Publ. Math. Debrecen
38 / 3–4 (1991), 255–262

Additive functions on the Gaussian integers

By MARIJKE van ROSSUM–WIJSMULLER (Philadelphia)

Introduction

In this paper we generalize a result of KÁTAI on completely additive functions by defining the functions on the ring of Gaussian integers. In [1] I. KÁTAI considers the following problem: Let F_0, F_1, F_2 and F_3 be real–valued functions, that are completely additive and which are defined on the positive integers. Under the assumption that the sum $[F_0(n) + F_1(n+1) + F_2(n+2) + F_3(n+3)]$ is integer–valued for all n, KÁTAI proves that F_0, F_1, F_2 and F_3 are integer–valued functions.

A completely additive function is determined by its values on the primes and this suggests that one could define such a function on a unique factorization domain if one takes care to define the functions properly at 0 to preserve additivity. In this paper we consider complex–valued functions defined on the ring of Gaussian integers and prove three theorems.

Results

Let the ring of Gaussian integers be denoted by $\mathbf{G}$. For $j = 0, 1, 2, 3$, let F_j be a complex–valued function defined on $\mathbf{G}$ with the additional requirement that $F_j(0) = \infty$. We require the functions to be completely additive, which means that

$$F_j(\alpha\beta) = F_j(\alpha) + F_j(\beta) \quad \text{for all } \alpha \text{ and } \beta \text{ in } \mathbf{G}.$$

We use the symbol $\mathbf{G}^\star$ for the ring of Gaussian integers together with ∞.

Because $\mathbf{G}$ has four units and four associates of each nonzero element, the additivity of the functions F_j requires:

$$(1) \qquad F_j(1) = F_j(-1) = F_j(i) = F_j(-i) = 0.$$

$$(2) \qquad F_j(\alpha) = F_j(-\alpha) = F_j(i\alpha) = F_j(-i\alpha) \qquad \text{for all } \alpha \text{ in } \mathbf{G}.$$

As the first theorem we state the following generalization of KÁTAI's result:

Theorem 1. *If the sum* $[F_0(\alpha)+F_1(\alpha+1)+F_2(\alpha+2)+F_3(\alpha+3)]$ *is in* $\mathbf{G}^\star$*, for all* α *in* $\mathbf{G}$*, then* $F_j(\alpha)$ *is in* $\mathbf{G}^\star$*, for all* α *in* $\mathbf{G}$*, and* $j=0,1,2,3$.

The second theorem is a somewhat stronger result and can be stated as follows:

Theorem 2. *If for some fixed, nonzero* β *in* $\mathbf{G}$ *and all* α *in* $\mathbf{G}$ *the sum* $[F_0(\alpha)+F_1(\alpha+\beta)+F_2(\alpha+2\beta)+F_3(\alpha+3\beta)]$ *is an element of* $\mathbf{G}^\star$ *then* $F_j(\alpha)$ *is an element of* $\mathbf{G}^\star$ *for all* α *in* $\mathbf{G}$ *and* $j=0,1,2,3$.

In contrast to the integers, $\mathbf{G}$ is two–dimensional and this is the motivation of Theorem 3.

Theorem 3. *If the sum* $[F_0(\alpha)+F_1(\alpha+i)+F_2(\alpha+1)+F_3(\alpha-i)]$ *is an element of* $\mathbf{G}^\star$ *for all* α *in* $\mathbf{G}$*, then* $F_j(\alpha)$ *is an element of* $\mathbf{G}^\star$ *for all* α *in* $\mathbf{G}$ *and* $j=0,1,2,3$.

PROOF OF THEOREM 1. The proof in [1] is equally valid for functions that are defined on the Gaussian integers. Hence by KÁTAI's result, $F_j(n)\in\mathbf{G}$ for all positive integers n. Therefore, by (2), $F_j(-n)$ and $F_j(\pm ni)$ are elements of $\mathbf{G}$. In addition, if a Gaussian integer α is divisible by a rational integer n then $F_j(\alpha)\in\mathbf{G}^\star$ if and only if $F_j(\alpha/n)\in\mathbf{G}^\star$.

Because $\alpha\overline{\alpha}$ is a positive integer for every nonzero $\alpha\in\mathbf{G}$ we have $F_j(\alpha\overline{\alpha})\in\mathbf{G}$. It follows that $F_j(\alpha)\in\mathbf{G}$ if and only if $F_j(\overline{\alpha})\in\mathbf{G}$. Together with (2) this means that $F_j(a+bi)\in\mathbf{G}$ implies $F_j(\pm a\pm bi)\in\mathbf{G}$ and also $F_j(\pm b\pm ai)\in\mathbf{G}$.

From (2) it follows immediately that the following four statements are equivalent.

(3) $[F_0(\alpha)+F_1(\alpha+1)+F_2(\alpha+2)+F_3(\alpha+3)]\in\mathbf{G}^\star$ for all α.

(4) $[F_0(\alpha)+F_1(\alpha-1)+F_2(\alpha-2)+F_3(\alpha-3)]\in\mathbf{G}^\star$ for all α.

(5) $[F_0(\alpha)+F_1(\alpha+i)+F_2(\alpha+2i)+F_3(\alpha+3i)]\in\mathbf{G}^\star$ for all α.

(6) $[F_0(\alpha)+F_1(\alpha-i)+F_2(\alpha-2i)+F_3(\alpha-3i)]\in\mathbf{G}^{\star\prime}$ for all α.

The equivalence of (3) and (4) and the equivalence of (5) and (6) results in a useful symmetry, which is more apparent when we rewrite (4) and (6) as

(4a) $[F_0(\alpha+3)+F_1(\alpha+2)+F_2(\alpha+1)+F_3(\alpha)]\in\mathbf{G}^\star$ for all α.

(6a) $[F_0(\alpha+3i)+F_1(\alpha+2i)+F_2(\alpha+i)+F_3(\alpha)]\in\mathbf{G}^\star$ for all α.

Therefore, if a linear combination of the four functions F_j is in $\mathbf{G}^\star$ then it follows that a similar linear combination, obtained by interchanging F_0 with F_3 and F_1 with F_2 simultaneously, will also be in $\mathbf{G}^\star$. In particular,

$F_0(\alpha) \in \mathbf{G}^\star$ if and only if $F_3(\alpha) \in \mathbf{G}^\star$ and $F_1(\alpha) \in \mathbf{G}^\star$ if and only if $F_2(\alpha) \in \mathbf{G}^\star$.

Next we show that $F_j(\alpha) \in \mathbf{G}^\star$ for all α with norm less or equal to 13. Because of symmetry it is sufficient to show that $F_1(3+2i)$, $F_1(1+i)$, $F_1(2+i)$, $F_0(1+i)$, $F_0(2+i)$ and $F_0(3+2i)$ are in $\mathbf{G}$.

When we denote the sums in (3), (4), (5) and (6) by L_α^{+1}, L_α^{-1}, L_α^{+i} and L_α^{-i} respectively, it is easy to verify that our claim follows from the following six equations:

$$\text{(i)}\quad L_{4+2i}^{-1} + L_{1+i}^{-i} + L_{2-2i}^{+i} - L_{-1+i}^{+1} - L_{3-i}^{-1} = F_0(4) + F_1(3+2i) + F_2(4) + F_3(5)$$

$$\text{(ii)}\quad L_{3+4i}^{-i} + L_{2-i}^{+i} - L_i^{+1} - L_{2+i}^{-1} = F_0(5) + F_1(3)(1+i) + F_2(3+2i) + F_3(2)$$

$$\text{(iii)}\quad L_{2+2i}^{-i} - L_{1-i}^{+i} = F_0(2) + F_1(2+i) + F_2(1+i)$$

$$\text{(iv)}\quad L_{3+3i}^{-i} = F_0(3)(1+i) + F_1(3+2i) + F_2(2-i)(1+i) + F_3(3)$$

$$\text{(v)}\quad L_{2+i}^{-1} = F_0(2+i) + F_1(1+i) + F_3(1+i)$$

$$\text{(vi)}\quad L_{3+2i}^{-1} = F_0(3+2i) + F_1(2)(1+i) + F_2(1+2i) + F_3(2i)$$

We finish the proof by induction on the norm of α. We assume that there is some α with smallest norm for which the theorem is not true. Without loss of generality, we may assume that $\alpha = (a+bi)$ is a prime in $\mathbf{G}$ and $a \not\equiv b \pmod 2$. Also we may assume $a > b > 0$, since otherwise an associate of α or $\overline{\alpha}$ will have this property. Since $\alpha\overline{\alpha} > 13$, a must be strictly greater than 3. From $L_\alpha^{-1} \in \mathbf{G}$ it then follows at $F_0(\alpha) \in \mathbf{G}$ and thus also $F_3(\alpha) \in \mathbf{G}$. If a is odd, $(a+1)$ is even and $(\alpha+1)$ is divisible by 2. If a is even, $(a+1)$ is odd and $(\alpha+1)$ is divisible by $(1+i)$. In both cases $F_0(\alpha+1) \in \mathbf{G}$ and, since $L_{\alpha+1}^{-1} \in \mathbf{G}$, we conclude that $F_1(\alpha) \in \mathbf{G}$ and therefore also $F_2(\alpha) \in \mathbf{G}$. This finishes the proof of Theorem 1.

PROOF OF THEOREM 2. As was the case in Theorem 1, there are four equivalent forms of the hypothesis of Theorem 2, but we will only make use of 2 of these and therefore we will not state the other two.

Because of (2), the following are equivalent statements:

$$\text{(7)}\quad F_0(\alpha) + F_1(\alpha+\beta) + F_2(\alpha+2\beta) + F_3(\alpha+3\beta) \in \mathbf{G}^\star \quad \text{for all } \alpha.$$

$$\text{(8)}\quad F_0(\alpha) + F_1(\alpha-\beta) + F_2(\alpha-2\beta) + F_3(\alpha-3\beta) \in \mathbf{G}^\star \quad \text{for all } \alpha.$$

We will denote the sums in (7) and (8) by L_α^+ and L_α^-. Since L_α^+ is an element of $\mathbf{G}^\star$ for all α, so is $L_{\alpha\beta}^+$. But

$$L_{\alpha\beta}^+ = [F_0(\alpha) + F_1(\alpha+1) + F_2(\alpha+2) + F_3(\alpha+3)] + [F_0(\beta) + F_1(\beta) + F_2(\beta) + F_3(\beta)].$$

If we can show that

$$[F_0(\beta) + F_1(\beta) + F_2(\beta) + F_3(\beta)] \in \mathbf{G}, \tag{9}$$

Theorem 2. will follow from Theorem 1.

We will therefore set out to prove (9). We will use an idea of KÁTAI and consider linear combinations of $F_j(2)$ and $F_j(3)$ for $j = 0, 1, 2, 3$. These are obtained by simplifying the following expressions.

$$L^+_{32\beta} + L^+_{3\beta} + L^-_{5\beta} + L^-_{6\beta} - L^-_{12\beta} - L^+_{15\beta} - L^+_{2\beta} - L^+_{4\beta} \tag{10}$$

$$L^-_{35\beta} + L^-_{6\beta} + L^+_{2\beta} + L^+_{3\beta} - L^+_{9\beta} - L^-_{18\beta} - L^-_{5\beta} - L^-_{7\beta} \tag{10a}$$

$$L^+_{3\beta} + L^-_{11\beta} + L^-_{18\beta} + L^+_{5\beta} - L^+_{33\beta} - L^-_{6\beta} - L^+_{2\beta} - L^-_{5\beta} \tag{11}$$

$$L^-_{6\beta} + L^+_{8\beta} + L^+_{15\beta} + L^-_{8\beta} - L^-_{36\beta} - L^+_{3\beta} - L^-_{5\beta} - L^+_{2\beta} \tag{11a}$$

$$L^+_{1\beta} - L^-_{4\beta} \tag{12}$$

$$L^+_{3\beta} + L^+_{7\beta} - L^-_{7\beta} - L^+_{2\beta} \tag{13}$$

$$L^-_{6\beta} + L^-_{10\beta} - L^+_{4\beta} - L^-_{5\beta} \tag{13a}$$

$$L^-_{343\beta} + L^-_{5\beta} + L^-_{11\beta} + 3L^+_{3\beta} - L^+_{9\beta} - L^-_{20\beta} - L^-_{33\beta} - 3L^-_{7\beta} \tag{14}$$

$$L^+_{340\beta} + L^+_{2\beta} + L^+_{8\beta} + 3L^-_{6\beta} - L^-_{12\beta} - L^+_{17\beta} - L^+_{30\beta} - 3L^+_{4\beta} \tag{14a}$$

By hypothesis, each sum is an element of $\mathbf{G}$. The nine equations can be expressed in matrix form by $MR = G$, where R is the transpose of the row vector $(F_0(2), F_0(3), F_1(2), F_1(3), F_2(2), F_2(3), F_3(2), F_3(3))$, M is the coefficient matrix which is equal to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 1 & -2 \\ 1 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 & 4 & -1 \\ 4 & -1 & 0 & 1 & 1 & 0 & -1 & 1 \\ -2 & 0 & 1 & -1 & -1 & 1 & 2 & 0 \\ -1 & 1 & 4 & -2 & -2 & 2 & 0 & 1 \\ 0 & 1 & -2 & 2 & 4 & -2 & -1 & 1 \\ -2 & 0 & 1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 1 & -1 & -2 & 0 \end{bmatrix}$$

and G is a column vector with entries in $\mathbf{G}$. M contains eight linearly independent rows. Using Gaussian elimination over the integers, it follows that $F_j(2) \in \mathbf{G}$ and $F_j(3) \in \mathbf{G}$ for $j = 0, 1, 2, 3$. Since $L^+_{1\beta}$ is an element of $\mathbf{G}$, it now follows that $[F_0(\beta) + F_1(\beta) + F_2(\beta) + F_3(\beta)]$ is an element of $\mathbf{G}$, which finishes the proof of Theorem 2.

PROOF OF THEOREM 3. Because of (2) the following statements are equivalent:

(15) $F_0(\alpha) + F_1(\alpha + i) + F_2(\alpha + 1) + F_3(\alpha - i) \in \mathbf{G}^\star$ for all α.

(16) $F_0(\alpha) + F_1(\alpha + 1) + F_2(\alpha - i) + F_3(\alpha - 1) \in \mathbf{G}^\star$ for all α.

(17) $F_0(\alpha) + F_1(\alpha - i) + F_2(\alpha - 1) + F_3(\alpha + i) \in \mathbf{G}^\star$ for all α.

(18) $F_0(\alpha) + F_1(\alpha - 1) + F_2(\alpha + i) + F_3(\alpha + 1) \in \mathbf{G}^\star$ for all α.

We will denote the sums in (15), (16), (17) and (18) by respectively S_α^{+i}, S_α^{+1}, S_α^{-i} and S_α^{-1}.

We will first prove that $F_j(\alpha) \in \mathbf{G}^\star$ for all α with norm less or equal to 5. We will do this in three Lemma's:

Lemma 1. *If S_α^{+i} is in $\mathbf{G}^\star$ for all α, then $F_j(2)$ is in $\mathbf{G}$ for $j = 0, 1, 2, 3$.*

PROOF. Because $F_2(3) = [S_2^{+i} - S_{1+i}^{+1} - S_{1-i}^{-1}]$, it follows that $F_2(3)$ is an element of $\mathbf{G}$. Since

$$[S_2^{+i} + S_2^{-i}] - [S_{1+i}^{+1} + S_{1+i}^{-1} + S_{1+i}^{+i} + S_{1+i}^{-i}] = F_2(3) - F_2(5)$$

we can conclude that $F_2(5)$ is in $\mathbf{G}$. The Lemma now follows from the following four equations:

(i) $S_3^{+i} + S_2^{+i} - S_3^{-i} - S_2^{-i} = F_2(2) + F_2(3)$

(ii) $S_{1-2i}^{+i} + S_{1+i}^{+1} + S_{1+2i}^{+1} + S_1^{+i} - S_{1+2i}^{+i} - S_{1-i}^{-1} - S_{1-2i}^{-1} =$
$F_1(2) + F_2(2)$

(iii) $2S_1^{+i} = F_1(2) + 2F_2(2) + F_3(2)$

(iv) $S_2^{+i} + S_{1-2i}^{+1} + S_{1+2i}^{-1} - S_{1+2i}^{+i} - S_{1-2i}^{+i} - S_1^{+i} =$
$F_0(2) + F_1(2) + F_2(3)(5) - 3F_2(2) + F_3(2)$

Lemma 2. *If S_α^{+i} is in $\mathbf{G}^\star$ for all α, then $F_j(3 \pm i) \in \mathbf{G}$ for $j = 0, 1, 2, 3$.*

PROOF. As part of the proof in Lemma 1. it was shown that $F_2(5) \in \mathbf{G}$ and that $F_2(3) \in \mathbf{G}$. The following eight equations now prove the Lemma; first for $j = 2$, then for $j = 1$, next for $j = 3$ and finally for $j = 0$.

(i) $S_3^{+1} + S_2^{+i} + S_1^{+i} - S_3^{-i} = F_0(2) + 2F_2(2) + F_2(3)(3 - i) + F_3(2)$

(ii) $F_2(3 - i) + F_2(3 + i) = F_2(2) + F_2(5)$

(iii) $S^{+1}_{2+i} - S^{+i}_{1} - S^{+i}_{2+i} = F_1(3+i) - 2F_1(2) - F_2(3+i) - F_3(2)$

(iv) $S^{+1}_{2-i} - S^{-i}_{2-i} - S^{+i}_{1} = F_1(3-i) - 2F_1(2) - F_3(2)$

(v) $S^{-i}_{2} + S^{+i}_{1} = F_0(2) + F_1(3+i) + F_2(2) + F_3(3-i)$

(vi) $S^{+i}_{2} + S^{+i}_{1} = F_0(2) + F_1(3-i) + F_2(2)(3) + F_3(3+i)$

(vii) $S^{+1}_{2+i} + S^{-1}_{1-i} = F_0(3-i) + F_1(3+i) + F_2(2) + F_3(3+i)$

(viii) $S^{+1}_{2-i} + S^{-1}_{1+i} = F_0(3+i) + F_1(3-i) + F_2(2)(3+i) + F_3(3-i)$

Lemma 3. *If $S^{+i}_{\alpha} \in \mathbf{G}^{\star}$ for all α, then $F_j(1+i) \in \mathbf{G}$ for $j = 0,1,2,3$.*

PROOF. By Lemma 2, if $F_j(1+i) \in \mathbf{G}$ then $F_j(2 \pm i) \in \mathbf{G}$. It also follows from Lemma 2. that $F_j(10) \in \mathbf{G}$ and hence, by Lemma 1, that $F_j(5) \in \mathbf{G}$ for $j = 0,1,2,3$. But then $F_j(3 \pm 4i) \in \mathbf{G}$ because

$$F_j(3+4i) = F_j(3-i) - F_j(3+i) + F_j(5) \quad \text{and}$$

$$F_j(3-4i) = F_j(3+i) - F_j(3-i) + F_j(5).$$

The Lemma now follows from the following four equations:

(i) $S^{+1}_{2+i} + S^{+1}_{2-i} - 2S^{+i}_{1} = F_0(5) + F_1(5) + F_2(1+i)$

(ii) $S^{+i}_{3+3i} + S^{+1}_{2} - S^{-i}_{3+i} - S^{-1}_{3} =$
$[F_0(2)(1+i) + F_1(3+4i) + F_2(4+3i)] -$
$[F_0(3+i) + F_1(2) + F_2(1+3i) + 2F_3(2)]$

(iii) $S^{+1}_{2+i} = F_0(2+i) + F_1(3+i) + F_2(2) + F_3(1+i)$

(iv) $S^{+i}_{1} = F_1(1+i) + F_2(2) + F_3(1-i)$

This finishes the proof of Lemma 3.

By Lemma 2. and 3, $F_j(2 \pm i) \in \mathbf{G}$ for $j = 0,1,2,3$. It has therefore been shown that $F_j(\alpha) \in \mathbf{G}^{\star}$ for all α with norm ≤ 8.

We now finish the proof of Theorem 3. by induction on the norm of α. We denote the norm of α by $\mathcal{N}(\alpha)$.

If the Theorem is false, there is some α in $\mathbf{G}$ with smallest norm, for which $F_j(\alpha)$ is not in $\mathbf{G}$ for some $j = 0,1,2$ or 3. Clearly one may assume α to be a prime element of $\mathbf{G}$. Let $\alpha = (a+bi)$. There is no loss in generality if one assumes that $a \geq 1$ and $b \geq 0$ since either α or one of its associates will have this property.

Since $\mathcal{N}(\alpha) > 8$, α is not equal to $(1 \pm i)$. Therefore $(\alpha \pm 1)$ and $(\alpha \pm i)$ are all composite elements of $\mathbf{G}$. In addition, $\mathcal{N}(\alpha \pm 1)$ and $\mathcal{N}(\alpha \pm i)$ are both strictly less than $2\mathcal{N}(\alpha)$. Therefore

$$(19) \qquad F_j(\alpha \pm 1) \in \mathbf{G} \text{ and } F_j(\alpha \pm i) \in \mathbf{G}, \quad \text{for } j = 0, 1, 2, 3.$$

While $(\alpha - 1 - i)$ need not be composite, for all α under consideration the norm of $(\alpha - 1 - i)$ is strictly less than the norm of α. Therefore

$$(20) \qquad F_j(\alpha - 1 - i) \in \mathbf{G}, \quad \text{for } j = 0, 1, 2, 3.$$

Since $S_\alpha^{+i} = F_0(\alpha) + F_1(\alpha + i) + F_2(\alpha + 1) + F_3(\alpha - i)$, by (19) and the fact that $S_\alpha^{+i} \in \mathbf{G}$, it follows that $F_0(\alpha) \in \mathbf{G}$.

When considering $F_1(\alpha)$ we distringuish two cases.

Case 1: $a > 1$. Then $\mathcal{N}(\alpha - 2) < \mathcal{N}(\alpha)$ and therefore $F_3(\alpha - 2) \in \mathbf{G}$. Since $S_{\alpha-1}^{+1} = F_0(\alpha - 1) + F_1(\alpha) + F_2(\alpha - 1 - i) + F_3(\alpha - 2)$ it follows from (16), (19) and (20) that $F_1(\alpha) \in \mathbf{G}$.

Case 2: $a = 1$. Then b must be greater than 3. Therefore $\mathcal{N}(\alpha + 1 - i)$ as well as $\mathcal{N}(\alpha - 2i)$ are strictly less than $\mathcal{N}(\alpha)$ and $F_2(\alpha + 1 - i)$ as well as $F_3(\alpha - 2i)$ are elements of $\mathbf{G}$. Since $S_{\alpha-i}^{+i}$ is in $\mathbf{G}$ we conclude that also in this case $F_1(\alpha) \in \mathbf{G}$.

To show that $F_2(\alpha) \in \mathbf{G}$, we consider separately the case $b \geq (a + 1)$ and the case $b \leq (a - 1)$, which is sufficient since $a \neq b$ because α is a prime.

Case 1: $b \geq (a + 1)$. The fact that $S_{\alpha-i}^{-1} \in \mathbf{G}$, together with (19) and (20) implies that $F_2(\alpha) \in \mathbf{G}$ if and only if $F_3(\alpha + 1 - i) \in \mathbf{G}$.
If $b > (a + 1)$ then $\mathcal{N}(\alpha + 1 - i) < \mathcal{N}(\alpha)$ and $F_3(\alpha + 1 - i) \in \mathbf{G}$.
If $b = a + 1$ then $\mathcal{N}(\alpha + 1 - i) = \mathcal{N}(\alpha)$. But since $\mathcal{N}(\alpha) > 8$, b is strictly greater than 2 and therefore $(\alpha + 1 - 2i)$, $(\alpha - 2i)$ and $(\alpha + 1 - 3i)$ all have norm less than $\mathcal{N}(\alpha)$. From $S_{\alpha+1-2i}^{-i} \in \mathbf{G}$ it follows that $F_3(\alpha + 1 - i) \in \mathbf{G}$.

Case 2: $b \leq (a - 1)$. The argument is similar. Considering $S_{\alpha-1}^{+i}$ we see that $F_2(\alpha) \in \mathbf{G}$ if and only if $F_1(\alpha - 1 + i) \in \mathbf{G}$ which is true when $b < (a - 1)$. When $b = (a - 1)$, a is strictly greater than 2 and therefore $\mathcal{N}(\alpha - 2)$, $\mathcal{N}(\alpha - 3 + i)$ and $\mathcal{N}(\alpha - 2 + i)$ are all less than $\mathcal{N}(\alpha)$. By $[S_{\alpha-2+i}^{+1} \in \mathbf{G}]$ it then follows that $F_1(\alpha - 1 + i)$ is an element of $\mathbf{G}$.

In both cases it follows that $F_2(\alpha) \in \mathbf{G}$.

Remains to show that $F_3(\alpha) \in \mathbf{G}$.

Case 1: $b \geq 2$. We conclude from $[S_{\alpha-i}^{-i} \in \mathbf{G}]$, (19) and (20) that $F_3(\alpha) \in \mathbf{G}$ because $\mathcal{N}(\alpha - 2i) < \mathcal{N}(\alpha)$.

Case 2: $b < 2$, e.g. $b = 0$ or 1. Since $a > 3$, it follows that $\mathcal{N}(\alpha - 2)$ and $\mathcal{N}(\alpha - 1 + i)$ both are less than $\mathcal{N}(\alpha)$. We conclude from $[S_{\alpha-1}^{-1} \in \mathbf{G}]$ and (19) that $F_3(\alpha) \in \mathbf{G}$.

This finishes the proof of Theorem 3.

An extension of the results of Theorem 1. to more than four functions seems difficult to prove, just as it is in the case considered in [1]. At the same time, the complete additivity of the functions F_j thwarts the attempts to construct a counter example and leaves one with the strong impression that a similar Theorem holds for an arbitrary number of functions.

References

[1] I. KÁTAI, On additive functions satisfying a congruence, *Acta Sci. Math.* **47** (1984), 85–92.

MARIJKE VAN ROSSUM-WIJSMULLER
LA SALLE UNIVERSITY
MATHEMATICAL SCIENCES DEPT.
PHILADELPHIA, PA 19141

(Received July 7, 1988)