\infty x^\alpha e^{-(1-\mu)x}dx =: c(\alpha,\mu).$$

Now

$$\int_0^\infty x^\alpha e^{-x}L_n(f,x)^2dx = \int_0^\infty x^\alpha e^{-x}\left[\sum_{k=1}^n f(x_k)\ell_k(x)\right]^2 dx =$$
$$= \int_0^\infty x^\alpha e^{-x}\sum_{k=1}^n f(x_k)^2\ell_k^2(x)dx \le$$
$$\le \max_{1\le k\le n} f(x_k)^2 e^{-\mu x_k}\int_0^\infty x^\alpha e^{-x}\sum_{k=1}^n e^{\mu x_k}\ell_k^2(x)dx =$$
$$= \max_{1\le k\le n} f(x_k)^2 e^{-\mu x_k}\int_0^\infty x^\alpha e^{-x}\sum_{k=1}^n e^{\mu x_k}\frac{x_k(x_k-\alpha)+x(\alpha+1-x_k)}{x_k}$$

since

$$\sum_{k=1}^{n} e^{\mu x_k}\ell_k^2(x) - \sum_{k=1}^{n} e^{\mu x_k}\frac{x_k(x_k-\alpha)+x(\alpha+1-x_k)}{x_k}\ell_k^2(x) =$$
$$= \sum_{k=1}^{n} e^{\mu x_k}\frac{(\alpha+1-x_k)(x_k-x)}{x_k}\ell_k^2(x);$$

further

$$\int_0^\infty x^\alpha e^{-x}|H_n(f,x)|dx \le \int_0^\infty x^\alpha e^{-x}\left|\sum_{k=1}^{n} f(x_k)\frac{x_k(x_k-\alpha)+x(\alpha+1-x_k)}{x_k}\right.$$
$$\left.\ell_k^2(x)\right|dx + \int_0^\infty x^\alpha e^{-x}\left|\sum_{k=1}^{n} f'(x_k)(x-x_k)\ell_k^2(x)\right|dx = I_1 + I_2,$$

by orthogonality

$$I_1 \le \max_{1\le k\le n}|f(x_k)|e^{-\mu x_k}\int_0^\infty x^\alpha e^{-x}\sum_{k=1}^{n} e^{\mu x_k}\left[(x_k+|\alpha|)+\left(1+\alpha\frac{x}{x_k}+x\right)\right]$$
$$\ell_k^2(x)dx \le \max_{1\le k\le n}|f(x_k)|e^{-\mu x_k}\left\{\int_0^\infty x^\alpha e^{-x}\sum_{k=1}^{n} e^{(\mu+1)x_k/2}\ell_k^2(x)dx+\right.$$
$$+|\alpha|\int_0^\infty x^\alpha e^{-x}\sum_{k=1}^{n} e^{\mu x_k}\ell_k^2(x)dx + (1+\alpha)\int_0^\infty x^\alpha e^{-x}\sum_{k=1}^{n} e^{\mu x_k}\ell_k^2(x)dx+$$
$$\left.+\int_0^\infty x^\alpha e^{-x}\sum_{k=1}^{n} e^{\mu x_k}x_k\ell_k^2(x)dx\right\} \le$$
$$\le \max_{1\le k\le n}|f(x_k)|e^{-\mu x_k}\left\{2\int_0^\infty x^\alpha e^{-x}\sum_{k=1}^{n} e^{\frac{\mu+1}{2}x_k}\ell_k^2(x)dx+\right.$$
$$\left.+(1+\alpha+|\alpha|)\int_0^\infty x^\alpha e^{-x}\sum_{k=1}^{n} e^{\mu x_k}\ell_k^2(x)dx\right\} \le$$
$$\le \max_{1\le k\le n}|f(x_k)|e^{-\mu x_k}\left\{2c(\alpha,\frac{\mu+1}{2})+(1+\alpha+|\alpha|)\,c\,(\alpha,\mu)\right\} =$$
$$= \max_{1\le k\le n}|f(x_k)|e^{-\mu x_k}c^\star(\alpha,\mu)$$

and

$$I_2 \leq \max_{1\leq k\leq n} |f'(x_k)|e^{-\mu x_k} \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\mu x_k}(x+x_k)\ell_k^2(x)dx =$$

$$=2 \max_{1\leq k\leq n} |f'(x_k)|e^{-\mu x_k} \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\mu x_k} x_k \ell_k^2(x)dx \leq$$

$$\leq 2 \max_{1\leq k\leq n} |f'(x_k)|e^{-\mu x_k} \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\frac{\mu+1}{2}x_k} \ell_k^2(x)dx \leq$$

$$\leq 2c\left(\alpha, \frac{\mu+1}{2}\right) \max_{1\leq k\leq n} |f'(x_k)|e^{-\mu x_k}.$$

Summarizing our estimates we get (4) with $\bar{c} = \max\{c^\star(\alpha,\mu), 2c(\alpha, \frac{\mu+1}{2})\}$. □

PROOF of the Theorem.
Let $p_n \in \prod_{n-1}$ be such that $\max_{x\geq 0} |f(x) - p_n(x)|e^{-\lambda x} \leq E_{n-1}^\lambda(f)$, then applying (3) we get for $\mu = 2\lambda$

$$\int_0^\infty x^\alpha e^{-x}|f(x) - L_n(f,x)|^2 dx \leq 2\int_0^\infty x^\alpha e^{-x}|f(x) - p_n(x)|^2 dx +$$

$$+2\int_0^\infty x^\alpha e^{-x}|L_n(f-p_n,x)|^2 dx \leq 2E_{n-1}^2(f)\int_0^\infty x^\alpha e^{-(1-2\lambda)x}dx +$$

$$+2c^2(\alpha, \frac{\mu}{2}) \max_{1\leq k\leq n} |(f-p_n)(x_k)|^2 e^{-\mu x_k} \leq c_1 E_{n-1}^2(f) + c_2 E_{n-1}^2(f),$$

hence (1) follows. For the proof of (2) pick a polynomial $p_n \in \prod_{2n-1}$ such that $\max_{x\geq 0} |f'(x) - p_n'(x)|e^{-\lambda x} \leq E_{2n-2}^\lambda(f')$. We can further assume that $f(1) = p_n(1)$, hence

$$|f(x) - p_n(x)| = \left|\int_1^x \Big(f'(t) - p_n'(t)\Big)dt\right| \leq \int_1^x |f' - p_n'| \leq E_{2n-2}(f') \cdot$$

$$\cdot \int_1^x e^{\lambda t}dt = O\left(e^{\lambda x}E_{2n-2}(f')\right) \ (x \geq 0)$$

i.e. $\max_{x\geq 0} |f(x) - p_n(x)|e^{-\lambda x} \leq c_3 E_{2n-2}(f') \ (n \geq 1)$.

Hence, applying also (4) for $\mu = \lambda$:

$$\int_0^\infty x^\alpha e^{-x}|f(x) - H_n(f,x)|dx \leq \int_0^\infty x^\alpha e^{-x}|f(x) - p_n(x)|dx +$$

$$+ \int_0^\infty x^\alpha e^{-x}|H_n(f - p_n, x)|dx \leq c_3 E_{2n-2}(f') \int_0^\infty x^\alpha e^{-(1-\lambda)x}dx +$$

$$+ \bar{c} \max_{1\leq k\leq n} |f(x_k) - p_n(x_k)|e^{-\lambda x_k} + \bar{c} \max_{1\leq k\leq n} |f'(x_k) - p_n'(x_k)|e^{-\lambda x_k} =$$

$$= O(E_{2n-2}(f')). \quad \square$$

Remark. For the sake of completeness we have to investigate the quantity $E_n(f)$ defined here. We return to this in a subsequent paper. Taking into account the Stone–Weierstraß Theorem, we have at least: $E_n(f) \to 0 \quad (n \to \infty)$.

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